Approximation of maps on C*-algebras by completely contractive elementary operators

Bojan Magajna

Abstract. Let $\overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$ be the closure in the point-norm topology of the set of all completely contractive elementary operators on a C*-algebra A. If $\psi \leq \phi$ are completely positive maps on A and $\phi \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$, then $\psi \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$. A completely positive contraction ϕ on a von Neumann algebra R is in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$ if and only if the normal and the singular part of ϕ are both in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$. Maps on R admitting pointwise approximation by *sequences* of elementary complete contractions may have additional properties that are not shared by all maps in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$. A specific example on $\mathrm{B}(\mathcal{H})$ is also studied.

Mathematics Subject Classification (2000). Primary 46L07; Secondary 47B47.

Keywords. Elementary operators, complete contractions, C*-algebras, von Neumann algebras.

1. Introduction and notation

Given a Banach algebra A, we may ask which operators on A can be approximated by elementary operators, either uniformly or pointwise. Clearly all such operators must preserve ideals of A (where by an ideal we mean a closed two-sided ideal) and for C*-algebras this necessary condition is also sufficient for the pointwise approximation [2], [13]. Here by an operator we shall always mean a linear bounded operator.

A variant of the above problem is to characterize all Banach or all C*-algebras A such that every operator on A that preserves ideals can be approximated uniformly by elementary operators. Separable such C*-algebras are characterized in [17] (they are just finite direct sums of homogeneous C* algebras of finite type), but for nonseparable C*-algebras and more general Banach algebras the problem

Partially supported by the Ministry of Science and Education of Slovenia.
The author is grateful to Milan Hladnik and Aleksej Turnšek for their remarks concerning the paper.

has not yet been studied, as far as we know. Since the set of all elementary operators on a (unital) C*-algebra A coincides with the range of the natural map from the algebraic tensor product $A \otimes_Z A$ (over the center Z of A) to A, this question is in some sense dual to the problem of characterizing all C* algebras A for which the multiplication map is (completely) isometric when $A \otimes_Z A$ is equipped with the Haagerup (semi-)norm. The latter problem received much attention (see [8], [20], [6], [1], [14], [2] and the references there) and a definitive solution has been obtained recently by Somerset [21] and Archbold, Somerset and Timoney [3].

On a C*-algebra every elementary operator ψ is a *completely bounded* map in the sense that $\|\psi\|_{\rm cb} = \sup_n \|\psi_n\| < \infty$, where for each $n \in \mathbb{N}$ we denote by ψ_n the map on $\mathrm{M}_n(A)$ obtained by applying ψ entrywise. So, in this case we may ask a sharper question: which complete contractions ϕ on A (that is, $\|\phi\|_{\rm cb} \leq 1$) that preserve closed two-sided ideals, can be approximated pointwise by elementary complete contractions (that is, by completely contractive elementary operators) ? We note that the approximation on singletons is always possible [15], the problem is the simultaneous approximation on more general finite subsets of a C*-algebra. It turns out that this question is also related to C*-tensor products of C*-algebras [16]. (However, here we will not study the relation to tensor products of C*-algebras and no knowledge of the theory of tensor products of C*-algebras is needed to read this paper.)

Since the Calkin algebra $\mathrm{C}(\mathcal{H}) = \mathrm{B}(\mathcal{H})/\mathrm{K}(\mathcal{H})$ of a separable Hilbert space is algebraically simple, it is a consequence of the classical Jacobson density theorem that every linear operator ϕ on $\mathrm{C}(\mathcal{H})$ coincides on each finite dimensional subspace V of $\mathrm{C}(\mathcal{H})$ with some elementary operator ψ_V. But if $\|\phi\|_{\rm cb} < 1$, in general ψ_V can not be chosen to be completely contractive; namely, as observed in [16], there is a lifting obstruction for such an approximation. Thus, the requirement that the approximands should be completely contractive is essential for the character of the approximation problem studied here.

In Section 2 we will reformulate the above question and provide a different approach to a part of the main result of [16].

In Section 3 we will deduce some consequences for completely positive maps of the result of Section 2, needed later. In particular, for completely positive contractions $\psi \leq \phi$ on a C*-algebra A we show that $\phi \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$ implies $\psi \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$, where $\overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$ denotes the point-norm closure of the set $\mathrm{E}_1(A)$ of all elementary operators on a C*-algebra A that can be represented by tensors in $A \otimes A$ with the Haagerup norm at most 1. (By Theorem 2.1 below this is the same as the point-norm closure of all completely contractive elementary operators on A.)

Using these we will show in Section 4 that a completely positive contraction on a von Neumann algebra R is in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$ if and only if its normal and its singular part are both in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$. We will also note that if a map ϕ on R can be approximated pointwise by a *sequence* of completely contractive elementary operators, then its second adjoint ϕ^{**} on R^{**} is a module homomorphism over the center of R^{**}, and that this property does not hold in general for all maps in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$.

We do not know of any example of a non-nuclear C*-algebra A on which all complete contractions that preserve ideals are in $\overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$. Even the situation on $\mathrm{B}(\mathcal{H})$ is not completely clear to the author. Recall that every bounded linear map ϕ on $\mathrm{B}(\mathcal{H})$ can be decomposed uniquely as $\phi = \phi_{sing} + \phi_{nor}$, where ϕ_{nor} is normal (meaning weak* continuous) and ϕ_{sing} is *singular* (which is equivalent to $\phi_{sing}(\mathrm{K}(\mathcal{H})) = 0$) . If ϕ is completely contractive, Voiculescu's theorem [4], [23] can be used to approximate the singular part ϕ_{sing} by two-sided multiplications of the form $x \mapsto axb$, where a and b are contractions on $\mathcal{H}$, and the approximation problem can be reduced to normal maps (Section 5). It is well known that all normal completely bounded maps on $\mathrm{B}(\mathcal{H})$ ($\mathcal{H}$ separable) are of the form $\phi(x) = \sum_{i=1}^{\infty} a_i^* x b_i$, where a_i and b_i are elements of $\mathrm{B}(\mathcal{H})$ such that the two infinite columns $a = [a_1, a_2, \ldots]^T$ and $b = [b_1, b_2, \ldots]^T$ represent bounded operators from $\mathcal{H}$ into $\mathcal{H}^\infty$ (in other words, the two sums $\sum a_i^* a_i$ and $\sum b_i^* b_i$ are convergent in the strong operator topology). Occasionally we shall denote such a map ϕ by $a^* \odot b$. But such maps do not always preserve compact operators: for example, if e_{ij} are the matrix units of $\mathrm{B}(\mathcal{H})$ (with respect to some orthonormal basis of $\mathcal{H}$), the map $x \mapsto \sum e_{i1} x e_{1i}$ sends the rank 1 projection e_{11} into the identity operator, which is not compact. In Section 5 we will consider a special class of normal complete contractions on $\mathrm{B}(\mathcal{H})$ that do preserve the ideal $\mathrm{K}(\mathcal{H})$ of compact operators and show that they can not always be approximated pointwise by *sequences* of elementary complete contractions. (Since $\mathrm{B}(\mathcal{H})$ is not separable in norm, this does not imply that such maps are not in $\overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$.) Of course, the approximation in the point-weak* topology is always possible in $\mathrm{B}(\mathcal{H})$. More generally it follows from results of Chatterjee and Smith [6] that each complete contraction on an injective von Neumann algebra R which preserves weak* closed two-sided ideals of R can be approximated by elementary complete contractions in the point-weak* topology.

We shall denote by Φ the universal representation of a von Neumann (or a C*-algebra) R. Then the weak* closure $\overline{\Phi(R)}$ of $\Phi(R)$ (in $\mathrm{B}(\mathcal{K})$, where $\mathcal{K}$ is the Hilbert space of Φ) is naturally identified with the bidual R^{**} of R. Each functional in R^* can be extended uniquely to a normal functional on R^{**} and the map $\Phi^{-1} : \Phi(R) \to R$ has the weak* continuous extension $\overline{\Phi^{-1}} : R^{**} \to R$ (or to $\overline{R}$ if R is merely a C*-algebra represented on a Hilbert space). Since $\overline{\Phi^{-1}}$ is a $*$-homomorphism, the kernel of $\overline{\Phi^{-1}}$ is a weak* closed ideal in R^{**}, hence of the form $P^\perp R^{**}$ for a central projection $P \in R^{**}$. Since the map ϕ^{**} on R^{**} is weak* continuous, the composite $\overline{\phi} := \overline{\Phi^{-1}}\phi^{**}$ is also weak* continuous and is the unique weak* continuous extension of $\phi\Phi^{-1}$ to a map $R^{**} \to R$. The normal and the singular part of ϕ are then given by $\phi_{nor}(x) = \overline{\phi}(P\Phi(x))$ and $\phi_{sing}(x) = \overline{\phi}(P^\perp\Phi(x))$ $(x \in R)$. More explanation on this can be found e.g. in [12, Section 10.1].

2. A preliminary result

The following result is a reformulation of a part of [16, Theorem 2.1] (with the additional observation, that it suffices to consider the universal representations). We will give a proof, different from that in [16].

Theorem 2.1. *Let ϕ be a complete contraction on a C*-algebra A such that $\phi(J) \subseteq J$ for each closed two-sided ideal J of A. Then ϕ can be approximated pointwise by a net of elementary complete contractions ψ_k if and only if for each representation $\pi : A \to \mathrm{B}(\mathcal{H})$ and all finite subsets $\{x_1, \ldots, x_n\}$ of A and $\{x'_1, \ldots, x'_n\}$ of $\pi(A)'$ (the commutant of $\pi(A)$) the inequality*

$$\|\sum_{i=1}^{n} x'_i \pi(\phi(x_i))\| \leq \|\sum_{i=1}^{n} x'_i \pi(x_i)\| \tag{2.1}$$

holds. Moreover, it suffices that (2.1) holds for the universal representation of A, and then the approximating operators ψ_k can be chosen to be of the form $\psi_k(x) = \sum_i a_{i,k} x b_{i,k}$ $(a_{i,k}, b_{i,k} \in A)$ with $\|\sum_i a_{i,k} a^*_{i,k}\| \leq 1$, $\|\sum_i b^*_{i,k} b_{i,k}\| \leq 1$ (that is, the Haagerup norms of tensors corresponding to the maps ψ_k are at most 1).

Proof. Suppose first that ϕ is an elementary operator on A with $\|\phi\|_{\mathrm{cb}} < 1$. Let $\pi : A \to \mathrm{B}(\mathcal{H})$ be a representation of A and ϕ_π the elementary operator induced by ϕ on the weak* closure R of $\pi(A)$. Since $\|\phi_\pi\|_{\mathrm{cb}}$ is equal to the norm of the tensor corresponding to ϕ_π in the central Haagerup tensor product $R \otimes^h_Z R$ (see [6]), we have that ϕ_π is of the form $\phi_\pi(x) = \sum_j c^*_j x d_j$ $(x \in R)$, where $c_j,\ d_j \in R$ and the columns $c := (c_1, \ldots, c_m)^T$ and $d := (d_1, \ldots, d_m)^T$ have norms less than 1. Then for any $x_i \in A$ and $x'_i \in \pi(A)'$ $(i = 1, \ldots, n)$ we compute (denoting by $x^{(m)}$ the $m \times m$ diagonal matrix with x along the diagonal) that

$$\begin{aligned}
\|\textstyle\sum_i x'_i \pi(\phi(x_i))\| &= \|\textstyle\sum_i x'_i \phi_\pi(\pi(x_i))\| &&= \|\textstyle\sum_i x'_i \sum_j c^*_j \pi(x_i) d_j\| \\
&= \|c^*(\textstyle\sum_i x'_i \pi(x_i))^{(m)} d\| &&\leq \|c\| \|\textstyle\sum_i x'_i \pi(x_i)\| \|d\| \\
&\leq \|\textstyle\sum_i x'_i \pi(x_i)\|.
\end{aligned}$$

This proves (2.1) in the case ϕ is elementary with $\|\phi\|_{\mathrm{cb}} < 1$, the general case follows by an approximation.

Suppose now that (2.1) holds for the universal representation of A. We will regard A as a C*-subalgebra of its universal von Neumann envelope $A'' = A^{**}$ (and so regard the universal representation as the inclusion). So the condition (2.1) simply says that

$$\|\sum_{i=1}^{n} \phi(x_i) x'_i\| \leq \|\sum_{i=1}^{n} x_i x'_i\|$$

for all finite subsets $\{x_1, \ldots, x_n\}$ of A and $\{x'_1, \ldots, x'_n\}$ of A', which means that ϕ extends to a contractive A'-bimodule homomorphism $\tilde{\phi}$ of the C*-algebra $\overline{\overline{AA'}}$ generated by A and A' inside $\mathrm{B}(\mathcal{K})$, where $\mathcal{K}$ is the Hilbert space of the universal representation of A. Since every normal state on A'' (that is, every state on A) is a vector state [12, Section 10.1], every finite subset of $\mathcal{K}$ is contained in a cyclic subspace for A' by [20, Lemma 2.3] and it follows that $\tilde{\phi}$ is completely contractive by [20, Theorem 2.1 and Remark 2.2]. Then by an appropriate variant of the Wittstock extension theorem [5], [18, p. 116] ϕ can be extended to a completely contractive A'-bimodule homomorphism ψ on $\mathrm{B}(\mathcal{K})$. By [8] the space $CB_{A'}(\mathrm{B}(\mathcal{K}))_{A'}$ of all completely bounded A'-bimodule homomorphisms on $\mathrm{B}(\mathcal{K})$ can be identified with the second dual $(A \overset{h}{\otimes} A)^{**}$ of the Haagerup tensor product $A \overset{h}{\otimes} A$ completely isometrically and weak* homeomorphically by the map which extends the well known complete isometry $A \overset{h}{\otimes} A \to \mathrm{B}(\mathcal{K})$. (Perhaps a simpler proof of this is in [7]: on the level of preduals the corresponding map is the result of the following identifications: $\mathcal{K}^* \overset{h}{\otimes}_{A'} \mathrm{B}(\mathcal{K}) \overset{h}{\otimes}_{A'} \mathcal{K} = \mathcal{K}^* \overset{eh}{\otimes}_{A'} (\mathcal{K} \overset{eh}{\otimes} \mathcal{K}^*) \overset{eh}{\otimes}_{A'} \mathcal{K} = (\mathcal{K}^* \overset{h}{\otimes}_{A'} \mathcal{K}) \overset{eh}{\otimes} (\mathcal{K}^* \overset{h}{\otimes}_{A'} \mathcal{K}) = A^* \overset{eh}{\otimes} A^*$. Here $\overset{eh}{\otimes}$ denotes the extended Haagerup tensor product and we have used some relations from [7].) It follows that ψ is the point-weak* limit of a net ψ_k of elementary complete contractions on $\mathrm{B}(\mathcal{K})$ of the form $\psi_k(x) = \sum_i a_{i,k} x b_{i,k}$, where $a_{i,k}, b_{i,k} \in A$, for each k the sum is finite and $\sum_i a_{i,k} a_{i,k}^* \leq 1$, $\sum_i b_{i,k}^* b_{i,k} \leq 1$. Since such elementary operators map A into A and each functional in A^* extends to a normal functional on $\mathrm{B}(\mathcal{K})$, it follows that the restrictions $\psi_k|A$ converge to $\phi = \psi|A$ in the point-weak topology. Since the point-weak and the point-norm topology have the same continuous linear functionals, it follows by a well known convexity argument that ϕ can be approximated by elementary complete contractions of the required form. □

3. Pointwise approximation of completely positive maps by completely contractive elementary operators

Proposition 3.1. *Let ϕ and ψ be completely positive maps on a C^*-algebra A such that $\psi \leq \phi$ (that is, $\phi - \psi$ is completely positive). If $\phi \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$, then $\psi \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$.*

Proof. We shall use Theorem 2.1. Given a representation π of A and finite subsets $\{x_1, \ldots, x_n\}$ of A and $\{x_1', \ldots, x_n'\}$ of $\pi(A)'$, we denote by x the $n \times n$ matrix which has the first row $(x_1, \ldots, x_n)$ and the remaining rows all 0, and by x' the matrix with the first column $(x_1', \ldots, x_n')^T$ and the remaining columns 0. We denote by π, ϕ and ψ also the amplifications of these maps to the matrix algebra $\mathrm{M}_n(A)$. Then, using the Schwarz inequality for completely positive maps, we compute that

$$\begin{array}{rcl} \|\sum_j \pi(\psi(x_j))x_j'\|^2 & = & \|\sum_{i,j} {x_i'}^* \pi(\psi(x_i)^* \psi(x_j))x_j'\| \quad = \quad \|{x'}^* \pi(\psi(x)^* \psi(x))x'\| \\ & \leq & \|{x'}^* \pi(\psi(x^* x))x'\| \qquad\qquad\qquad \leq \quad \|{x'}^* \pi(\phi(x^* x))x'\| \\ & = & \|\sum_{i,j} \pi(\phi(x_i^* x_j)){x'}_i^* x_j'\| \qquad\qquad \leq \quad \|\sum_{i,j} \pi(x_i^* x_j){x'}_i^* x_j'\| \\ & = & \|\sum_j \pi(x_j)x_j'\|^2, \end{array}$$

where for the last inequality we have used the condition (2.1) for the map ϕ and the subsets $\{x_i^* x_j : i, j = 1, \ldots, n\}$ and $\{{x'}_i^* x_j' : i, j = 1, \ldots, n\}$. □

Lemma 3.2. *Let ψ and θ be completely positive maps on a unital C*-algebra A and let $h, k \in A^+$ be such that $h^2 + k^2 \leq 1$. If for all $t \in (0, 1)$ the two maps $h^t\psi h^t$ and $k^t\theta k^t$ are in $\overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$ then $\phi := h\psi h + k\theta k$ is also in $\overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$.*

Proof. Let $t \in (0, 1)$. For each $x \in A$ we may write

$$\phi(x) = \begin{bmatrix} h^{1-t} & k^{1-t} \end{bmatrix} \begin{bmatrix} h^t\psi(x)h^t & 0 \\ 0 & k^t\theta(x)k^t \end{bmatrix} \begin{bmatrix} h^{1-t} \\ k^{1-t} \end{bmatrix}.$$

If ψ_k and θ_k are nets of maps in $\mathrm{E}_1(A)$ converging pointwise to $h^t\psi h^t$ and $k^t\theta k^t$, respectively, then the elementary operators $\phi_k := h^{1-t}\psi_k h^{1-t} + k^{1-t}\theta_k k^{1-t}$ converge pointwise to ϕ. Further, $\|\phi_k\|_{\mathrm{cb}} \leq \|h^{2(1-t)} + k^{2(1-t)}\|$, so now it suffices to observe that $\limsup_{t\to 0} \|h^{2(1-t)} + k^{2(1-t)}\| \leq 1$. This follows from the relations $\lim_{t\to 0} \|h^{2(1-t)} - h^2\| = 0 = \lim_{t\to 0} \|k^{2(1-t)} - k^2\|$, which can be proved by the functional calculus. □

Lemma 3.3. *Let ψ be a completely positive contraction on a unital C*-algebra A and $h \in A$, $0 \leq h \leq 1$. Suppose that $h\psi h \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$. Then $h^t\psi h^t \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$ for all $t > 0$; more generally, $f(h)^*\psi f(h) \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$ for any continuous function f on the spectrum of h such that $\|f\| \leq 1$ and (if h is not invertible) $f(0) = 0$.*

Proof. Since $h\psi h \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$, $h\psi h$ satisfies the condition (2.1) of Theorem 2.1, hence for each positive constant $c \in \mathbb{R}$ the map $\theta_c := (h + c)^{-1}hf(h)^*\psi f(h)h(h + c)^{-1}$ satisfies the following condition: for every unital representation π of A and every finite subsets $\{x_1, \ldots, x_n\}$ of A and $\{x'_1, \ldots, x'_n\}$ of $\pi(A)'$ the inequality

$$\|\sum_i \pi(\theta_c(x_i))x'_i\| \leq \kappa_c \|\sum_i \pi(x_i)x'_i\|$$

holds, where $\kappa_c = \|f(h)h(h + c)^{-1}\|^2$. This means that the map $\sum_i \pi(x_i)x'_i \mapsto \sum_i \pi(\theta_c(x_i))x'_i$ extends to a bounded map $\tilde{\theta}_c$ on the C*-algebra $\overline{\overline{\pi(A)\pi(A)'}}$. Since ψ is completely positive, it is not hard to verify that

$$\tilde{\theta}_c((\sum_i \pi(x_i)x'_i)^*(\sum_i \pi(x_i)x'_i)) \geq 0,$$

hence $\tilde{\theta}_c$ is positive. A similar argument shows that $\tilde{\theta}_c$ is completely positive, so $\|\tilde{\theta}_c\|_{\mathrm{cb}} = \|\tilde{\theta}_c(1)\| \leq \|\theta_c(1)\| = \|f(h)h(h + c)^{-1}\|^2 \leq 1$. Therefore

$$\|\sum_i \pi(\theta_c(x_i))x'_i\| \leq \|\sum_i \pi(x_i)x'_i\|$$

for any finite sets (x_i) and (x'_i) and representation π as in Theorem 2.1, hence $\theta_c \in \overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$. This holds for every choice of c, so it suffices now to observe that θ_c tends uniformly to $f(h)^*\psi f(h)$ as $c \to 0$ since $\lim_{c\to 0} \|f(h)h(h+c)^{-1} - f(h)\| = 0$ by the functional calculus. □

With a similar proof as that of Lemma 3.3 we can show the following proposition.

Proposition 3.4. *Let A be a unital C^*-algebra. If a completely positive map ϕ is in $\overline{\mathrm{E}_1(\mathrm{A})}^{p.n.}$, then ϕ is in the pointwise closure of a net of elementary completely positive contractions σ_k. If, in addition, ϕ is unital, σ_k can be chosen to be unital.*

Proof. Let $h = \sqrt{\phi(1)}$. If h is invertible, we consider the unital completely positive map $\psi = h^{-1}\phi h^{-1}$ and show that it satisfies (2.1) by using complete positivity (in the same way as in the proof of Lemma 3.3). So by Theorem 2.1 there is a net $\psi_k = a_k^* \odot b_k$ in $\mathrm{E}_1(A)$ converging pointwise to ψ such that $\|a_k\|, \|b_k\| \leq 1$. Since

$$\|a_k - b_k\|^2 = \|(a_k - b_k)^*(a_k - b_k)\| \leq 2\|1 - \mathrm{Re}(a^*b)\| \to 0, \tag{3.1}$$

the net of completely positive contractions $\theta_k = a_k^* \odot a_k$ also converges pointwise to ψ. Hence the net of maps $\sigma_k := h\theta_k h = c_k^* \odot c_k$, where $c_k := a_k h$, converges to $h\psi h = \phi$. If $\phi(1) = 1$,we may replace each σ_k by the unital map $\tau_k(x) = \sigma_k(x) + \sqrt{1 - c_k^* c_k} x \sqrt{1 - c_k^* c_k}$. In general (if h is not invertible), we consider first the maps $(1-t)\phi + t\,\mathrm{id}$ (where id is the identity map on A) and then let $t \to 0$. □

4. Stability of maps in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$ under the decomposition into the singular and the normal part

The following lemma is perhaps well - known.

Lemma 4.1. *Let $\phi : R \to S$ be a completely positive map between von Neumann algebras and denote $\phi(1) = h^2$, where $h \geq 0$. Then there exists a unique completely positive map $\psi : R \to S$ such that $\psi(1)$ is the range projection p of h and $\phi(x) = h\psi(x)h$ for all $x \in R$. Moreover, ψ is normal (singular) if and only if ϕ is normal (singular).*

Proof. We may assume that $\|\phi\|_{\mathrm{cb}} = 1$, so that $\|h\| = 1$. The sequence of completely positive maps

$$\psi_n(x) = (h + \frac{1}{n})^{-1}\phi(x)(h + \frac{1}{n})^{-1} \quad (n \in \mathbb{N})$$

is bounded since $\|\psi_n\|_{\mathrm{cb}} = \|\psi_n(1)\| \leq 1$, hence it has a limit point ψ in the point-weak* topology. From $\phi(x) = (h+\frac{1}{n})\psi_n(x)(h+\frac{1}{n})$ we conclude that $\phi(x) = h\psi(x)h$ for all $x \in R$. Since the sequence $\psi_n(1) = (h+\frac{1}{n})^{-1}h^2(h+\frac{1}{n})^{-1}$ converges strongly to the range projection p of h, we have that $\psi(1) = p$. This proves the existence of ψ. The uniqueness of ψ is a consequence of the fact that the range of ψ must be contained in pSp (since $0 \leq \psi(x) \leq \psi(1) = p$ if $0 \leq x \leq 1$). Namely, if $\tilde{\psi}$ is another such map, then the difference $\theta := \tilde{\psi} - \psi$ satisfies $h\theta(x)h = \phi(x) - \phi(x) = 0$. But, since $\theta(x) = p\theta(x)p$ and the operator $h = ph = hp$ is injective on the range of p, this implies that $\theta(x) = 0$ for all (positive) $x \in R$.

We shall now use the notation from the last paragraph of the Introduction. From $\phi = h\psi h$ we have that $\overline{\phi} = h\overline{\psi}h$, hence, if ϕ is normal, then $h\psi(x)h = \phi(x) = \overline{\phi}(P\Phi(x)) = h\overline{\psi}(P\Phi(x))h$ for all $x \in R$. Since ψ and $\overline{\psi}$ both have their ranges contained in pSp, this implies that $\psi(x) = \overline{\psi}(\Phi(x)P)$ $(x \in R)$, so ψ is normal. The converse and the statement about singular maps is proved similarly. □

Corollary 4.2. *A completely positive contraction ϕ on a von Neumann algebra R is in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$ if and only if its normal part ϕ_{nor} and its singular part ϕ_{sing} are both in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$.*

Proof. If $\phi \in \overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$, Proposition 3.1 implies that ϕ_{nor}, $\phi_{sing} \in \overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$. To prove the converse, by Lemma 4.1 we write $\phi_{nor} = h\psi h$ and $\phi_{sing} = k\theta k$ where ψ is normal, θ is singular, both are completely positive contractions on R and $h, k \in R^+$ satisfy $h^2 = \phi_{nor}(1)$, $k^2 = \phi_{sing}(1)$, hence $h^2 + k^2 = \phi(1) \leq 1$. Since ϕ_{nor} and ϕ_{sing} are in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$, it follows now from Lemma 3.3 that $h^t\psi h^t$ and $k^t\theta k^t$ are in $\overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$ for all $t > 0$. Then Lemma 3.2 implies that $\phi \in \overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$. □

The author does not know if Corollary 4.2 can be extended to all completely contractive maps. Note, however, that if ϕ is a point-norm limit of a *sequence* of elementary operators (or more generally, of a sequence of normal maps) ϕ_k on a von Neumann algebra R, then ϕ is necessarily normal. Indeed, for each ω in the predual R_* of R the sequence $(\omega \circ \phi_k)$ in R_* converges weakly to $\omega \circ \phi$, but R_* is weakly sequentially complete [22, 5.2], so $\omega \circ \phi \in R_*$ and ϕ is normal. Here is a somewhat related result.

Proposition 4.3. *If ϕ is a bounded linear map on a von Neumann algebra R which preserves norm closed two-sided ideals of R, then ϕ_{nor} and ϕ_{sing} also preserve such ideals.*

Proof. We shall use again the notation from the last paragraph of the Introduction. If J is a norm closed ideal in R, then J^{**} coincides with the weak* closure of $\Phi(J)$ in R^{**}, hence is of the form $J^{**} = pR^{**}$ for a central projection $p \in R^{**}$. From $\phi(J) \subseteq J$ we have that $\phi^{**}(J^{**}) \subseteq J^{**}$, hence $\phi^{**}(py) = p\phi^{**}(py)$ for all $y \in R^{**}$. Let $x \in J$ and apply the last equality to $y = P\Phi(x) = Pp\Phi(x) = pP\Phi(x)$ to conclude that $\phi_{nor}(x) = \overline{\phi}(P\Phi(x)) = \overline{\Phi^{-1}}(\phi^{**}(pP\Phi(x))) = \overline{\Phi^{-1}}(p\phi^{**}(pP\Phi(x))) \in \overline{\Phi^{-1}}(J^{**}) \subseteq J$, since $\overline{\Phi^{-1}}$ is weak* continuous and $\Phi^{-1}(\Phi(J)) = J$. Then $\phi_{sing}(J) \subseteq J$ since $\phi_{sing} = \phi - \phi_{nor}$. □

Proposition 4.4. *If a bounded linear map ϕ on a von Neumann algebra R is a point-norm limit of a sequence of elementary operators on R then ϕ^{**} is a module homomorphism over the center of R^{**}. This conclusion does not hold necessarily if we assume only that $\phi \in \overline{\mathrm{E}_1(\mathrm{R})}^{p.n.}$.*

Proof. If ϕ_k is a sequence of elementary operators converging pointwise to ϕ, then for each functional $\rho \in R^*$ the sequence $(\phi_k^*(\rho))$ converges to $\phi^*(\rho) \in R^*$ in the weak* topology, hence, since R is a Grothendieck space [19], the convergence is in the weak topology of R^*. This means that the maps ϕ_k^{**} on R^{**} converge to ϕ^{**} in the point-weak* topology. Note that each ϕ_k^{**} is an elementary operator on R^{**}, hence a module map over the center Z of R^{**}, and that $\rho z \in R^*$ for each $z \in R^{**}$ and $\rho \in R^*$. (Here ρz is defined by $(\rho z)(x) = \overline{\rho}(zx)$, where $\overline{\rho}$ is the unique weak* continuous extension of ρ to a normal functional on R^{**}, and then ρz can be regarded as a normal functional on R^{**}.) For each $y \in R^{**}$ and $z \in Z$ we compute that

$$\langle \phi^{**}(zy), \rho \rangle = \lim_{k\to\infty} \langle \phi_k^{**}(zy), \rho \rangle = \lim_{k\to\infty} \langle z\phi_k^{**}(y), \rho \rangle = \langle \phi^{**}(y), \rho z \rangle = \langle z\phi^{**}(y), \rho \rangle.$$

Thus, ϕ^{**} is a Z-module map.

To show that the approximation by general nets is not sufficient for the above conclusion, let ω_{ξ_k} be a net of vector states on $\mathrm{B}(\mathcal{H})$ (ξ_k a unit vector in $\mathcal{H}$) weak* converging to a non-normal state ω (by a result of Glimm [10] the weak* closure of vector states on $\mathrm{B}(\mathcal{H})$ consists precisely of states of the form $t\rho + (1-t)\theta$, where $t \in [0,1]$, ρ is a vector state and θ is a state annihilating $\mathrm{K}(\mathcal{H})$). Let $x_0 \in \mathrm{K}(\mathcal{H})$ be the rank 1 operator $\eta\otimes\eta^*$, where $\eta \in \mathcal{H}$ is a unit vector. Then the net (ϕ_k) of complete contractions on $\mathrm{B}(\mathcal{H})$ defined by $\phi_k(x) := \omega_{\xi_k}(x)x_0$ $(x \in \mathrm{B}(\mathcal{H}))$ converges pointwise to the map ϕ defined by $\phi(x) = \omega(x)x_0$. Note that each ϕ_k is an elementary operator (which acts as $x \mapsto (\eta\otimes\xi_k^*)x(\xi_k\otimes\eta^*)$). Nevertheless, ϕ^{**} is not a module map over the center Z of $\mathrm{B}(\mathcal{H})^{**}$. To see this, we shall again use the notation from the last paragraph of the Introduction, with $R = \mathrm{B}(\mathcal{H})$. If ϕ^{**} were a $Z-$module map, then in particular $\phi^{**}(Py) = P\phi^{**}(y)$ for all $y \in \mathrm{B}(\mathcal{H})^{**}$, hence the map $\overline{\phi} = \overline{\Phi^{-1}}\phi^{**}$ would satisfy $\overline{\phi}(Py) = \overline{\Phi^{-1}}(P\phi^{**}(y)) = \overline{\Phi^{-1}}(\phi^{**}(y) = \overline{\phi}(y)$. But then ϕ would be weak* continuous on $\mathrm{B}(\mathcal{H})$ [12, p. 721], a contradiction. □

5. Maps on $\mathrm{B}(\mathcal{H})$

In this section let $\mathcal{H}$ be a separable Hilbert space and $\mathrm{K}(\mathcal{H})$ the ideal of compact operators in $\mathrm{B}(\mathcal{H})$.

Proposition 5.1. *If there exists a complete contraction on $\mathrm{B}(\mathcal{H})$ which preserves $\mathrm{K}(\mathcal{H})$ and is not in $\overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$ then there exists a normal such map. All singular complete contractions on $\mathrm{B}(\mathcal{H})$ are in $\overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$.*

Proof. Suppose that ϕ is a complete contraction on $\mathrm{B}(\mathcal{H})$ such that $\phi(\mathrm{K}(\mathcal{H})) \subseteq \mathrm{K}(\mathcal{H})$. Given any separable unital C*-subalgebra A of $\mathrm{B}(\mathcal{H})$ with $A \supseteq \mathrm{K}(\mathcal{H})$, by the Stinespring theorem we may represent $\phi|A$ as $\phi|A = V^*\pi W$, where π is a representation of A on a separable Hilbert space $\mathcal{K}$ and $V, W : \mathcal{H} \to \mathcal{K}$ are contractions. Let $\pi = \theta \oplus \sigma$ be the decomposition of π into the normal part θ (a multiple of the identity representation) and the singular part σ which annihilates $\mathrm{K}(\mathcal{H})$. By Voiculescu's theorem [4], [23] π is approximately unitarily equivalent to θ modulo $\mathrm{K}(\mathcal{H})$ (if $\theta \neq 0$), which means that there is a sequence of unitary operators $U_j : \mathcal{K} \to \mathcal{H}_\theta$ (where $\mathcal{H}_\theta$ is the Hilbert space of θ) such that $\lim_{j\to\infty} \|\pi(x) - U_j^*\theta(x)U_j\| = 0$ and that $\pi(x) - U_j^*\theta(x)U_j$ is compact for all $x \in A$. Then $V^*U_j^*\theta(x)U_jW - \phi(x) = V^*(U_j^*\theta(x)U_j - \pi(x))W$ is compact for all $x \in A$, hence for each j the map $\psi_j := V^*U_j^*\theta U_jW$ preserves $\mathrm{K}(\mathcal{H})$ (since $\phi(\mathrm{K}(\mathcal{H})) \subseteq \mathrm{K}(\mathcal{H})$). So, if all normal complete contractions on $\mathrm{B}(\mathcal{H})$ that preserve $\mathrm{K}(\mathcal{H})$ are in $\overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$, then in particular the (unique) weak* continuous extension $\tilde{\psi}_j$ of ψ_j to $\mathrm{B}(\mathcal{H})$ is in $\overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$. Since $\lim_{j\to\infty} \|\phi(x) - \psi_j(x)\| = 0$ for all $x \in A$, it follows that $\phi|A$ can be approximated pointwise on A by a sequence of elementary complete contractions with coefficients in $\mathrm{B}(\mathcal{H})$. But, as we may choose A to contain any given finite subset of $\mathrm{B}(\mathcal{H})$, this implies that $\phi \in \overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$.

If ϕ is singular, then π can be taken singular in the above argument (by the Stinespring construction) and then by (another) Voiculescu's theorem [4], [23] there is a sequence of isometries $V_j : \mathcal{K} \to \mathcal{H}$ such that $\|\pi(x) - V_k^* x V_k\| = 0$ for all $x \in A$, so that $\phi|A$ is in the point norm closure of the sequence of operators $x \mapsto V^* V_k^* x V_k W$ and consequently $\phi \in \overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$. □

As noted already in the Introduction, a general normal complete contraction on $\mathrm{B}(\mathcal{H})$ does not necessarily preserve the ideal $\mathrm{K}(\mathcal{H})$ and therefore can not be approximated pointwise in norm by elementary operators. However, for every orthogonal sequence of projections $p_i \in \mathrm{B}(\mathcal{H})$ the completely positive contraction ϕ defined by

$$\phi(x) = \sum_{i=1}^{\infty} p_i x p_i \quad (x \in \mathrm{B}(\mathcal{H})) \tag{5.1}$$

preserves $\mathrm{K}(\mathcal{H})$, for ϕ is just the block-diagonal compression, that is, it sends x to the direct sum of operators $p_i x p_i$. Although the finite partial sums $\sum_{i=1}^n p_i x p_i$ usually do not converge in norm to the infinite sum in (5.1), this does not imply that ϕ can not be approximated pointwise in norm by elementary complete contractions. In fact, if the ranks of the projections p_i are bounded, the range of ϕ is contained in a nuclear C^* algebra and $\phi \in \overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$ by [16]. On the other hand, if the ranks of the projections p_i (are finite, but) increase to the infinity (or, if infinitely many of the p_i's have infinite rank), then perhaps $\phi \notin \overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$, but we will only prove a weaker result. For simplicity we assume that $\sum p_i = 1$, so that ϕ is unital. The range of ϕ is a von Neumann algebra, which we denote by B_ϕ, and ϕ is an idempotent B_ϕ-bimodule map. We suspect that such maps are critical for the approximation problem on $\mathrm{B}(\mathcal{H})$ in the sense that if all maps of the form (5.1) could be approximated pointwise by completely contractive elementary operators then the same would hold for all completely positive contractions that preserve $\mathrm{K}(\mathcal{H})$. We show first that, although ϕ is a B_ϕ-bimodule map, ϕ in general can not be approximated pointwise by completely contractive elementary operators which are B_ϕ-module maps. For this, we need the following finite-dimensional lemma.

Lemma 5.2. *Let $M := \mathrm{M}_n(\mathbb{C})^n$ be embedded diagonally into $N := \mathrm{M}_n(\mathrm{M}_n(\mathbb{C}))$, denote $A := M' \cong \ell_\infty(n)$ (the commutant of M in N) and let $\phi : N \to M$ be the projection $\phi([x_{ij}]) = \oplus_{j=1}^n x_{jj}$. With (e_{ij}) the usual matrix units in $\mathrm{M}_n(\mathbb{C})$, let y be the unitary element in N defined by*

$$y = [e_{ji}] \quad \text{(that is, the entry on the position } (i,j) \text{ is } e_{ji}). \tag{5.2}$$

For each $m \in \mathbb{N}$ denote by $d_{m,n}$ the distance of $\phi(y)$ to the set $S_m := \{\psi(y) : \psi \in \mathrm{EUCP}_M^m(N)\}$, where $\mathrm{EUCP}_M^m(N)$ is the set of all elementary unital completely *positive M-bimodule maps of the form*

$$\psi(x)=\sum_{k=1}^{m}a_k^*xa_k,\ \textit{where } a_k\in A \textit{ and } \sum_{k=1}^{m}a_k^*a_k=1. \tag{5.3}$$

Then for each fixed m

$$\lim_{n\to\infty} d_{m,n}=1.$$

Proof. Given ψ of the form (5.3), noting that each a_k is a diagonal matrix, $a_k=\operatorname{diag}(\alpha_{kj}1)$ ($\alpha_{kj}\in\mathbb{C}$), we compute that the matrix $\psi(y)$ is

$$\psi(y)=\left[\sum_{k=1}^{m}\overline{\alpha}_{ki}e_{ji}\alpha_{kj}\right].$$

Thus, denoting for each i by $\alpha_i\in\mathbb{C}^m$ the vector $\alpha_i=(\alpha_{1i},\dots,\alpha_{mi})$, we have

$$\delta:=\|\psi(y)-\phi(y)\|=\|\,[(\langle\alpha_i,\alpha_j\rangle-\delta_{ij})e_{ji}]\,\|. \tag{5.4}$$

From $\psi(1)=1$ we compute that $\langle\alpha_i,\alpha_i\rangle=1$ for all $i=1,\dots,n$. From (5.4) we have that $|\langle\alpha_i,\alpha_j\rangle|\leq\delta$ if $i\neq j$, hence

$$\|\alpha_i-\alpha_j\|^2=2(1-\operatorname{Re}\langle\alpha_i,\alpha_j\rangle)\geq 2(1-\delta)=:r^2,\ \text{if } i\neq j,\ i,j\in\{1,\dots,n\}. \tag{5.5}$$

This implies that any two of the n balls with centers α_i and radius $r/2$ in $\mathbb{C}^m$ have at most one common point. Since all these balls are contained in the ball of $\mathbb{C}^m$ with center 0 and radius $1+\frac{r}{2}$, we conclude by considering the Euclidean volume that $n(\frac{r}{2})^{2m+1}\leq(1+\frac{r}{2})^{2m+1}$, hence from (5.5)

$$\delta=1-\frac{r^2}{2}\geq 1-2(n^{\frac{1}{2m+1}}-1)^{-2}.$$

Since this holds for all ψ, $d_{m,n}\geq 1-2(n^{\frac{1}{2m+1}}-1)^{-2}$, hence (since $d_{m,n}\leq 1$) $d_{m,n}$ necessarily tends to 1 as $n\to\infty$. □

For notational simplicity we shall consider now only a special kind of maps of the form (5.1). Given a (strictly) increasing sequence (n_i) in $\mathbb{N}$, let $\mathcal{H}=\oplus_{i=1}^{\infty}(\mathbb{C}^{n_i})^{n_i}$ and consider the diagonal unital embeddings

$$B:=\oplus_{i=1}^{\infty}\mathrm{M}_{n_i}(\mathbb{C})^{n_i}\subseteq\oplus_{i=1}^{\infty}N_i\subseteq\mathrm{B}(\mathcal{H}),\ \text{where } N_i=\mathrm{M}_{n_i}(\mathrm{M}_{n_i}(\mathbb{C})). \tag{5.6}$$

For each i let $y_i\in N_i:=\mathrm{M}_{n_i}(\mathrm{M}_{n_i}(\mathbb{C}))$ be the unitary defined by (5.2), so that

$$y:=\oplus_{i=1}^{\infty}y_i \tag{5.7}$$

is a unitary in $\mathrm{B}(\mathcal{H})$. Finally, let ϕ be the projection of $\mathrm{B}(\mathcal{H})$ onto B, so that

$$\phi(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{n_i}p_k^ixp_k^i\quad(x\in\mathrm{B}(\mathcal{H})), \tag{5.8}$$

where p_k^i is the unit of the k-th summand $\mathrm{M}_{n_i}(\mathbb{C})$ of $\mathrm{M}_{n_i}(\mathbb{C})^{n_i}$ inside N_i (a projection in $\mathrm{B}(\mathcal{H})$). Note that

$$B=\operatorname{im}\phi\ \text{ and }\ A:=B'=\oplus_{i=1}^{\infty}\oplus_{k=1}^{n_i}\mathbb{C}p_k^i=\text{the center of } B.$$

Proposition 5.3. *The inequality* $\|\phi(y)-\psi(y)\| \geq 1$ *holds for every unital completely positive map* ψ *on* $\mathrm{B}(\mathcal{H})$ *of the form*

$$\psi(x) = \sum_{j=1}^{m} a_j^* x a_j, \text{ where } a_j \in A \text{ and } m \in \mathbb{N}. \tag{5.9}$$

Further, ϕ *can not be approximated pointwise by elementary complete contractions of the form* $x \mapsto \sum_{j=1}^{m} a_j x b_j$*, where all* a_j *or all* b_j *are in* A*.*

Proof. The first statement follows from Lemma 5.2 by considering the restrictions of ϕ to subalgebras N_i. Then the second statement follows by noting that the estimate (3.1) would allow us to replace completely contractive elementary approximands by unital completely positive elementary approximands as in the proof of Proposition 3.4. □

It can be shown by methods of [20] or [14] that all completely contractive elementary left B-module homomorphisms on $\mathrm{B}(\mathcal{H})$ are of the form $x \mapsto \sum_j a_j x b_j$, where $a_j \in B' = A$ and $b_j \in \mathrm{B}(\mathcal{H})$, hence by Proposition 5.3 the map ϕ can not be approximated pointwise in norm by such homomorphisms.

The question of whether the conditional expectation from $\mathrm{B}(\mathcal{H})$ onto an atomic maximal abelian self-adjoint subalgebra A_0 of $\mathrm{B}(\mathcal{H})$ can be approximated pointwise in norm by completely contractive A_0-bimodule homomorphisms, turns out to be equivalent to the well known paving problem (hence, to the Kadison - Singer problem), but we shall not present a proof here.

Lemma 5.4. *Let* $E : \mathrm{B}(\mathcal{H}) \to A$ *be the diagonal projection,* θ *a unital completely positive map on* $\mathrm{B}(\mathcal{H})$ *of the form*

$$\theta(x) = \sum_{j=1}^{m} b_j^* x b_j, \text{ where } b_j \in \mathrm{B}(\mathcal{H}) \text{ and } m \in \mathbb{N},$$

set $b := (b_1, \ldots, b_m)^T$ *and* $a := (E(b_1), \ldots, E(b_m))^T =: (a_1, \ldots, a_m)^T$*. If* $\|\theta(y) - \phi(y)\| \leq 1/4$ *(where* y *and* ϕ *are as above, defined by (5.7) and (5.8)), then* $\|a-b\| > 1/8$*.*

Proof. Assume the contrary, that $\|a - b\| \leq 1/8$ and define a map ψ on $\mathrm{B}(\mathcal{H})$ by (5.9) with $a_j = E(b_j)$. By the Schwarz inequality for completely positive maps

$$a^*a = \sum_{j=1}^{m} E(b_j)^* E(b_j) \leq E(\sum_{j=1}^{m} b_j^* b_j) = 1.$$

It follows (since $\theta(y) = b^*yb$ and $\psi(y) = a^*ya$)

$$\begin{aligned} \|\phi(y) - \psi(y)\| &\leq \|\phi(y) - \theta(y)\| + \|b^*yb - a^*ya\| \\ &\leq \tfrac{1}{4} + 2\|b - a\| \leq \tfrac{1}{2}. \end{aligned} \tag{5.10}$$

Further, $\|1 - a^*a\| = \|b^*b - a^*a\| \leq 2\|b - a\| \leq 1/4$. Define a unital completely positive map ψ_0 on $\mathrm{B}(\mathcal{H})$ by $\psi_0(x) = \psi(x) + \sqrt{1 - a^*a}x\sqrt{1 - a^*a}$. Then $\|\psi_0 -$

$\psi\|_{\rm cb} \leq \|1 - a^*a\| \leq 1/4$, hence (5.10) implies that $\|\phi(y) - \psi_0(y)\| \leq 3/4$, which contradicts Proposition 5.3. □

Proposition 5.5. *The map* ϕ *on* $\mathrm{B}(\mathcal{H})$ *defined by (*5.8*) can not be approximated pointwise by any sequence of elementary complete contractions.*

Proof. Suppose the contrary, that a sequence (θ_k) of elementary complete contractions on $\mathrm{B}(\mathcal{H})$ converges pointwise to ϕ. By a simple adjustment of coefficients (as in the proof of Proposition 3.4) we may assume that the maps θ_k are unital and completely positive of the form $\theta_k(x) = \sum_{j=1}^{m_k} b_{k,j}^* x b_{k,j}$, hence the columns $b_k := (b_{k,1}, \dots, b_{k,m_k})^T \in \mathrm{B}(\mathcal{H})^{m_k}$ satisfy $b_k^* b_k = 1$.

Let $\mathbb{N} = \cup_{j=1}^{\infty} \mathbb{N}_j$ be a partition of $\mathbb{N}$ into a sequence of infinite subsets $\mathbb{N}_j$ and for each j let

$$P_j := \sum_{i \in \mathbb{N}_j} p^i,$$

where for each i the projection $p^i \in \mathrm{B}(\mathcal{H})$ is the unit of N_i (N_i is defined in (5.6)). Then $\|\phi(x) - P_j\theta_k(x)P_j\| = \|P_j(\phi(x) - \theta_k(x))P_j\| \stackrel{k\to\infty}{\longrightarrow} 0$ for each $x \in P_j\mathrm{B}(\mathcal{H})P_j$. If $E : \mathrm{B}(\mathcal{H}) \to A$ is the diagonal projection, then $E|P_j\mathrm{B}(\mathcal{H})P_j$ is the diagonal projection onto AP_j and $E(P_jxP_j) = E(x)P_j$ since $P_j \in A$. Applying Lemma 5.4 to the restriction $\phi|P_1\mathrm{B}(\mathcal{H})P_1$ (with P_1y instead of the unitary y used in the lemma) it follows that

$$\|P_1b_kP_1 - E(b_k)P_1\| > \frac{1}{8} \tag{5.11}$$

for all large enough k, say, for $k \geq k_1$. Now recall that (since $A = B'$ is abelian, hence equal to the weak* closure of a union of an increasing net of finite dimensional subalgebras) E (= the composition of the conditional expectation ϕ : $\mathrm{B}(\mathcal{H}) \to B$ followed by the central trace $B \to A$) can be approximated in the point-weak* topology by convex combinations of maps of the form $x \mapsto uxu^*$, where $u \in B$ are unitary [12, p. 523 and p. 571]. So, it follows from (5.11) that there exists a unitary $u_1 \in P_1B$ such that $\|P_1b_{k_1}P_1 - u_1b_{k_1}u_1^*P_1\| > 1/8$, that is

$$\|P_1(b_{k_1}u_1 - u_1b_{k_1})P_1\| > \frac{1}{8}.$$

Similarly, applying Lemma 5.4 to the restriction $\phi|P_2\mathrm{B}(\mathcal{H})P_2$, there exists $k_2 > k_1$ such that $\|P_2b_kP_2 - E(b_k)P_2\| > 1/8$ for all $k \geq k_2$ and consequently there exists a unitary $u_2 \in P_2B$ such that

$$\|P_2(b_{k_2}u_2 - u_2b_{k_2})P_2\| > \frac{1}{8}.$$

Continuing in this way, we find a subsequence (b_{k_j}) of (b_k) and a sequence of unitary elements $u_j \in P_jB$ such that

$$\|P_j(b_{k_j}u_j - u_jb_{k_j})P_j\| > \frac{1}{8}.$$

Then $u := \sum_{j=1}^{n} u_j$ is a unitary in B and for each i

$$\begin{aligned} \|b_{k_i}u - ub_{k_i}\| &\geq \|P_i(b_{k_i}u - ub_{k_i})P_i\| \\ &= \|P_i(b_{k_i}uP_i - P_iub_{k_i})P_i\| \\ &= \|P_i(b_{k_i}u_i - u_ib_{k_i})P_i\| > \tfrac{1}{8}. \end{aligned}$$

Since

$$\begin{aligned} \|b_{k_i}u - ub_{k_i}\|^2 &= \|(u^*b_{k_i}^* - b_{k_i}^*u^*)(b_{k_i}u - ub_{k_i})\| \\ &= \|2 - b_{k_i}^*u^*b_{k_i}u - u^*b_{k_i}^*ub_{ki}\| \\ &\leq \|(u - b_{k_i}^*ub_{k_i})^*u\| + \|u^*(u - b_{k_i}^*ub_{k_i})\| \\ &= 2\|u - b_{k_i}^*ub_{k_i}\|, \end{aligned}$$

it follows that

$$\|u - b_{k_i}^*ub_{k_i}\| \geq \frac{1}{2 \cdot 8^2}.$$

But this is a contradiction since $b_{k_i}^*ub_{k_i}$ must converge to $\phi(u) = u$ as $i \to \infty$ (because $u \in B$). $\square$

Elliott [9] proved that pointwise convergence to the identity of a sequence of completely positive maps on a W*-algebra implies uniform convergence. Our initial proof of Proposition 5.5 was based on techniques of [9]. Later we have found a simpler proof presented above, for which the inspiration of constructing the appropriate unitary u comes from [11].

Problem. Is the map ϕ (defined by (5.8)) in $\overline{\mathrm{E}_1(\mathrm{B}(\mathcal{H}))}^{p.n.}$?

References

[1] P. Ara and M. Mathieu, *On the central Haagerup tensor product,* Proc. Edinburgh Math. Soc. **37** (1994), 161–174.

[2] P. Ara and M. Mathieu, *Local multipliers of C*-algebras,* Springer Monographs in Math., Springer-Verlag, Berlin, 2003.

[3] R. J. Archbold, D. W. B. Somerset and R. M. Timoney, *On the central Haagerup tensor product and completely bounded mappings of a C*-algebra*, J. Funct. Anal. **226** (2005), 406–428.

[4] W. B. Arveson, *Notes on extensions of C*-algebras*, Duke. Math. J. **44** (1977), 329–355.

[5] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules,* L.M.S. Monographs, New Series **30**, Clarendon Press, Oxford, 2004.

[6] A. Chatterjee and R. R. Smith, *The central Haagerup tensor product and maps between von Neumann algebras,* J. Funct. Anal. **112** (1993), 97–120.

[7] E. G. Effros and R. Exel, *On multilinear double commutation theorems,* Operator algebras and appl., Vol. 1, LMS Lecture Note Series, Vol. 135, Cambridge Univ. Press, Cambridge, 1988, pp. 81–94.

[8] E. G. Effros and A. Kishimoto, *Module maps and Hochschild – Johnson cohomology,* Indiana Univ. Math. J. **36** (1987), 257–276.

[9] G. A. Elliott, *On the convergence of a sequence of completely positive maps to the identity,* J. Austral. Math. Soc. Ser. A **68** (2000), 340–348.

[10] J. Glimm, *A Stone-Weierstrass theorem for C^*-algebras,* Ann. of Math. **72** (1960), 216–244.

[11] B. E. Johnson and S. K. Parrott, *Operators commuting with a von Neumann algebra modulo the set of compact operators,* J. Funct. Anal. **11** (1972), 39–61.

[12] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras, Vols. 1, 2*, Academic Press, London, 1983, 1986.

[13] B. Magajna, *A transitivity theorem for algebras of elementary operators,* Proc. Amer. Math. Soc. **118** (1993), 119–127.

[14] B. Magajna, *The Haagerup norm on the tensor product of operator modules,* J. Funct. Anal. **129** (1995), 325–348.

[15] B. Magajna, *A transitivity problem for completely bounded mappings,* Houston J. Math. **23** (1997), 109–120.

[16] B. Magajna, *Pointwise approximation by elementary complete contractions,* Proc. Amer. Math. Soc. **137** (2009), 2375–2385.

[17] B. Magajna, *Uniform approximation by elementary operators,* Proc. Edinburgh Math. Soc. **52** (2009), 731–749.

[18] V. I. Paulsen, *Completely bounded maps and operator algebras,* Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, Cambridge, 2002.

[19] H. Pfitzner, *Weak compactness in the dual of a C^*-algebra is determined commutatively,* Math. Ann. **298** (1994), 349–371.

[20] R. R. Smith, Completely bounded module maps and the Haagerup tensor product, *J. Funct. Anal.* **102** (1991), 156–175.

[21] D. W. B. Somerset, *The central Haagerup tensor product of a C^*-algebra*, J. Operator Theory **39** (1998), 113–121.

[22] M. Takesaki, *Theory of operator algebras I,* Springer-Verlag, New-York, 1979.

[23] D. V. Voiculescu, *A non - commutative Weyl - von Neumann theorem,* Rev. Roum. Math. Pures Appl. **21** (1976), 97–113.

Bojan Magajna
Department of Mathematics
University of Ljubljana
Jadranska 21
1000 Ljubljana
Slovenia
e-mail: `Bojan.Magajna@fmf.uni-lj.si`