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Cones of completely bounded maps

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Abstract By analogy with the Choi matrix we associate an operator $C_\varphi \in \mathrm{B}(\mathscr{H})$ to each weak* continuous $\mathscr{A}$-bimodule map $\varphi : \mathrm{B}(\mathscr{K}) \to \mathrm{B}(\mathscr{H})$, where $\mathscr{K}$ and $\mathscr{H}$ are normal Hilbert modules over a von Neumann algebra $\mathscr{A}$ and $\mathscr{K}$ contains a cyclic vector for $\mathscr{A}$. If $\mathscr{A} \subseteq \mathrm{B}(\mathscr{K})$ has no central summands of type I ($\mathscr{K}$ cyclic), every normal $\mathscr{A}$-bimodule map on $\mathrm{B}(\mathscr{K})$, which is positive on $\mathscr{A}'$, is shown to be completely positive on $\mathscr{Z}'$, where $\mathscr{A}'$ and $\mathscr{Z}'$ are the commutant of $\mathscr{A}$ and of the center $\mathscr{Z}$ of $\mathscr{A}$. We investigate cones of bimodule maps, introduce the corresponding dual cones of operators and show that in an appropriate context these notions reduce to those studied earlier by Størmer. We also consider positive maps relative to a mapping cone and positivity in operator projective tensor product of suitable operator bimodules.

Keywords C^*-algebra · von Neumann algebra · Completely bounded map · Positive map · Mapping cone

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1 Introduction

The Choi matrix [3] of a map $\varphi : \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_m(\mathbb{C})$, defined by $Ch_\varphi = \sum_{i,j=1}^n e_{i,j} \otimes \varphi(e_{i,j})$, where $e_{i,j}$ are the usual matrix units in $\mathrm{M}_n(\mathbb{C})$, has been of fundamental importance in studying positive maps [26]. In fact, such matrices were introduced already by Pillis in [21] and Jamiolkowski in [8] and the isomorphism from the linear maps $L(\mathrm{M}_n(\mathbb{C}), \mathrm{M}_m(\mathbb{C}))$ to the Choi Matrices is called Jamiolkowski-Choi isomorphism. Different analogues of the Choi matrix for maps between algebras of all bounded operators on infinite-dimensional Hilbert spaces were introduced by Holevo [7], Li and Du [12], and by Størmer [27] for special classes of maps on von Neumann algebras. Here we will study completely bounded normal

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$\mathscr{A}$-bimodule maps $\varphi : \mathrm{B}(\mathscr{K}) \to \mathrm{B}(\mathscr{H})$, where $\mathscr{K}$ and $\mathscr{H}$ are normal Hilbert modules over a von Neumann algebra $\mathscr{A}$ (that is, Hilbert spaces with normal representations of $\mathscr{A}$). In the case $\mathscr{H} = \mathscr{K}$ any such map sends $\mathscr{A}'$ (the commutant of $\mathscr{A}$) into itself and can be represented by certain weak* convergent sum (Kraus decomposition [11]), so that the class of such maps include those considered by Størmer in [27]. If $\mathscr{K}$ contains a cyclic vector Ω for $\mathscr{A}$, then, as a normal bounded $\mathscr{A}$-bimodule map, φ is determined by the operator

$$C_\varphi := \varphi(\Omega \otimes \Omega^*) \in \mathrm{B}(\mathscr{K}),$$

where $\Omega \otimes \Omega^*$ denotes the rank-one operator, $(\Omega \otimes \Omega^*)\xi := \langle \xi, \Omega \rangle \Omega$. This naturally generalizes the notion of the Choi matrix; namely, in the case, $\mathscr{A} = \mathrm{M}_n(\mathbb{C})$, we may take $\mathscr{K} = \mathbb{C}^n \otimes \mathbb{C}^n$ and $\Omega = \sum_{j=1}^n \varepsilon_j \otimes \varepsilon_j$, where $(\varepsilon_j)_{j=1}^n$ is the standard orthonormal basis of $\mathbb{C}^n$. Then the classical Choi matrix of of a linear map $\varphi : \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_m(\mathbb{C})$ is just $C_{\mathrm{id}_{\mathrm{M}_n(\mathbb{C})} \otimes \varphi}$.

Just as in the finite-dimensional case, also in the wider context of $\mathscr{A}$-bimodule maps $\varphi : \mathrm{B}(\mathscr{K}) \to \mathrm{B}(\mathscr{K})$ the operator C_φ encodes the properties of φ and of the restriction $\varphi|\mathscr{A}'$. After demonstrating this in Section 3, we will study in Section 4 mapping cones of completely bounded $\mathscr{A}$-bimodule maps from operator $\mathscr{A}$-systems X to $\mathrm{B}(\mathscr{K})$, where $\mathscr{K}$ is a normal Hilbert $\mathscr{A}$-module containing a cyclic and separating vector for $\mathscr{A}$. For positive maps such cones were introduced by Størmer [24], but perhaps surprisingly, many results can be proved without assuming the maps to be positive, although the case of positive maps is the most important. We show that there is a natural duality between such mapping cones and certain cones in X (if X is strong). As shown in Section 5, when $\mathscr{A} = \mathrm{B}(\mathscr{L})$ for a Hilbert space $\mathscr{L}$, this duality reduces to the type of duality of mapping cones studied by Størmer [25], but in general in infinite dimensions it can not be described solely in terms of the Choi operators of maps. For a mapping cone $\mathscr{C}$ consisting of positive completely bounded A-bimodule maps on a von Neumann algebra $\mathscr{R}$, where A is a C*-subalgebra of $\mathscr{R}$, and an operator A-system X, we consider in Section 7 the notion of a $\mathscr{C}$-positive map from X to $\mathscr{R}$, which in the case $A = \mathbb{C}$ and $\mathscr{R} = \mathrm{B}(\mathscr{H})$ essentially coincides (at least in the case of symmetric mapping cones) with the notion introduced by Størmer [24] . We will also prove an extension theorem for such maps provided that $\mathscr{R}$ is the W*-envelope of a nuclear C*algebra. (In the special case $A = \mathbb{C}$ and $\mathscr{R} = \mathrm{B}(\mathscr{H})$ this reduces to the Størmer extension theorem only in the case of symmetric mapping cones, but does not cover the more general cones studied in [24], [26], [29].) To investigate such mapping cones, we need a suitable notion of positivity for elements of projective tensor products of the form $X_A \hat{\otimes}_A \mathscr{R}_\sharp$, where X is an operator system over a C*-subalgebra A of a von Neumann algebra $\mathscr{R}$ and $\mathscr{R}_\sharp$ is the pre-dual of $\mathscr{R}$. This notion, introduced in Section 6, is based on completely positive A-bimodule maps from X to $\mathscr{R}$, $\mathrm{CP}_A(X,\mathscr{R})$. If $\mathscr{R}_\sharp$ is the dual space of a C*-algebra R (thus $\mathscr{R} = R^{\sharp\sharp}$, the W*-envelope of R), then it is natural to ask, when is the set $\mathrm{CP}_A(X,R)$ weak* dense in $\mathrm{CP}_A(X,R^{\sharp\sharp})$ (for then the smaller set $\mathrm{CP}_A(X,R)$ can be used in the definition of positivity in $X_A \hat{\otimes}_A \mathscr{R}_\sharp$). We consider briefly this question in Section 8.

2 Preliminaries

All C*-algebras considered here are assumed to be unital, unless stated otherwise, and are denoted by $A, B, C, \ldots$, while W*-algebras are usually denoted by $\mathscr{A}, \mathscr{B}, \mathscr{R}\ldots$.

By a *representation* of A on a Hilbert space $\mathscr{H}$ we mean a unital $*$-representation $\pi : A \to \mathrm{B}(\mathscr{H})$; this makes $\mathscr{H}$ a *Hilbert A-module* by $a\xi := \pi(a)\xi$ ($a \in A$, $\xi \in \mathscr{H}$). A Hilbert A-module $\mathscr{K}$ is called *cyclic* if there exists a vector $\Omega \in \mathscr{K}$ such that $\mathscr{H} = [A\Omega]$ (where [.] denotes the closure of the linear span). The set of all bounded A-module maps on a Hilbert

A-module $\mathscr{H}$ is denoted by $\mathrm{B}_A(\mathscr{H})$. (This is just the commutant A' of A if $A \subseteq \mathrm{B}(\mathscr{H})$.) Then $\mathrm{B}(\mathscr{H})$ is an A-bimodule, and a norm closed A-subbimodule of such a $\mathrm{B}(\mathscr{H})$ is called an *operator A-bimodule.* (Such bimodules can be characterized abstractly [1], [20].) If in addition X is an operator system (that is, X is closed under the involution $*$ on $\mathrm{B}(\mathscr{H})$ and contains the identity operator), then X is called an *operator A-system*.

A Hilbert module $\mathscr{H}$ over a von Neumann algebra $\mathscr{A}$ is called *normal* if the underlying representation $\mathscr{A} \to \mathrm{B}(\mathscr{H})$ is normal. Then a norm closed $\mathscr{A}$ subbimodule X of $\mathrm{B}(\mathscr{H})$ is called a *normal operator $\mathscr{A}$-bimodule*; if in addition X is weak* closed, then X is called a *normal dual operator $\mathscr{A}$-bimodule*. A *normal dual operator $\mathscr{A}$-system* is a normal dual operator $\mathscr{A}$-bimodule which is also an operator $\mathscr{A}$-system.

For operator A-bimodules X and Y we will denote by $\mathrm{CB}_A(X,Y)_A$ the space of all completely bounded (c.b.) A-bimodule maps from X to Y, and by $\mathrm{CP}_A(X,Y)$ the subset of all completely positive such maps. If X and Y are dual operator $\mathscr{A}$-bimodules (or operator $\mathscr{A}$-systems), we denote by $\mathrm{NCB}_{\mathscr{A}}(X,Y)_{\mathscr{A}}$ (or $\mathrm{NCP}_{\mathscr{A}}(X,Y)$) the space of all weak* continuous maps in $\mathrm{CB}_{\mathscr{A}}(X,Y)_{\mathscr{A}}$ (in $\mathrm{CP}_{\mathscr{A}}(X,Y)$, respectively). The corresponding spaces of all bounded and all bounded normal A-bimodule maps will be denoted by $\mathrm{B}_A(X,Y)$ and $\mathrm{NB}_A(X,Y)$, respectively.

By $X^\sharp$ we denote the dual of a Banach X, which carries the canonical operator space structure if X is an operator space [1], [6], [22]. If X is a (normed) A-bimodule, then so is $X^\sharp$ by $(a\rho b)(x) := \rho(bxa)$. If X is a dual space, then $X_\sharp$ denotes its predual.

For a von Neumann subalgebra $\mathscr{A}$ in $\mathrm{B}(\mathscr{H})$ there is a so called $\mathscr{A},\mathscr{A}$-topology on $\mathrm{B}(\mathscr{H})$ that is in between the strong operator and the norm topology [14], [15], but in this paper we will only need to know (in Section 4) which convex sets are closed in this topology, so it suffices to describe continuous linear functionals. If X is a normal operator $\mathscr{A}$-bimodule, a functional $\rho \in X^\sharp$ is $\mathscr{A},\mathscr{A}$-continuous if for each $x \in X$ the maps $\mathscr{A} \to \mathbb{C}$, defined by $a \mapsto \rho(ax)$ and $a \mapsto \rho(xa)$, are weak* continuous. The $\mathscr{A}$-subbimodules of $\mathrm{B}(\mathscr{H})$ closed in this topology are called *strong*. For example, if $\mathscr{A} = \mathrm{B}(\mathscr{H})$ is identified with the space of bounded matrices $\mathrm{M}_J(\mathbb{C})$, where J is the cardinality of an orthonormal basis of $\mathscr{H}$, then strong $\mathscr{A}$-bimodules turn out to be precisely $\mathrm{M}_J(\mathbb{C})$-bimodules of the form $\mathrm{M}_J(V)$, where V is a norm-complete operator space.

The *operator projective tensor product* $X\hat{\otimes}Y$ of operator spaces X, Y is defined so that its dual space $(X\hat{\otimes}Y)^\sharp$ is completely isometric to $\mathrm{CB}(X,Y^\sharp)$ under the natural map which sends $\theta \in (X\hat{\otimes}Y)^\sharp$ to $\check{\theta} \in \mathrm{CB}(X,Y^\sharp)$ defined by $\langle \check{\theta}(x), y\rangle = \theta(x \otimes y)$ [1],[6]. If X and Y are operator A-bimodules, then $X\hat{\otimes}Y$ is a Banach A-bimodule by $a(x\otimes y)b := (ax)\otimes(yb)$. Let $N(X,Y)$ be the closed A-subimodule of $X\hat{\otimes}Y$ generated by the set

$$\{axb \otimes y - x \otimes bya : a,b \in A,\ x \in X,\ y \in Y\}.$$

The quotient space $(X\hat{\otimes}Y)/N(X,Y)$ is denoted by

$$X_A\hat{\otimes}_A Y$$

and the coset of $x\otimes y \in X\hat{\otimes}Y$ in $X_A\hat{\otimes}_A Y$ by $x_A\otimes_A y$. The dual space of $X_A\hat{\otimes}_A Y$ consists of functionals in $(X\hat{\otimes}Y)^\sharp$ that annihilate $N(X,Y)$; under the identification $(X\hat{\otimes}Y)^\sharp = \mathrm{CB}(X,Y^\sharp)$ these correspond to A-bimodule maps from X to $Y^\sharp$. Thus

$$(X_A\hat{\otimes}_A Y)^\sharp = \mathrm{CB}_A(X,Y^\sharp)_A.$$

$\mathrm{T}(\mathscr{H})$ denotes the trace-class operators on a Hilbert space $\mathscr{H}$. Given a subset A of $\mathrm{B}(\mathscr{H})$ and cardinals m, n, we denote by $\mathrm{M}_{m,n}(A)$ the set of all those $m \times n$ matrices with the entries in A that represent bounded operators from $\mathscr{H}^n$ to $\mathscr{H}^m$.

3 The Choi operator of a map

Throughout this section $\mathscr{H}$ and $\mathscr{K}$ are normal Hilbert modules over a von Neumann algebra $\mathscr{A}$ and $\mathscr{K}$ contains a cyclic vector Ω for $\mathscr{A}$.

Definition 3.1 The *Choi operator* of a map $\varphi \in \mathrm{NB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}),\mathrm{B}(\mathscr{H}))_{\mathscr{A}}$ or of a map $\varphi \in \mathrm{B}_{\mathscr{A}}(\mathrm{T}(\mathscr{K}),\mathrm{T}(\mathscr{H}))$ is

$$C_\varphi = \varphi(\Omega \otimes \Omega^*).$$

Example 3.1 Let $\mathscr{A} = \mathrm{B}(\mathscr{L})$ for a separable Hilbert space $\mathscr{L}$, $\mathscr{K} = \mathscr{L}\otimes\mathscr{L}$, (ε_i) an orthonormal basis of $\mathscr{L}$ and $e_{i,j} := \varepsilon_i \otimes \varepsilon_j^*$ the corresponding matrix units. Any normal Hilbert $\mathscr{A}$-module $\mathscr{H}$ is of the form $\mathscr{H}_0 \otimes \mathscr{L}$ for a Hilbert space $\mathscr{H}_0$ [9, 10.4.7]. For any normal completely bounded map $\theta : \mathrm{B}(\mathscr{L}) \to \mathrm{B}(\mathscr{H}_0)$ the map $\varphi := \theta\overline{\otimes}1_{\mathrm{B}(\mathscr{L})} : \mathrm{B}(\mathscr{K}) \to \mathrm{B}(\mathscr{H})$ is in $\mathrm{NB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}),\mathrm{B}(\mathscr{H}))_{\mathscr{A}}$. For any sequence $\lambda = (\lambda_i) \in \ell^2$, with all $\lambda_i \neq 0$, the vector $\Omega := \sum_i \lambda_i \varepsilon_i \otimes \varepsilon_i \in \mathscr{K}$ is cyclic for $\mathscr{A}$. The rank one operator $\Omega\otimes\Omega^*$ on $\mathscr{K}$ can be expressed (by an easy computation) as

$$\Omega\otimes\Omega^* = \sum_{i,j} \lambda_i \overline{\lambda}_j (\varepsilon_i \otimes \varepsilon_j^*) \otimes (\varepsilon_i \otimes \varepsilon_j^*) = \sum_{i,j} \lambda_i \overline{\lambda}_j e_{i,j} \otimes e_{i,j}. \tag{3.1}$$

Hence

$$C_\varphi = \varphi(\Omega\otimes\Omega^*) = \sum_{i,j} \lambda_i \overline{\lambda}_j \theta(e_{i,j}) \otimes e_{i,j},$$

which is just (a slight modification of) the operator considered in [12, 1.3, 1.4]. If $\dim \mathscr{L} < \infty$, we may choose all λ_i equal to 1 and in this way we obtain that $C_{\theta\overline{\otimes}1_{\mathrm{B}(\mathscr{L})}}$ is essentially the Choi matrix of θ (see [26] or [3]).

Remark 3.1 The maps on a factor $\mathscr{M} \subseteq \mathrm{B}(\mathscr{K})$ considered by Størmer [27] can be regarded as $\mathscr{M}'$-bimodule maps on $\mathrm{B}(\mathscr{K})$, and in this way the above definition of the Choi operator can be seen as extension of that in [27].

Normal maps into a general W^*-algebra $\mathscr{R}$ usually can not be described explicitly. Therefore it is worth to state the following generalization of a theorem of Choi [3] and Kraus [11]. We denote by $\mathrm{CCP}_A(X,Y)$ the set of all A-bimodule c.p. contractions from X to Y.

Theorem 3.1 *Let $\sigma : \mathscr{A} \to \mathscr{R}$ be a normal $*$-homomorphism between W^*-algebras (so that $\mathscr{R}$ is an $\mathscr{A}$-bimodule) and $\mathscr{L}$ a separable Hilbert space. Each weak* continuous map $\varphi \in \mathrm{CCP}_{\mathscr{A}}(\mathrm{B}(\mathscr{L})\overline{\otimes}\mathscr{A},\mathscr{R}) = \mathrm{CCP}_{\mathscr{A}}(\mathrm{M}_\infty(\mathscr{A}),\mathscr{R})$ is of the form*

$$\varphi([x_{i,j}]) = \sum_{k=1}^{\infty} r_k^* [\sigma(x_{i,j})] r_k \quad ([x_{i,j}] \in \mathrm{M}_\infty(\mathscr{A})) \tag{3.2}$$

for some columns $r_k \in \mathrm{M}_{\infty,1}(C_{\mathscr{A}}(\mathscr{R}))$ with $\sum_{k=1}^\infty r_k^* r_k \leq 1$, where $C_{\mathscr{A}}(\mathscr{R}) = \{r \in \mathscr{R} : r\sigma(a) = \sigma(a)r \ \forall a \in \mathscr{A}\}$.

Proof Let $\varphi \in \mathrm{CCP}_{\mathscr{A}}(\mathrm{M}_\infty(\mathscr{A}),\mathscr{R})$ be weak* continuous, (ε_j) an orthonormal basis of $\mathscr{L}$, $e_{i,j} = \varepsilon_i \otimes \varepsilon_j^*$ the corresponding matrix units in $\mathrm{B}(\mathscr{L})$ and $\Omega := \sum_{j=1}^\infty \lambda_j \varepsilon_j \otimes \varepsilon_j \in \mathscr{L}\otimes\mathscr{L}$, where all $\lambda_j \neq 0$. Note that $\mathrm{M}_\infty(\mathbb{C}) = \mathrm{B}(\mathscr{L})$ is contained in $\mathrm{M}_\infty(\mathscr{A})$. So $\varphi(e_{i,j})$ is defined and $\sigma(a)\varphi(e_{i,j}) = \varphi(ae_{i,j}) = \varphi(e_{i,j}a) = \varphi(e_{i,j})\sigma(a)$ shows that $\varphi(e_{i,j}) \in C_{\mathscr{A}}(\mathscr{R})$. Since $\Omega \otimes \Omega^*$ can be expressed as in (3.1), we have

$$(\varphi \overline{\otimes} 1_{\mathrm{B}(\mathscr{L})})(\Omega \otimes \Omega^*) = \sum_{i,j=1}^{\infty} \lambda_i \overline{\lambda}_j \varphi(e_{i,j}) \otimes e_{i,j}.$$

This is a positive element of $\mathrm{M}_\infty(C_{\mathscr{A}}(\mathscr{R}))$, hence of the form

$$(\varphi \overline{\otimes} 1_{\mathrm{B}(\mathscr{L})})(\Omega \otimes \Omega^*) = [T_{i,k}][T_{k,j}]^* = \sum_{i,j} \sum_{k} T_{i,k} T_{j,k}^* \otimes e_{i,j}$$

for a matrix $[T_{i,j}] = \sum_{i,j} T_{i,j} \otimes e_{i,j} \in \mathrm{M}_\infty(C_{\mathscr{A}}(\mathscr{R}))$. Thus

$$\varphi(e_{i,j}) = \frac{1}{\lambda_i \overline{\lambda}_j} \sum_{k=1}^{\infty} T_{i,k} T_{j,k}^*.$$

This can also be written as

$$\varphi(e_{i,j}) = \sum_{k=1}^{\infty} T_k \Lambda e_{i,j} \Lambda^* T_k^*, \tag{3.3}$$

where Λ is the diagonal matrix with (λ_i^{-1}) along the diagonal and

$$T_k := [T_{1,k}, T_{2,k}, \ldots], \quad T_{j,k} \in C_{\mathscr{A}}(\mathscr{R}).$$

From (3.3) we have that

$$\sum_{k=1}^{\infty} (T_k \Lambda)(T_k \Lambda)^* = \sum_{k,i=1}^{\infty} (T_k \Lambda) e_{i,i} (T_k \Lambda)^* = \sum_{i=1}^{\infty} \varphi(e_{i,i}) = \varphi(1) \leq 1,$$

hence each $T_k \Lambda$ is in $\mathrm{M}_{1,\infty}(C_{\mathscr{A}}(\mathscr{R}))$ and the map

$$\mathrm{M}_\infty(\mathscr{A}) \to \mathscr{R}, \quad [x_{i,j}] \mapsto \sum_{k=1}^{\infty} (T_k \Lambda)[\sigma(x_{i,j})](T_k \Lambda)^* \tag{3.4}$$

is weak* continuous. By (3.3) this map agrees with φ on matrix units $e_{i,j}$, hence by weak* continuity and $\mathscr{A}$-bimodule property the two maps must be identical. So

$$\varphi([x_{i,j}]) = \sum_{k=1}^{\infty} (T_k \Lambda)[\sigma(x_{i,j})](T_k \Lambda)^* = \sum_{k=1}^{\infty} r_k^* [\sigma(x_{i,j})] r_k,$$

where $r_k = (T_k \Lambda)^*$. □

Just as properties of a map between matrix algebras can be read from the properties of its Choi matrix ([8], [3], [26]), the same holds in our more general context.

Proposition 3.2 *A map $\varphi \in \mathrm{NB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}), \mathrm{B}(\mathscr{H}))_{\mathscr{A}}$ is completely positive (or positive) if and only if $C_\varphi \geq 0$.*

Proof We must verify that $C_\varphi \geq 0$ implies that the amplification

$$\varphi_n : \mathrm{M}_n(\mathrm{B}(\mathscr{K})) = \mathrm{B}(\mathscr{K}^n) \to \mathrm{B}(\mathscr{H}^n) = \mathrm{M}_n(\mathrm{B}(\mathscr{H}))$$

is positive for each $n = 1, 2, \ldots$. Since finite sums of rank one positive operators are weak*-dense in $\mathrm{B}(\mathscr{K}^n)_+$, it suffices to show that $\varphi_n(\xi \otimes \xi^*) \geq 0$ for each $\xi \in \mathscr{K}^n$. Since Ω is cyclic for $\mathscr{A}$, ξ can be approximated by vectors of the form $a\Omega$, where $a = [a_1, \ldots, a_n]^T \in \mathrm{M}_{n,1}(\mathscr{A})$. But for ξ of the form $\xi = a\Omega$ we have

$$\varphi_n(\xi \otimes \xi^*) = [\varphi(a_i\Omega \otimes \Omega^* a_j^*)] = a\varphi(\Omega \otimes \Omega^*)a^* \geq 0.$$

□

If the map $x \mapsto \sum_i a_i x b_i$ on $\mathscr{A}$, where a_i and b_i are fixed elements of $\mathscr{A}$, is positive, is then the map $x \mapsto \sum_i b_i x a_i$ also positive? In matrix algebras the trace enables us to see each of the two maps as the dual of the other, but if $\mathscr{A}$ is purely infinite, there is no trace and the following theorem helps to answer the question.

Theorem 3.3 *Denote by $\mathscr{Z}$ the center of $\mathscr{A} \subseteq \mathrm{B}(\mathscr{K})$, assume that $\mathscr{K}$ contains a cyclic vector for $\mathscr{A}$ and that $\mathscr{A}$ has no non-zero central parts of type I. If $\varphi \in \mathrm{NCB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))$ is such that $\varphi|\mathscr{A}'$ is positive, then $\varphi|\mathscr{Z}'$ is completely positive.*

Proof We may assume that $\|\varphi\| \leq 1$. Since $\mathscr{K}$ is cyclic for $\mathscr{A}$, by [23] φ is completely bounded and, as a normal $\mathscr{A}$-bimodule map, necessarily of the form

$$\varphi(x) = \sum_i a_i' x b_i' \quad (a_i', b_i' \in \mathscr{A}'), \tag{3.5}$$

where the sums

$$\sum_i a_i' a_i'^* \quad \text{and} \quad \sum_i b_i'^* b_i' \tag{3.6}$$

are convergent in the strong operator topology (s.o.t.). If the sums (3.6) are norm convergent, then $\|\varphi|\mathscr{A}'\|_{\mathrm{cb}} = \|\varphi|\mathscr{A}'\|$ by [13, 3.3] since $\mathscr{A}'$ has non non-zero central parts of type I. The same holds even if the sums (3.6) are only s.o.t. convergent, for by the non-commutative Egoroff theorem [30, II.4.13] (noting that $\sum_{i\in F} a_i' a_i'^*$ increases if F is enlarged) in each s.o.t. neighborhood of 1 there exists a projection $e' \in \mathscr{A}'$ such that the sums $\sum_i e' a_i' a_i'^* e'$ and $\sum_i e' b_i'^* b_i' e'$ are norm convergent. Let $\varphi_{e'}(x) := \sum_i e' a_i' x b_i' e'$. Since $\|\varphi_{e'}|\mathscr{A}'\|_{\mathrm{cb}} = \|\varphi_{e'}|\mathscr{A}'\| \leq \|\varphi|\mathscr{A}'\|$ and $\varphi_{e'}(x)$ converges to $\varphi(x)$ as e' converges to 1 in the s.o.t., it follows that $\|\varphi|\mathscr{A}'\|_{\mathrm{cb}} = \|\varphi|\mathscr{A}'\|$.

If $\varphi(1) = 1$, then $\varphi|\mathscr{A}'$ is a unital complete contraction, hence completely positive [6, 5.2.1]. More generally, if $\varphi(1)$ is invertible, we may write $\varphi = a'\psi a'$, where $a' := \varphi(1)^{1/2} \in \mathscr{A}'$ and $\psi(x) := a'^{-1}\varphi(x)a'^{-1}$. Since $\psi|\mathscr{A}'$ is positive, $\|\psi|\mathscr{A}'\| = \|\psi(1)\| = 1$ [20, 2.9], hence $\psi|\mathscr{A}'$ is a unital contraction, hence a unital complete contraction (by previous paragraph), thus completely positive. But then $\varphi|\mathscr{A}'$ is also completely positive. If $\varphi(1)$ is not invertible, then we apply the argument just given to the maps $\varphi_k := \varphi + \frac{1}{k}\mathrm{id}$ $(k = 1, 2, \ldots)$ and let $k \to \infty$. So $\varphi|\mathscr{A}'$ is completely positive.

Thus the map $(\varphi|\mathscr{A}') \otimes \mathrm{id}_{\mathscr{A}} : \mathscr{A}' \overset{\max}{\otimes} \mathscr{A} \to \mathscr{A}' \overset{\max}{\otimes} \mathscr{A}$ on the maximal C*-algebraic tensor product is completely positive [22, 11.3]. Since φ is an $\mathscr{A}$-bimodule map, the diagram

$$\begin{array}{ccc} \mathscr{A}' \overset{\max}{\otimes} \mathscr{A} & \xrightarrow{(\varphi|\mathscr{A}')\otimes \mathrm{id}_{\mathscr{A}}} & \mathscr{A}' \overset{\max}{\otimes} \mathscr{A} \\ \downarrow q & & \downarrow q \\ \overline{\overline{\mathscr{A}'\mathscr{A}}} & \xrightarrow{\quad\varphi\quad} & \overline{\overline{\mathscr{A}'\mathscr{A}}} \end{array},$$

where q is the natural $*$-epimorphism $a'\otimes a\mapsto a'a$, commutes. Since each positive element in $\mathrm{M}_n(\overline{\overline{\mathscr{A}'\mathscr{A}}})$ can be lifted to a positive element in $\mathrm{M}_n(\mathscr{A}'\overset{\max}{\otimes}\mathscr{A})$, it follows that $\varphi|\overline{\overline{\mathscr{A}'\mathscr{A}}}$ is completely positive. Then by weak* continuity the same holds also for $\varphi|\overline{\mathscr{A}'\mathscr{A}}$. But $\overline{\mathscr{A}'\mathscr{A}}=\mathscr{Z}'$. □

The following proposition generalizes the well-known criterion for positivity of linear maps between matrix algebras [8], [26, 4.1.11] and also for maps on factors considered in [27, Th. 10].

Proposition 3.4 *(i) For each $\varphi\in\mathrm{NB}_{\mathscr{A}}(\mathrm{B}(\mathscr{H}),\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ and all $a\in\mathscr{A}$, $a'\in\mathrm{B}_{\mathscr{A}}(\mathscr{H})$ the equality*

$$\langle\varphi(a')a\Omega,\Omega\rangle=\mathrm{Tr}(C_{\varphi_\sharp}a'a)$$

holds, hence $\varphi|\mathrm{B}_{\mathscr{A}}(\mathscr{H})$ is positive if and only if $\mathrm{Tr}(C_{\varphi_\sharp}a'a)\geq 0$ for all $a'\in\mathrm{B}_{\mathscr{A}}(\mathscr{H})_+$ and $a\in\mathscr{A}_+$.

(ii) If $\varphi\in\mathrm{NB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}),\mathrm{B}(\mathscr{H}))_{\mathscr{A}}$ is such that $\varphi(\mathrm{T}(\mathscr{K}))\subseteq\mathrm{T}(\mathscr{H})$, then $\varphi_\sharp$ can be extended uniquely to a weak continuous map in $\mathrm{B}(\mathrm{B}(\mathscr{H}),\mathrm{B}(\mathscr{K}))$, namely to $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp\in\mathrm{NB}_{\mathscr{A}}(\mathrm{B}(\mathscr{H}),\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$, and*

$$\langle((\varphi|\mathrm{T}(\mathscr{K}))^\sharp)(a')a\Omega,\Omega\rangle=\mathrm{Tr}(C_\varphi a'a)$$

for all $a\in\mathscr{A}$ and $a'\in\mathrm{B}_{\mathscr{A}}(\mathscr{H})$, hence $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp|\mathrm{B}_{\mathscr{A}}(\mathscr{H})$ is positive if and only if $\mathrm{Tr}(C_\varphi a'a)\geq 0$ for all $a\in\mathscr{A}_+$ and $a'\in\mathrm{B}_{\mathscr{A}}(\mathscr{H})_+$.

(iii) In the same situation as in (ii), if $\mathscr{H}=\mathscr{K}$ (so that C_φ and $C_{\varphi_\sharp}$ are both defined), then we have that $\mathrm{Tr}(C_{\varphi_\sharp}a'a)\geq 0$ for all $a\in\mathscr{A}_+$ and $a'\in\mathrm{B}_{\mathscr{A}}(\mathscr{K})_+$ if and only if $\mathrm{Tr}(C_\varphi a'a)\geq 0$ for all $a\in\mathscr{A}_+$ and $a'\in\mathrm{B}_{\mathscr{A}}(\mathscr{K})_+$.

Proof (i) We compute that

$$\mathrm{Tr}(C_{\varphi_\sharp}a'a)=\mathrm{Tr}(\varphi_\sharp(\Omega\otimes\Omega^*)a'a)=\mathrm{Tr}(\varphi(a')a(\Omega\otimes\Omega^*))=\langle\varphi(a')a\Omega,\Omega\rangle.$$

Applying this to a^*a instead of a and noting that $\varphi(a')\in\mathrm{B}_{\mathscr{A}}(\mathscr{K})$ (since $a\varphi(a')=\varphi(aa')=\varphi(a'a)=\varphi(a')a$ for all $a\in\mathscr{A}$) we get $\langle\varphi(a')a\Omega,a\Omega\rangle=\mathrm{Tr}(C_{\varphi_\sharp}a'a^*a)$. This proves (i), since $[\mathscr{A}\Omega]=\mathscr{K}$ and all positive elements in $\mathscr{A}$ are of the form a^*a.

(ii) Since $\mathrm{T}(\mathscr{H})$ is weak* dense in $\mathrm{B}(\mathscr{H})$, the uniqueness of the extension is obvious. To prove the existence, we first observe (by an easy application of the closed graph theorem) that the map $\varphi|\mathrm{T}(\mathscr{K}):\mathrm{T}(\mathscr{K})\to\mathrm{T}(\mathscr{H})$ is bounded also in the trace norms $\|\cdot\|_1$ on $\mathrm{T}(\mathscr{K})$ and $\mathrm{T}(\mathscr{H})$. Hence the adjoint $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp$ exists, and clearly it is in $\mathrm{NB}_{\mathscr{A}}(\mathrm{B}(\mathscr{H}),\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$. Evidently $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp|\mathrm{T}(\mathscr{H})=\varphi_\sharp$, hence $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp$ is a weak* continuous extension of $\varphi_\sharp$. A similar computation as in (i) proves the equality stated in (ii).

(iii) By (i) and (ii) it suffices to show that $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp|\mathscr{A}'$ is positive if and only if $\varphi|\mathscr{A}'$ is positive, where $\mathscr{A}'=\mathrm{B}_{\mathscr{A}}(\mathscr{K})$. First we consider the case, when $\mathscr{A}'$ is finite. Let τ' be a faithful normal trace on $\mathscr{A}'$, extended to $L^1(\mathscr{A}')$. Recall that $\mathscr{A}'\subseteq L^1(\mathscr{A}')=(\mathscr{A}')_\sharp$, where each $a'\in\mathscr{A}'$ corresponds to the functional $\rho_{a'}\in(\mathscr{A}')_\sharp$ by $\rho_{a'}(x')=\tau'(a'x')$ (see [30, V.2.18]). We will show that

$$(\varphi|\mathrm{T}(\mathscr{K}))^\sharp|\mathscr{A}'=(\varphi|\mathscr{A}')_\sharp|\mathscr{A}'. \tag{3.7}$$

This will imply (since a map is positive if and only if its dual map is positive and $\mathscr{A}'_+$ is dense in $L^1(\mathscr{A}')_+$) that $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp|\mathscr{A}'$ is positive if and only if $(\varphi|\mathscr{A}')_\sharp$ is positive on $L^1(\mathscr{A}')$ $(=(\mathscr{A}')_\sharp)$, which is the case if and only if $\varphi|\mathscr{A}'$ is positive. To prove (3.7), recall that the predual $(\mathscr{A}')_\sharp$ of $\mathscr{A}'$ is the quotient $(\mathscr{A}')_\sharp=\mathrm{T}(\mathscr{K})/\mathscr{A}'_\perp$, where $\mathscr{A}'_\perp:=\{t\in\mathrm{T}(\mathscr{K}):$

$\mathrm{Tr}(ta') = 0\ \forall a' \in \mathscr{A}'\}$. Observe that $(\mathscr{A}')_\perp$ is invariant under φ, since $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp(\mathscr{A}') \subseteq \mathscr{A}'$ by the $\mathscr{A}$-bimodule property of $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp$. Thus φ induces a map $\dot{\varphi} : (\mathscr{A}')_\sharp \to (\mathscr{A}')_\sharp$ and it is easy to verify that $(\dot{\varphi})^\sharp = (\varphi|\mathrm{T}(\mathscr{K}))^\sharp|\mathscr{A}'$. So, to prove (3.7), we need to show that $(\dot{\varphi})^\sharp = (\varphi|\mathscr{A}')_\sharp|\mathscr{A}'$ or (equivalently by density of $\mathscr{A}'$ in $L^1(\mathscr{A}')$), that

$$\tau'((\dot{\varphi})^\sharp(a')b') = \tau'((\varphi|\mathscr{A}')_\sharp(a')b') \ \ (a', b' \in \mathscr{A}'). \tag{3.8}$$

Extending τ' to a normal state on $\mathrm{B}(\mathscr{K})$, let $t \in \mathrm{T}(\mathscr{K})$ be the operator representing τ', that is

$$\tau'(x') = \mathrm{Tr}(x't) \ \ \forall x' \in \mathscr{A}'. \tag{3.9}$$

The left side of (3.8) can be expressed as

$$\tau'((\dot{\varphi})^\sharp(a')b') = \tau'(a'\dot{\varphi}(b')), \tag{3.10}$$

where on the right-hand side of this identity b' is regarded as an element of $L^1(\mathscr{A}') = \mathscr{A}'_\sharp = \mathrm{T}(\mathscr{K})/\mathscr{A}'_\perp$ and $\dot{\varphi}(b')$ is the class $\varphi(t_{b'})^\cdot$ of $\varphi(t_{b'})$ in $(\mathscr{A}')_\sharp$, where $t_{b'} \in \mathrm{T}(\mathscr{K})$ is the operator representing the functional $\rho_{b'}$, that is, $\tau'(x'b') = \mathrm{Tr}(x't_{b'})$ for all $x' \in \mathscr{A}'$. Using (3.9) we now have

$$\mathrm{Tr}(x't_{b'}) = \tau'(x'b') = \mathrm{Tr}(x'b't) \ \ \forall x' \in \mathscr{A}',$$

hence we may take $t_{b'} = b't$. Therefore (3.10) can be rewritten as

$$\tau'((\dot{\varphi})^\sharp(a')b') = \tau'(a'\varphi(t_{b'})^\cdot) = \tau'(a'\varphi(b't)^\cdot). \tag{3.11}$$

Recall that since φ and hence $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp$ is automatically completely bounded by [23] and a normal $\mathscr{A}$-bimodule map by hypothesis, it is necessarily of the form

$$(\varphi|\mathrm{T}(\mathscr{K}))^\sharp(x) = \sum_i b'_i x a'_i,$$

where $a'_i, b'_i \in \mathscr{A}'$ are such that the sums $\sum_i a'^*_i a'_i$ and $\sum_i b'_i b'^*_i$ are convergent in the strong operator topology (see [23] or [13]). Then it follows readily that $\varphi|\mathrm{T}(\mathscr{K})$ must be of the form $\varphi(x) = \sum_i a'_i x b'_i$, where the sum is convergent in the trace norm for each $x \in \mathrm{T}(\mathscr{K})$. So it follows now from (3.11) that

$$\tau'((\dot{\varphi})^\sharp(a')b') = \tau'((\sum_i a'a'_i b' t b'_i)^\cdot) = \tau'(\sum_i a'a'_i b' \dot{t} b'_i). \tag{3.12}$$

Now observe that $\dot{t}$ is just the identity element $1_{\mathscr{A}'} \in \mathscr{A}' \subseteq L^1(\mathscr{A}') = \mathscr{A}'_\sharp = \mathrm{T}(\mathscr{K})/\mathscr{A}'_\perp$ (as can be seen by applying (3.12) to the identity map $x \mapsto 1x1$ in place of φ). Thus (3.12) can be rewritten as

$$\tau'((\dot{\varphi})^\sharp(a')b') = \tau'(\sum_i a'a'_i b' b'_i) = \tau'(a'\varphi(b')) = \tau'((\varphi|\mathscr{A}')_\sharp(a')b'),$$

which agrees with the right-hand side of (3.8).

When $\mathscr{A}'$ is semi-finite (but not necessarily finite), we chose a net (p'_F) of finite projections in $\mathscr{A}'$ increasing to 1 and consider the maps $\varphi_F : \mathrm{B}(p'_F\mathscr{K}) \to \mathrm{B}(p'_F\mathscr{K}) \cong p'_F\mathrm{B}(\mathscr{K})p'_F$ defined by $\varphi_F(x) = p'_F\varphi(x)p'_F$. Since $\mathrm{B}_{\mathscr{A}p'_F}(p'_F\mathscr{K}) = p'_F\mathscr{A}'p'_F$ is finite, we have by what we have already proved that $\mathrm{Tr}(C_{\varphi_F} p'_F a' p'_F a)$ is positive for all $a' \in \mathscr{A}'$ and $a \in \mathscr{A}$ if and only if the same holds for $\mathrm{Tr}(C_{(\varphi_F)_\sharp} p'_F a' p'_F a)$, where $C_{\varphi_F} = (p'_F\Omega) \otimes (p'_F\Omega)^*$. But $C_{\varphi_F} = p'_F C_\varphi p'_F$ and $C_{(\varphi_F)_\sharp} = p'_F C_{\varphi_\sharp} p'_F$, hence we get the desired conclusion in the limit as $p'_F \to 1$.

By the central decomposition of $\mathscr{A}$ the proof can now evidently be reduced to the case when $\mathscr{A}$ (and hence also $\mathscr{A}'$) is of type III. Assume that $\varphi|\mathscr{A}'$ is positive, hence $\varphi|\mathscr{Z}'$ is c.p. by Theorem 3.3. Then $\varphi|\mathscr{Z}'$ is necessarily of the form $\psi(x) = c'^*xc'$ for an (infinite) column $c' = (c'_i)$ with the entries in $\mathscr{A}'$ such that the sum $\sum_i c_i'^* c_i'$ is s.o.t. convergent (the proof of this is the same as in [13, 1.2]). If $\mathscr{A}$ is a factor, $\mathscr{Z}' = \mathrm{B}(\mathscr{K}) \supseteq \mathrm{T}(\mathscr{K})$, so we may take x of the form $\xi \otimes \xi^*$ ($\xi \in \mathscr{K}$). Then $\|\varphi(x)\|_1 = \mathrm{Tr}(\varphi(x)) = \sum_i \langle c'_i c_i'^* \xi, \xi\rangle$, hence by polarization the sum $\sum_i c'_i c_i'^*$ converges (recall that $\varphi|\mathrm{T}(\mathscr{K}) : \mathrm{T}(\mathscr{K}) \to \mathrm{T}(\mathscr{K})$ is bounded). It follows that $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp$ is of the form $x \mapsto \sum_i c'_i x c_i'^*$, which is (completely) positive, hence so is also its restriction to $\mathscr{A}'$. If $\mathscr{A}$ is not necessarily a factor, then by using the fact that $\mathscr{Z}$ is unitarily equivalent to a direct sum of algebras of the form $L^\infty(\mu) \otimes 1$ acting on $L^2(\mu) \otimes \mathscr{L}$ for a positive measure μ and a Hilbert space $\mathscr{L}$, it can be shown that the sum $\sum_i c_i c_i^*$ is weak* convergent and consequently the map $\psi : \mathrm{B}(\mathscr{K}) \to \mathrm{B}(\mathscr{K})$, $\psi(x) = c'^*xc'$ (which coincides with φ on $\mathscr{Z}'$) maps $\mathrm{T}(\mathscr{K})$ into itself. Moreover, since $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp$ and $(\psi|\mathrm{T}(\mathscr{K}))^\sharp$ are both $\mathscr{A}$-bimodule maps (hence also $\mathscr{Z}$-bimodule maps), they preserve $\mathscr{Z}'$, hence $\varphi(\mathscr{Z}'_\perp) \subseteq \mathscr{Z}'_\perp$ and $\psi(\mathscr{Z}'_\perp) \subseteq \mathscr{Z}'_\perp$, where $\mathscr{Z}'_\perp \subseteq \mathrm{T}(\mathscr{K})$ is the annihilator of $\mathscr{Z}'$. Thus we have the induced maps

$$\varphi_{\mathscr{Z}}, \psi_{\mathscr{Z}} : (\mathscr{Z}')_\sharp = \mathrm{T}(\mathscr{K})/\mathscr{Z}'_\perp \to (\mathscr{Z}')_\sharp.$$

The intersection $(\mathscr{Z}')_\sharp \cap \mathscr{Z}'$ is dense in $(\mathscr{Z}')_\sharp$ and weak* dense in $\mathscr{Z}'$. On this intersection the maps $\varphi_{\mathscr{Z}}$ and $\psi_{\mathscr{Z}}$ are just the restrictions of φ and ψ, respectively. Since $\varphi|\mathscr{Z}' = \psi|\mathscr{Z}'$, it follows that $\varphi_{\mathscr{Z}} = \psi_{\mathscr{Z}}$. Hence $\varphi^\sharp_{\mathscr{Z}} : \mathscr{Z}' \to \mathscr{Z}'$ is given by $\varphi^\sharp_{\mathscr{Z}}(z') = \psi^\sharp_{\mathscr{Z}}(z') = c'z'c'^*$, which is a (completely) positive map. Since $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp|\mathscr{A}' = \varphi^\sharp_{\mathscr{Z}}|\mathscr{A}'$, this proves that the positivity of $\varphi|\mathscr{A}'$ implies the positivity of $(\varphi|\mathrm{T}(\mathscr{K}))^\sharp|\mathscr{A}'$. The proof of the converse follows by essentially the same arguments. □

Proposition 3.5 *If $\mathscr{A} \subseteq \mathrm{B}(\mathscr{K})$ is a factor, $\Omega \in \mathscr{K}$ is a cyclic trace vector for $\mathscr{A}$ and $\mathscr{H}$ is a normal Hilbert $\mathscr{A}$-module such that $\mathrm{B}_{\mathscr{A}}(\mathscr{H})$ is finite, then $\varphi(\Omega \otimes \Omega^*) \in \mathrm{T}(\mathscr{H})$ for any map $\varphi \in \mathrm{NCB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}), \mathrm{B}(\mathscr{H}))_{\mathscr{A}}$. Hence $\varphi(\mathrm{K}(\mathscr{K})) \subseteq \mathrm{K}(\mathscr{H})$.*

Proof Note that $\mathscr{A}$ is finite and Ω is a trace vector also for $\mathscr{A}'$ and a separating vector for $\mathscr{A}$ [9, 7.2.14]. By [30, V.18] $\mathscr{H}$ is isometrically isomorphic to a module of the form $q'\mathscr{K}^m$ for a cardinal m and a projection $q' \in \mathrm{M}_m(\mathscr{A}')$, where $\mathscr{A}' = \mathrm{B}_{\mathscr{A}}(\mathscr{K})$. We claim that m can be taken to be finite. To show this, let let τ' be the semi-finite normal faithful trace on $\mathrm{M}_m(\mathscr{A}')$ (defined by $\tau'([a'_{i,j}]) = \sum_i \tau'_0(a_{i,i})$, where τ'_0 is the canonical trace on the factor $\mathscr{A}'$, $\tau'_0(a') = \langle a'\Omega, \Omega\rangle$). Since $q'\mathrm{M}_m(\mathscr{A}')q' \cong \mathrm{B}_{\mathscr{A}}(\mathscr{H})$ is finite by assumption, $\tau'(q') < \infty$. Let $n \in \mathbb{N}$ be such that $(n-1)\tau'(1_{\mathscr{A}'}) \le \tau'(q') \le n\tau'(1_{\mathscr{A}'})$, where $1_{\mathscr{A}'}$ has been identified with the diagonal projection $1_{\mathscr{A}'} \oplus 0 \oplus 0 \oplus \ldots$ in $\mathrm{M}_m(\mathscr{A}')$. Let $r' \in \mathscr{A}'$ be a projection such that $\tau'(r') = \tau'(q') - (n-1)\tau'(1_{\mathscr{A}'})$. Then q' is equivalent to the diagonal projection $p' := 1_{\mathscr{A}'} \oplus \ldots \oplus 1_{\mathscr{A}'} \oplus r' \oplus 0 \oplus \ldots$ since $\tau'(p') = \tau'(q')$. This implies that $\mathscr{H} \cong q'\mathscr{K}^m$ is isomorphic (as a Hilbert $\mathscr{A}$-module) to $p'\mathscr{K}^m$, which can be regarded as a Hilbert $\mathscr{A}$-submodule of $\mathscr{K}^n$.

Since φ is necessarily of the form (3.5, 3.6), but with $a_i'^*$ and b'_i in $\mathrm{B}_{\mathscr{A}}(\mathscr{H}, \mathscr{K})$ (instead of $\mathscr{A}'$), we have $\varphi(\Omega \otimes \Omega^*) = \sum_i a'_i\Omega \otimes (b_i'^*\Omega)^*$, hence

$$\|\varphi(\Omega \otimes \Omega^*)\|_1 \le \sum_i \|a'_i\Omega\|\|b_i'^*\Omega\| \le (\sum_i \langle a_i'^* a'_i \Omega, \Omega\rangle)^{1/2} (\sum_i \langle b'_i b_i'^* \Omega, \Omega\rangle)^{1/2}. \tag{3.13}$$

Further, since $\mathscr{H} \subseteq \mathscr{K}^n$, $a_i'^*$ and b'_i are of the form

$$a_i'^* = [a_{i,1}'^*, \ldots, a_{i,n}'^*] \quad \text{and} \quad b'_i = [b'_{i,1}, \ldots, b'_{i,n}] \quad (a'_{i,j}, b'_{i,j} \in \mathscr{A}')$$

and the expression on the right side of (3.13) is equal to

$$(\sum_i \sum_{j=1}^{n} \langle a'^{*}_{i,j} a'_{i,j}\Omega,\Omega\rangle)^{1/2}(\sum_i \sum_{j=1}^{n} \langle b'_{i,j} b'^{*}_{i,j}\Omega,\Omega\rangle)^{1/2}.$$

Since Ω is a trace vector for $\mathscr{A}'$, it now follows that

$$\|\varphi(\Omega\otimes\Omega^*)\|_1 \leq (\sum_{j=1}^{n}\sum_i \langle a'_{i,j} a'^{*}_{i,j}\Omega,\Omega\rangle)^{1/2}(\sum_{j=1}^{n}\sum_i \langle b'^{*}_{i,j} b'_{i,j}\Omega,\Omega\rangle)^{1/2}. \tag{3.14}$$

Since $\sum_i a'_i a'^{*}_i = \sum_i [a'_{i,k} a'^{*}_{i,j}] \in \mathrm{M}_n(\mathscr{A}')$ and n is finite, the sum $\sum_{j=1}^{n}\sum_i a'_{i,j}a'^{*}_{i,j}$ is in $\mathscr{A}'$. The same applies also to the sum $\sum_{j=1}^{n}\sum_i b'^{*}_{i,j}b'_{i,j}$, hence the right side of (3.14) is finite and therefore $\varphi(\Omega\otimes\Omega^*)\in \mathrm{T}(\mathscr{H})$.

We still have to prove the inclusion $\varphi(\mathrm{K}(\mathscr{K}))\subseteq \mathrm{K}(\mathscr{H})$. Since every operator in $\mathrm{K}(\mathscr{K})$ can be approximated in norm by linear combinations of operators of the form $\xi\otimes\xi^*$ ($\xi\in\mathscr{K}$) and each $\xi\in\mathscr{K}$ can be approximated by vectors of the form $a\Omega$ ($a\in\mathscr{A}$), it suffices to note that $\varphi(a\Omega\otimes(a\Omega)^*) = a\varphi(\Omega\otimes\Omega^*)a^*\in \mathrm{T}(\mathscr{H})\subseteq \mathrm{K}(\mathscr{H})$. □

In Proposition 3.5 the assumption that $\mathscr{A}$ is a factor is not redundant, as shown by the following example.

Example 3.2 Let $\mathscr{A}=\ell^\infty$ act on $\mathscr{K}=\ell^2$ in the usual way. $\Omega := (1,\frac{1}{2},\frac{1}{3},\dots)\in\mathscr{K}$ is cyclic for $\mathscr{A}$. Let $\mathscr{L}=\ell^2\otimes\ell^2$ and let $\mathscr{A}$ act on $\mathscr{L}$ as $\mathscr{A}\otimes 1$, so that $\mathrm{B}_{\mathscr{A}}(\mathscr{L}) = \mathscr{A}\overline{\otimes}\mathrm{B}(\ell^2)\cong \mathrm{M}_\infty(\mathscr{A})\cong\ell^\infty(\mathrm{M}_\infty(\mathbb{C})) = \ell^\infty(\mathrm{B}(\ell^2))$. Let $e_i=(0,\dots,0,1,0,\dots)$ be the usual rank 1 projections in $\mathscr{A}$ and let $p'\in\ell^\infty(\mathrm{M}_\infty(\mathbb{C}))$ be the projection given by the sequence $p'=(q_1,q_2,\dots)$, where $q_j\in\mathrm{M}_\infty(\mathbb{C})$ is the diagonal projection with the first j entries along the diagonal equal to 1 and all the remaining entries equal to 0 (that is, $q_j=e_1+\dots+e_j$). Since each q_j is a finite projection and the q_j are centrally orthogonal in $\ell^\infty(\mathrm{M}_\infty(\mathbb{C}))$, p' is a finite projections in $\ell^\infty(\mathrm{M}_\infty(\mathbb{C}))$. Regarded as an element of $\mathrm{B}_{\mathscr{A}}(\mathscr{L})=\mathrm{M}_\infty(\ell^\infty)$, p' is the diagonal matrix, whose diagonal entries in ℓ^∞ are the sums $\sum_{k=j}^{\infty}e_k=(0,\dots,0,1,1,\dots)$. Let $a'_i\in\mathrm{B}_{\mathscr{A}}(\mathscr{K},\mathscr{L})$ ($i=1,2,\dots$) be defined as column $a'_i=(e_i,e_i,\dots,e_i,0,0,\dots)^T$, where the first i components are equal to e_i, and define $\varphi\in\mathrm{NCB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}),\mathrm{B}(\mathscr{L}))$ by $\varphi(x)=\sum_i a'_i x a'^{*}_i$. Note that $\sum_i a'_i a'^{*}_i = p'$, φ is a c.p. map and $p'a'_i=a'_i$ implies that the range of φ is contained in $p'\mathrm{B}(\mathscr{L})p'\cong\mathrm{B}(\mathscr{H})$, where $\mathscr{H}:=p'\mathscr{L}$. Since p' is finite in $\mathrm{B}_{\mathscr{A}}(\mathscr{L})$, $\mathrm{B}_{\mathscr{A}}(\mathscr{H})$ is finite. However, $\varphi(\Omega\otimes\Omega^*)=\sum_i a'_i\Omega\otimes(a'_i\Omega)^*$ is not in the trace class since

$$\mathrm{Tr}(\varphi(\Omega\otimes\Omega^*)) = \sum_i\langle a'_i\Omega,a'_i\Omega\rangle = \sum_i\langle a'^{*}_i a'_i\Omega,\Omega\rangle = \sum_i i\langle e_i\Omega,\Omega\rangle = \sum_{i=1}^{\infty}\frac{1}{i}=\infty.$$

Remark 3.2 Suppose that $\Omega\in\mathscr{K}$ is cyclic and separating for $\mathscr{A}$ and let S be the closure of the map $a\Omega\mapsto a^*\Omega$ (the well known operator from the Tomita-Takesaki theory [9], [30]). For each $\varphi\in\mathrm{NB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ there is a connection between $C_\varphi=\varphi(\Omega\otimes\Omega^*)$ and $C_{\varphi_\sharp}=\varphi_\sharp(\Omega\otimes\Omega^*)$ that can be expressed as follows:

The domain of the operators $S^*\varphi(\Omega\otimes\Omega^*)^*S$ contains $\mathscr{A}\Omega$ and

$$S^*\varphi(\Omega\otimes\Omega^*)^*S\subseteq\varphi_\sharp(\Omega\otimes\Omega^*). \tag{3.15}$$

Moreover, $\varphi(\Omega\otimes\Omega^*)^*S(\mathscr{A}\Omega)\subseteq\mathscr{A}'\Omega$.

We will omit the relatively simple proof of this fact (not needed in this article), which uses the fact that φ is of the form (3.5) .

4 Cones of maps and cones of operators

Definition 4.1 Let $\mathscr{R}$ be a W*-algebra, A a C*-algebra, $\pi : A \to \mathscr{R}$ a $*$-homomorphism (making $\mathscr{R}$ an A-bimodule) and X an operator A-system.

(i) A *mapping cone in* $\mathrm{CB}_A(X,\mathscr{R})_A$ is a weak* closed cone $\mathscr{C} \subseteq \mathrm{CB}_A(X,\mathscr{R})_A$ such that $\alpha\varphi \in \mathscr{C}$ for all $\varphi \in \mathscr{C}$ and $\alpha \in \mathrm{CP}_A(\mathscr{R})$. (We do not assume in general that $\varphi\beta \in \mathscr{C}$ for all $\beta \in \mathrm{CP}_A(X)$ and $\varphi \in \mathscr{C}$.)

(ii) An *operator cone in* X is a norm closed cone $\mathscr{D} \subseteq X$ such that $a^*xa \in \mathscr{D}$ for all $a \in A$ and $x \in \mathscr{D}$.

(iii) The *predual operator cone in* X of a mapping cone $\mathscr{C} \subseteq \mathrm{CB}_A(X,\mathscr{R})_A$ is

$$\mathscr{C}_\circ = \{x \in X : \operatorname{Re}\varphi(x) \geq 0 \ \forall \varphi \in \mathscr{C}\},$$

where $\operatorname{Re}\varphi(x) = \frac{1}{2}(\varphi(x) + \varphi(x)^*)$.

(iv) The *dual mapping cone in* $\mathrm{CB}_A(X,\mathrm{B}(\mathscr{K}))_A$ of an operator cone $\mathscr{D} \subseteq X$ is

$$\mathscr{D}^\circ = \{\varphi \in \mathrm{CB}_A(X,\mathrm{B}(\mathscr{K}))_A ; \operatorname{Re}\varphi(x) \geq 0 \ \forall x \in \mathscr{D}\}.$$

We will need the following consequence of [15, 3.8, 3.9]:

Proposition 4.1 *Let $\mathscr{C}$ be a cone in a strong operator $\mathscr{A}$-bimodule X such that $a^*ca \in \mathscr{C}$ for all $c \in \mathscr{C}$ and $a \in \mathscr{A}$ and let $\mathscr{K}$ be a normal Hilbert $\mathscr{A}$-module containing a separating and cyclic vector for $\mathscr{A}$. If $\mathscr{C}$ is closed in the $\mathscr{A},\mathscr{A}$-topology then for each $x \in X \setminus \mathscr{C}$ there exists a map $\rho \in \mathrm{CB}_{\mathscr{A}}(X,\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ such that $\operatorname{Re}\rho(c) \geq 0$ for all $c \in \mathscr{C}$ and $\operatorname{Re}\rho(x) \ngeq 0$. Moreover, if X is a dual normal operator $\mathscr{A}$-bimodule and $\mathscr{C}$ is weak* closed, we may take ρ to be weak*-continuous.*

Proof Since $\mathscr{K}$ has a separating vector for $\mathscr{A}$, all normal states on $\mathscr{A}$ are vector states [9, 7.2.3], hence $\mathscr{K}$ contains (up to isomorphisms) all cyclic normal Hilbert $\mathscr{A}$-modules and it follows from [15, 3.8, 3.9] (or from [17, 2.3]) that there exist $\rho \in \mathrm{CB}_{\mathscr{A}}(X,\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ (which can be taken weak* continuous if X is a normal-dual operator $\mathscr{A}$-bimodule and $\mathscr{C}$ is weak* closed) and $\alpha \in \mathbb{R}$ such that $\operatorname{Re}\rho(c) \geq \alpha 1_{\mathrm{B}(\mathscr{K})}$ for all $c \in \mathscr{C}$ and $\operatorname{Re}\rho(x) \ngeq \alpha 1_{\mathrm{B}(\mathscr{K})}$. Note that for each $\xi \in \mathscr{K}$ the set $\{\langle \operatorname{Re}\rho(c)\xi,\xi\rangle : c \in \mathscr{C}\}$ is a cone in $\mathbb{R}$, hence we may take $\alpha = 0$. □

Theorem 4.2 *Let $\mathscr{K}$ be a normal Hilbert $\mathscr{A}$-module with a unit vector $\Omega \in \mathscr{K}$ which is cyclic and separating for $\mathscr{A}$ and let X be a strong operator $\mathscr{A}$-system. Then:*

(i) $(\mathscr{C}_\circ)^\circ = \mathscr{C}$ for each mapping cone $\mathscr{C} \subseteq \mathrm{CB}_{\mathscr{A}}(X,\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$.

(ii) $(\mathscr{D}^\circ)_\circ = \mathscr{D}$ for each $\mathscr{A},\mathscr{A}$-closed operator cone $\mathscr{D} \subseteq X$.

Proof (i) Evidently $\mathscr{C} \subseteq (\mathscr{C}_\circ)^\circ$. To prove that equality holds here, assume the contrary, that there exists $\varphi \in (\mathscr{C}_\circ)^\circ \setminus \mathscr{C}$. Since $X^\natural := \mathrm{CB}_{\mathscr{A}}(X,\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ is a normal dual operator $\mathscr{A}'$-bimodule, where $\mathscr{A}' = \mathrm{B}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))$, and $\mathscr{C}$ is weak* closed, by Proposition 4.1 there exists $\rho \in \mathrm{NCB}_{\mathscr{A}'}(X^\natural,\mathrm{B}(\mathscr{K}))_{\mathscr{A}'}$ such that $\operatorname{Re}\rho(\psi) \geq 0$ for all $\psi \in \mathscr{C}$ and $\operatorname{Re}\rho(\varphi) \ngeq 0$. By [16, 5.1] each $\rho \in \mathrm{NCB}_{\mathscr{A}'}(X^\natural,\mathrm{B}(\mathscr{K}))_{\mathscr{A}'}$ is just the evaluation at some element $x \in X$, hence for a suitable $x \in X$ we have now $\operatorname{Re}\psi(x) \geq 0$ for all $\psi \in \mathscr{C}$ (hence $x \in \mathscr{C}_\circ$) and $\operatorname{Re}\varphi(x) \ngeq 0$, hence $\varphi \notin (\mathscr{C}_\circ)^\circ$, a contradiction.

(ii) Again, the inclusion $\mathscr{D} \subseteq (\mathscr{D}^\circ)_\circ$ is evident. If $x \in X \setminus \mathscr{D}$, then by Proposition 4.1 there exists $\vartheta \in \mathrm{CB}_{\mathscr{A}}(X,\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ such that $\operatorname{Re}\vartheta(d) \geq 0$ for all $d \in \mathscr{D}$ (thus $\vartheta \in \mathscr{D}^\circ$) and $\operatorname{Re}\vartheta(x) \ngeq 0$, hence $x \notin (\mathscr{D}^\circ)_\circ$. □

We remark that Theorem 4.2 holds (with the same proof) also for not necessarily σ-finite $\mathscr{A}$, if $\mathscr{K}$ is such that all normal states on $\mathscr{A}$ and on $\mathscr{A}'$ are vector states arising from vectors in $\mathscr{K}$. (This is so if $\mathscr{A}$ is in the standard form on $\mathscr{K}$ [30, IX]).

Example 4.1 We can describe all weak* closed operator cones $\mathscr{D}$ in $\mathscr{A}_+$. (Thus, here $X = \mathscr{A}$ for a W*-algebra $\mathscr{A}$.) Assume that $0 \neq x \in \mathscr{D}$. Since $a^*xa \in \mathscr{D}$ for all $a \in \mathscr{A}$, it follows easily (by considering those positive a that are functions of x) that $\mathscr{D}$ contains a non-zero projection $e \in \mathscr{A}$. Then, since the central carrier $p := C_e$ of e is a sum of projections in $\mathscr{A}$ that are equivalent to sub-projections of e and $\mathscr{D}$ is weak* closed, it follows readily that $p \in \mathscr{D}$. Similarly $\mathscr{D}$ contains the range projection of each $x \in \mathscr{D}$, hence a standard maximality argument shows that $\mathscr{D} = q\mathscr{A}_+$, where q is the largest central projection in $\mathscr{D}$.

We can also describe the dual cones. Since any $\mathscr{A}$-bimodule map $\varphi : \mathscr{A} \to \mathrm{B}(\mathscr{K})$ is determined by $a' := \varphi(1) \in \mathscr{A}'$, we have $\mathrm{CB}_{\mathscr{A}}(\mathscr{A}, \mathrm{B}(\mathscr{K}))_{\mathscr{A}} \cong \mathscr{A}'$, hence

$$\mathscr{D}^\circ = \{a' \in \mathscr{A}' : \mathrm{Re}\,(a'x) \geq 0\ \forall x \in \mathscr{D}\} = \{a' \in \mathscr{A}' : \mathrm{Re}\,(a'qa) \geq 0\ \forall a \in \mathscr{A}_+\}$$

$$= \{a' \in \mathscr{A}' : q\mathrm{Re}\,a' \geq 0\}.$$

Let $\mathscr{K}$ be a normal Hilbert $\mathscr{A}$-module with a cyclic and separating vector Ω for $\mathscr{A}$ and let $\mathscr{A}' = \mathrm{B}_{\mathscr{A}}(\mathscr{K})$. A *normal mapping cone* in $\mathrm{CB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ is a mapping cone $\mathscr{C} \subseteq \mathrm{CB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ (in the sense of Definition 4.1) such that the set $\mathscr{C}_{\mathrm{nor}}$ of all weak* continuous maps in $\mathscr{C}$ is weak* dense in $\mathscr{C}$. For $a' \in \mathscr{A}'$ let $M_{a'^*,a'} \in \mathrm{CB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ be defined by $M_{a'^*,a'}(x) = a'^*xa'$. Motivated by the finite-dimensional case [26], we could define the *predual cone* $\mathscr{C}_\diamond$ as

$$\mathscr{C}_\diamond := (\mathscr{C}_{\mathrm{nor}})_\diamond := \{C_\psi \in \mathrm{B}(\mathscr{K}) :\ \psi \in \mathrm{NCB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}}, \mathrm{Re}\,\mathrm{Tr}(C_\psi C_{\varphi_\sharp}) \geq 0\ \forall \varphi \in \mathscr{C}_{\mathrm{nor}}\},$$

where $C_{\varphi_\sharp} = \varphi_\sharp(\Omega \otimes \Omega^*) \in \mathrm{T}(\mathscr{K})$ and $C_\psi = \psi(\Omega \otimes \Omega^*)$. (Note that $\mathscr{C}_\diamond$ is not necessarily an operator cone in the sense of Definition 4.1 since $M_{a^*,a} \circ \psi$ ($a \in \mathscr{A}$) is not necessarily in $\mathscr{C}_\diamond$ because $M_{a^*,a}$ is not an $\mathscr{A}$-bimodule map (if $\mathscr{A}$ is not abelian).)

Lemma 4.1 $\mathscr{C}_\diamond = \{C_\psi :\ \psi \in \mathrm{NCB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}},\ \mathrm{Re}\,\varphi(C_\psi) \geq 0\ \forall \varphi \in \mathscr{C}_{\mathrm{nor}}\}$.

Proof Since Ω is cyclic for $\mathscr{A}'$, we have that $\mathrm{Re}\,\varphi(C_\psi) = \mathrm{Re}\,\varphi(\psi(\Omega \otimes \Omega^*)) \geq 0$ if and only if $\mathrm{Re}\,\langle \varphi\psi(\Omega \otimes \Omega^*)a'\Omega, a'\Omega\rangle \geq 0$ for all $a' \in \mathscr{A}'$. This can be written as

$$\mathrm{Re}\,\mathrm{Tr}(((M_{a'^*,a'}\varphi\psi)(\Omega \otimes \Omega^*))(\Omega \otimes \Omega^*)) \geq 0.$$

Since $M_{a'^*,a'}\varphi \in \mathscr{C}_{\mathrm{nor}}$ for all $a' \in \mathscr{A}'$ and $\varphi \in \mathscr{C}_{\mathrm{nor}}$, it follows that $\mathrm{Re}\,\varphi(C_\psi) \geq 0$ for all $\varphi \in \mathscr{C}_{\mathrm{nor}}$ if and only if $\mathrm{Re}\,\mathrm{Tr}((\varphi\psi(\Omega \otimes \Omega^*))(\Omega \otimes \Omega^*)) \geq 0$ for all $\varphi \in \mathscr{C}_{\mathrm{nor}}$, which can be expressed as $\mathrm{Re}\,\mathrm{Tr}(C_\psi C_{\varphi_\sharp}) \geq 0$. □

Lemma 4.2 *If $\mathscr{K}$ (and hence also $\mathscr{A}$) is finite-dimensional, then the map*

$$\gamma : \mathrm{CB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}} \to \mathrm{B}(\mathscr{K}),\ \ \gamma(\psi) = C_\psi = \psi(\Omega \otimes \Omega^*)$$

is an isomorphism.

Proof If $\gamma(\psi) = 0$, then $\psi(a\Omega \otimes (b\Omega)^*) = a\psi(\Omega \otimes \Omega^*)b^* = 0$ for all $a, b \in \mathscr{A}$, hence $\psi(x) = 0$ for all rank one operators $x \in \mathrm{B}(\mathscr{K})$, since $[\mathscr{A}\Omega] = \mathscr{K}$. Therefore $\psi = 0$ and γ is injective. To prove surjectivity, we only need to observe that for each $x \in \mathrm{B}(\mathscr{K})$ the map $\psi \in \mathrm{CB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$,

$$\psi(\sum_j a_j\Omega \otimes (b_j\Omega)^*) := \sum_j a_j x b_j^*$$

is well defined, for clearly ψ is then an $\mathscr{A}$-bimodule map on $\mathrm{B}(\mathscr{K})$ with $\psi(\Omega \otimes \Omega^*) = x$. For this, it suffices to note that $\sum_j a_j\Omega \otimes \Omega^* b_j^* = 0$ implies that $\sum_j a_j(a'\Omega \otimes (b'\Omega)^*)b_j^* = 0$ for all $a', b' \in \mathscr{A}'$, hence $\sum_j a_j x b_j^* = 0$ for all (rank one, at first) operators $x \in \mathrm{B}(\mathscr{K})$ since Ω is cyclic for $\mathscr{A}'$. □

Theorem 4.2(i) tells us that each mapping cone $\mathscr{C}$ in $\mathrm{CB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}}$ can be expressed as an intersection of cones of the form $\hat{\mathscr{C}_b} := \{\varphi \in \mathrm{CB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}} : \mathrm{Re}\,\varphi(b) \geq 0\}$ ($b \in \mathrm{B}(\mathscr{K})$), if $\mathscr{K}$ contains a cyclic and separating vector for $\mathscr{A}$. A similar argument shows that each mapping cone $\mathscr{C}$ in $\mathrm{NCB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}} = \mathrm{CB}_{\mathscr{A}}(\mathrm{K}(\mathscr{K}), \mathrm{B}(\mathscr{K}))_{\mathscr{A}} = (\mathrm{K}(\mathscr{K})_A \hat{\otimes}_A \mathrm{T}(\mathscr{K}))^\sharp$ is an intersection of cones of the form

$$\mathscr{C}_b := \{\varphi \in \mathrm{NCB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}} : \mathrm{Re}\,\varphi(b) \geq 0\}, \tag{4.1}$$

where b is in the smallest strong operator $\mathscr{A}$-subbimodule of $\mathrm{B}(\mathscr{K})$ containing $\mathrm{K}(\mathscr{K})$. If $\dim \mathscr{K} < \infty$, Lemma 4.2 (together with Theorem 4.2(i)) implies that $\mathscr{C}$ is an intersection of cones of the form $\mathscr{C}_b$, where b ranges over Choi matrices of maps, and then Lemma 4.1 implies that the duality of cones can be defined in terms of Choi matrices. But in infinite dimensions this is not true, as can be seen from Proposition 3.5 and the following remark.

Remark 4.1 If $b \notin \mathrm{T}(\mathscr{K})$, the cone $\mathscr{C}_b$ (defined by (4.1)) can not always be expressed as an intersection of cones of the form $\mathscr{C}_t$ ($t \in \mathrm{T}(\mathscr{K})$) (defined by (4.1) with b replaced by t). To prove this, it suffices to show that there exists a sequence of elements $a'_k \in \mathscr{A}'$ weak* converging to an element a' such that $\mathrm{Re}\,(a')^2 \not\leq 0$ and $(a'_k)^2 = 0$ for all k. Namely, then the sequence of maps

$$M_{a'_k, a'_k} \in \mathrm{NCB}_{\mathscr{A}}(\mathrm{B}(\mathscr{K}))_{\mathscr{A}}, \;\; M_{a'_k, a'_k}(x) := a'_k x a'_k,$$

is such that for each $t \in \mathrm{T}(\mathscr{K})$ the sequence $M_{a'_k, a'_k}(t)$ converges to $M_{a'a'}(t)$ in the weak operator topology. Hence the map $M_{a',a'}$ is in each cone of the form $\mathscr{C}_t$ ($t \in \mathrm{T}(\mathscr{K})$) that contains all the maps $M_{a'_k, a'_k}$. The cone $\mathscr{C}_{-1}$ contains all the maps $M_{a'_k, a'_k}$ (since $M_{a'_k, a'_k}(-1) = -(a'_k)^2 = 0$), but does not contain $M_{a',a'}$ (since $\mathrm{Re} M_{a',a'}(-1) = -\mathrm{Re}\,(a')^2 \not\geq 0$. Hence $\mathscr{C}_{-1} \neq \bigcap_{t \in \mathrm{T}(\mathscr{K}), \mathscr{C}_t \supseteq \mathscr{C}_{-1}} \mathscr{C}_t$.

To prove the existence of a sequence (a'_k) with the required properties in a II_1 factor $\mathscr{A}'$, let $p' \in \mathscr{A}'$ be a projection equivalent to $p'^{\perp} = 1 - p'$, hence $p'\mathscr{A}'p' \cong p'^{\perp}\mathscr{A}'p'^{\perp}$. Since in a factor of type II_1 there exists a sequence of unitaries weak* converging to 0 (this can be seen by taking (an injective) subfactor realized as $L(G)$ for an appropriate group G), there is a sequence $u'_k \in p'^{\perp}\mathscr{A}'p'$ weak* converging to 0 and such that $u'^*_k u'_k = p'$ and $u'_k u'^*_k = p'^{\perp}$. Then $a'_k = p' + u'_k - u'^*_k - p'^{\perp}$ satisfies $(a'_k)^2 = 0$, the a'_k weak* converge to $a' := p' - p'^{\perp}$ and $\mathrm{Re}\,(a')^2 = 1$.

The maps $M_{a'_k, a'_k}$ defined above are not completely positive, but replacing a'_k with $b'_k = a'_k \oplus a'^*_k$, $M_{a'_k, a'_k}$ with $M_{b'_k, b'^*_k}$ and -1 with the matrix $\begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$, we get an example of c.p. maps demonstrating the same phenomenon.

5 $\mathrm{B}(\mathscr{L})$-convex cones

The aim of this section is to show that when $\mathscr{A} = \mathrm{B}(\mathscr{L})$ the duality of cones considered in Section 4 reduces to the type of duality investigated by Størmer [26]. If $\mathscr{A} = \mathrm{B}(\mathscr{L})$ for a separable Hilbert space $\mathscr{L}$, the Hilbert space $\mathscr{K} := \mathscr{L} \otimes \mathscr{L}$ contains a cyclic and separating vector for $\mathscr{A} \cong 1_{\mathrm{B}(\mathscr{L})}\overline{\otimes}\mathscr{A}$. If we identify $\mathrm{B}(\mathscr{L})$ with $\mathrm{M}_m(\mathbb{C})$ (where $m = \dim \mathscr{L} \in \mathbb{N} \cup \{\infty\}$), then all strong operator $\mathrm{B}(\mathscr{L})$-systems are of the form $X = \mathrm{M}_m(V)$, where V is an operator system [18, 3.4]. We will use the notation $V\overline{\otimes}\mathrm{B}(\mathscr{L}) := \mathrm{M}_m(V)$. There is a natural completely isometric isomorphisms

$$\iota_V : V\overline{\otimes}\mathrm{B}(\mathscr{L}) \to \mathrm{CB}(\mathrm{T}(\mathscr{L}), V) \subseteq \mathrm{CB}(\mathrm{T}(\mathscr{L}), V^{\sharp\sharp}) = (V^{\sharp}\hat{\otimes}\mathrm{T}(\mathscr{L}))^{\sharp}, \tag{5.1}$$

[1, 1.5.14(10)], determined by

$$\langle (\iota_V(w))(t), v^{\sharp}\rangle = \langle w, v^{\sharp} \otimes t\rangle \quad (t \in \mathrm{T}(\mathscr{L}),\ v^{\sharp} \in V^{\sharp},\ w \in V\overline{\otimes}\mathrm{B}(\mathscr{L})),$$

which shows naturality, but it is more conveniently defined by choosing matrix units $e_{i,j} \in \mathrm{B}(\mathscr{L})$ and setting

$$\iota_V(\sum_{i,j} v_{i,j} \otimes e_{i,j})(t) = \sum_{i,j} \langle e_{i,j}, t\rangle v_{i,j} \ \ (t \in \mathrm{T}(\mathscr{L}),\ [v_{i,j}] \in \mathrm{M}_m(V) = V\overline{\otimes}\mathrm{B}(\mathscr{L})). \tag{5.2}$$

(To see that the sum on the right side of (5.2) is even norm convergent, set $v = \sum_{i,j}\langle e_{i,j}, t\rangle v_{i,j}$ and first note that for each $v^{\sharp} \in V^{\sharp}$ we have

$$|\langle v^{\sharp}, v\rangle| = \sum_{i,j}\langle e_{i,j}, t\rangle\langle v^{\sharp}, v_{i,j}\rangle| = |\mathrm{Tr}(t[\langle v^{\sharp}, v_{i,j}\rangle])| \leq \|t\|_1\|[v_{i,j}]\|\|v^{\sharp}\|.$$

Then apply this inequality to the tail of the series on the right side of (5.2), that is, to $t - p_n t p_n$ instead of t, where $p_n = \sum_{i=1}^{n} e_{i,i}$. Since $\lim_{n\to\infty}\|t - p_n t p_n\|_1 = 0$, it follows that the series on the right-hand side of (5.2) is norm-convergent.)

Using the natural identification

$$\mathrm{CB}_{\mathrm{B}(\mathscr{L})}(V\overline{\otimes}\mathrm{B}(\mathscr{L}), \mathrm{B}(\mathscr{L})\overline{\otimes}\mathrm{B}(\mathscr{L}))_{\mathrm{B}(\mathscr{L})} \cong \mathrm{CB}(V, \mathrm{B}(\mathscr{L})),\ \ \varphi\overline{\otimes}1_{\mathrm{B}(\mathscr{L})} \mapsto \varphi. \tag{5.3}$$

and also the identification $\mathrm{B}(\mathscr{L})\overline{\otimes}\mathrm{B}(\mathscr{L}) \cong \mathrm{CB}(\mathrm{T}(\mathscr{L}), \mathrm{B}(\mathscr{L}))$ (a special case of (5.1)), for each $w \in V\overline{\otimes}\mathrm{B}(\mathscr{L})$ the evaluation

$$e_w : \mathrm{CB}_{\mathrm{B}(\mathscr{L})}(V\overline{\otimes}\mathrm{B}(\mathscr{L}), \mathrm{B}(\mathscr{L})\overline{\otimes}\mathrm{B}(\mathscr{L}))_{\mathrm{B}(\mathscr{L})} \to \mathrm{B}(\mathscr{L})\overline{\otimes}\mathrm{B}(\mathscr{L}),$$

$$\varphi\overline{\otimes}1_{\mathrm{B}(\mathscr{L})} \overset{e_w}{\mapsto} (\varphi\overline{\otimes}1_{\mathrm{B}(\mathscr{L})})(w),$$

corresponds to the composition

$$c_w : \mathrm{CB}(V, \mathrm{B}(\mathscr{L})) \to \mathrm{CB}(\mathrm{T}(\mathscr{L}), \mathrm{B}(\mathscr{L})),\ c_w(\varphi) = \varphi \circ \iota_V(w);$$

this is the meaning of Lemma 5.1 below. We will use this fact to prove Corollary 5.1 below, which together with [26, 6.1.5, 6.1.6(iv)] shows that duality of cones of Section 4 essentially coincides, when $\mathscr{A} = \mathrm{B}(\mathscr{L})$, with the duality studied by Størmer [25], [26], [28] (at least for symmetric mapping cones as defined in [26]). We will omit the simple verification (using (5.2)) of the following lemma.

Lemma 5.1 *The following diagram commutes:*

$$\begin{array}{ccc} \mathrm{CB}_{\mathrm{B}(\mathscr{L})}(V\overline{\otimes}\mathrm{B}(\mathscr{L}),\mathrm{B}(\mathscr{L})\overline{\otimes}\mathrm{B}(\mathscr{L}))_{\mathrm{B}(\mathscr{L})} & \stackrel{\cong}{\longrightarrow} & \mathrm{CB}(V,\mathrm{B}(\mathscr{L})) \\ \downarrow e_w & & \downarrow c_w \\ \mathrm{B}(\mathscr{L})\overline{\otimes}\mathrm{B}(\mathscr{L}) & \stackrel{\iota_{\mathrm{T}(\mathscr{L})}}{\longrightarrow} & \mathrm{CB}(\mathrm{T}(\mathscr{L}),\mathrm{B}(\mathscr{L})) \end{array}.$$

For a map $\varphi : U \to V$ between operator spaces equipped with involutions the adjoint map φ^* is defined by

$$\varphi^*(u) = \varphi(u^*)^* \quad (u \in U).$$

For an operator system V the isomorphism $\iota_V : V\overline{\otimes}\mathrm{B}(\mathscr{L}) \to \mathrm{CB}(\mathrm{T}(\mathscr{L}),V)$ (defined by (5.2)) is easily seen to be involution preserving. But ι_V does not necessarily map positive elements in $V\overline{\otimes}\mathrm{B}(\mathscr{L})$ to completely positive maps and for this reason we state Lemma 5.2 below. By an *anti-automorphism* of a C*-algebra R we mean the involution preserving anti-automorphism, that is, a linear bijection τ on R such that $\tau(yx) = \tau(x)\tau(y)$ and $\tau(x^*) = \tau(x)^*$. For example, in the case $R = \mathrm{B}(\mathscr{K})$, choose an orthonormal basis of $\mathscr{K}$ and let τ be the transposition, $\tau(x) = x^T$, relative to this basis. For any two anti-automorphisms τ_1, τ_2 the composition $\tau_1\tau_2$ is an automorphism, so all anti-automorphisms are equivalent for our purposes.

Lemma 5.2 *Let τ be an anti-automorphism of $\mathrm{B}(\mathscr{L})$, V an operator system, $w \in V\overline{\otimes}\mathrm{B}(\mathscr{L})$ and $\varphi \in \mathrm{CB}(V,\mathrm{B}(\mathscr{L}))$. Then $(\varphi\overline{\otimes}1_{\mathrm{B}(\mathscr{L})})(w) \geq 0$ if and only if*

$$\tau \circ \varphi \circ (\iota_V(w)) \in \mathrm{CP}(\mathrm{T}(\mathscr{L}),\mathrm{B}(\mathscr{L})),$$

where $\mathrm{T}(\mathscr{L})$ inherits its matrix ordered structure from $\mathrm{B}(\mathscr{L})$.

Proof Denote $\iota = \iota_V$. Choose an orthonormal basis of $\mathscr{L}$, let $e_{i,j}$ be the corresponding matrix units, $p_n = \sum_{i=1}^n e_{i,i}$ and denote by M_{p_n,p_n} the two-sided multiplication $x \mapsto p_nxp_n$ on $\mathrm{T}(\mathscr{L})$ and on $\mathrm{B}(\mathscr{L})$. Since any anti-automorphism of $\mathrm{B}(\mathscr{L})$ is a composition of the transposition and an (inner) automorphism of $\mathrm{B}(\mathscr{L})$, the proof is reduced to the case when τ is the transposition relative to the given basis (so that τ commutes with the compressions $x \mapsto p_nxp_n$). For w of the form $w = \sum_{i,j} v_{i,j} \otimes e_{i,j}$ and $t \in \mathrm{T}(\mathscr{L})$ we have from (5.2) that $(\tau\circ\varphi\circ\iota(w))(t) = \sum_{i,j}\langle e_{i,j},t\rangle\tau\varphi(v_{i,j})$, from which we see that $\tau\circ\varphi\circ\iota(w)$ is c.p. if and only if all the maps $M_{p_n,p_n}\circ\tau\circ\varphi\circ\iota(w)\circ M_{p_n,p_n}$ are c.p.. Further, $(\varphi\overline{\otimes}1_{\mathrm{B}(\mathscr{L})})(w) \geq 0$ if and only if $M_{p_n,p_n}\circ\varphi\overline{\otimes}M_{p_n,p_n}(w) \geq 0$ for all n. In this way the proof is reduced to the case when $\dim\mathscr{L} < \infty$. Then, given $w = \sum_{i,j} v_{i,j}\otimes e_{i,j} \in V\otimes\mathrm{B}(\mathscr{L})$, we compute (using (5.2)) the Choi matrix of the map $\tau\circ\varphi\circ(\iota(w))$ to be equal to

$$\sum_{k,l}\tau\circ\varphi\circ(\iota(w))(e_{k,l})\otimes e_{k,l} = \sum_{k,l}\sum_{i,j}\langle e_{i,j},e_{k,l}\rangle\tau\varphi(v_{i,j})\otimes e_{k,l} = \sum_{k,l}\tau\varphi(v_{l,k})\otimes e_{k,l},$$

which is just the transpose of the matrix $\sum_{k,l}\varphi(v_{l,k})\otimes e_{l,k} = (\varphi\otimes 1_{\mathrm{B}(\mathscr{L})})(w)$. Since such a map is c.p. if and only if its Choi matrix is positive and positivity is preserved under transposition, this finishes the proof. □

All completely positive maps on $\mathrm{B}(\mathscr{L})$ can be weak* approximated by finite sums of maps of the form $x \mapsto a^*xa$ ($a \in \mathrm{B}(\mathscr{L})$) and all completely positive $1_{\mathrm{B}(\mathscr{L})}\otimes\mathrm{B}(\mathscr{L})$-bimodule maps on $\mathrm{B}(\mathscr{L})\overline{\otimes}\mathrm{B}(\mathscr{L})$ can be weak* approximated by finite sums of maps of the from $x \mapsto (b^*\otimes 1_{\mathrm{B}(\mathscr{L})})x(b\otimes 1_{\mathrm{B}(\mathscr{L})})$, where $b \in \mathrm{B}(\mathscr{L})$ (this follows from [5, 2.5]). Thus, if we define *a mapping cone* in $\mathrm{CB}(V,\mathrm{B}(\mathscr{L}))$ as a weak* closed cone $\mathscr{C}$ that is invariant under c.p. maps on $\mathrm{B}(\mathscr{L})$ (that is, $\theta\varphi \in \mathscr{C}$ for all $\varphi \in \mathscr{C}$ and all c.p. maps θ on $\mathrm{B}(\mathscr{L})$), then under isomorphism (5.3) such cones correspond precisely to the mapping cones in $\mathrm{CB}_{\mathrm{B}(\mathscr{L})}(V\overline{\otimes}\mathrm{B}(\mathscr{L}),\mathrm{B}(\mathscr{L})\overline{\otimes}\mathrm{B}(\mathscr{L}))_{\mathrm{B}(\mathscr{L})}$ as defined in Section 4.

The map $\iota_V : V\overline{\otimes}\mathrm{B}(\mathscr{L}) \to \mathrm{CB}(\mathrm{T}(\mathscr{L}),V)$ defined by (5.2) is an $\mathrm{B}(\mathscr{L})$-bimodule map, if we define the $\mathrm{B}(\mathscr{L})$-bimodule structure on $\mathrm{CB}(\mathrm{T}(\mathscr{L}),V)$ by $(a\psi b)(t) = \psi(bta)$. Therefore the cones $\mathscr{D}$ in $\mathrm{CB}(\mathrm{T}(\mathscr{L}),V)$ that correspond to operator cones in $V\overline{\otimes}\mathrm{B}(\mathscr{L})$ (as defined in Definition 4.1) have the property

$$\psi \circ M_{a^*,a} \in \mathscr{D} \;\; \forall \psi \in \mathscr{D} \text{ and } \forall\, a \in \mathrm{B}(\mathscr{L}),$$

where $M_{a^*,a}$ is the two-sided multiplication $t \mapsto a^*ta$ $(t \in \mathrm{T}(\mathscr{L}))$. We will call such cones *operator cones in* $\mathrm{CB}(\mathrm{T}(\mathscr{L}),V)$.

To interpret Theorem 4.2 in our present context, we still have to determine, which operator cones $\mathscr{D}$ in $\mathrm{CB}(\mathrm{T}(\mathscr{L}),V)$ are such that the corresponding cones $\iota_V^{-1}(\mathscr{D})$ in $V\overline{\otimes}\mathrm{B}(\mathscr{L})$ are closed in the $\mathrm{B}(\mathscr{L}),\mathrm{B}(\mathscr{L})$-topology, which is determined by all the functionals $\rho \in (V\overline{\otimes}\mathrm{B}(\mathscr{L}))^\sharp$ with the property that for each $x \in V\overline{\otimes}\mathrm{B}(\mathscr{L})$ the maps $a \mapsto \rho(ax)$ and $a \mapsto \rho(xa)$ are weak* continuous on $\mathrm{B}(\mathscr{L})$. We denote the space of all such functionals by $(V\overline{\otimes}\mathrm{B}(\mathscr{L}))^{\sharp_{\mathrm{B}(\mathscr{L})}}$.

Lemma 5.3 $(V\overline{\otimes}\mathrm{B}(\mathscr{L}))^{\sharp_{\mathrm{B}(\mathscr{L})}} \cong V^\sharp\hat{\otimes}\mathrm{T}(\mathscr{L})$.

Proof For each $\rho \in (V\overline{\otimes}\mathrm{B}(\mathscr{L}))^{\sharp_{\mathrm{B}(\mathscr{L})}}$ the restriction $\rho \mapsto \rho|(V\check{\otimes}\mathrm{K}(\mathscr{L}))$ is isometric since for a net (p_k) of finite-rank projections in $\mathrm{K}(\mathscr{L})$ increasing to the identity $1_{\mathrm{B}(\mathscr{L})}$ we have that $\rho(x) = \lim_{k,l}\rho(p_kxp_l)$ for all $x \in V\overline{\otimes}\mathrm{B}(\mathscr{L})$. On the other hand we have the completely isometric isomorphism $\kappa : V^\sharp\hat{\otimes}\mathrm{T}(\mathscr{L}) \to (V\check{\otimes}\mathrm{K}(\mathscr{L}))^\sharp$ (see [6, (10.1.9)]), given by

$$\kappa(v^\sharp \otimes t)(v\otimes c) = \langle v^\sharp, v\rangle\langle t, c\rangle \;\; (c \in \mathrm{K}(\mathscr{L}), t \in \mathrm{T}(\mathscr{L}), v^\sharp \in V^\sharp, v \in V). \tag{5.4}$$

If we show that for each $y \in V^\sharp\hat{\otimes}\mathrm{T}(\mathscr{L})$ the maps $a \mapsto a\kappa(y)$ and $a \mapsto \kappa(y)a$ are weak* continuous on $\mathrm{B}(\mathscr{L})$, then (since $V\check{\otimes}\mathrm{K}(\mathscr{L})$ is dense in $V\overline{\otimes}\mathrm{B}(\mathscr{L})$ in the $\mathrm{B}(\mathscr{L}),\mathrm{B}(\mathscr{L})$-topology) each $\kappa(y)$ has a unique extension to an element $\rho \in (V\overline{\otimes}\mathrm{B}(\mathscr{L}))^{\sharp_{\mathrm{B}(\mathscr{L})}}$, so this will prove the lemma. To prove the required continuity, we may assume that y is of the form $y = v^\sharp \otimes t$ $(v^\sharp \in V^\sharp, t \in \mathrm{T}(\mathscr{L}))$, since the space of normal functionals on a W*-algebra is norm-closed. Thus we need to verify that for each $z \in V\check{\otimes}\mathrm{K}(\mathscr{L})$ the map $a \mapsto (a\kappa(y))(z) = \kappa(y)(za)$ is weak* continuous on $\mathrm{B}(\mathscr{L})$ (and similarly for the other map) and again we may assume that z is of the form $z = v\otimes c$ $(v \in V,\ c \in \mathrm{K}(\mathscr{L}))$. In this case we have by (5.4) $\kappa(y)(za) = \kappa(v^\sharp \otimes t)(v\otimes ca) = \langle v^\sharp, v\rangle\langle t, ca\rangle$, which is clearly weak* continuous in a. □

Let $\mathrm{CB}_{\mathrm{sa}}(X,Y)$ denotes the subset of all self-adjoint maps in $\mathrm{CB}(X,Y)$, that is, maps φ satisfying $\varphi(x^*) = \varphi(x)^*$). (Note that for such maps $\mathrm{Re}\,\varphi(x) = \varphi(\mathrm{Re}x)$.) Since τ and ι_V are involution-preserving, in view of (5.3) and Lemmas 5.1, 5.2, 5.3 we can now reformulate for mapping cones $\mathscr{C} \subseteq \mathrm{CB}_{\mathrm{sa}}(V,\mathrm{B}(\mathscr{L}))$ and operator cones $\mathscr{D} \subseteq \mathrm{CB}_{\mathrm{sa}}(\mathrm{T}(\mathscr{L}),V)$ the definitions of $\mathscr{C}_\circ$ and $\mathscr{D}^\circ$ as

$$\mathscr{C}_{\circ\mathrm{sa}} := \{\sigma \in \mathrm{CB}_{\mathrm{sa}}(\mathrm{T}(\mathscr{L}),V) :\; \tau\varphi\sigma \text{ is completely positive } \forall \varphi \in \mathscr{C}\}$$

and

$$\mathscr{D}^{\circ\mathrm{sa}} := \{\varphi \in \mathrm{CB}_{\mathrm{sa}}(V,\mathrm{B}(\mathscr{L})) :\; \tau\varphi\sigma \text{ is completely positive } \forall \sigma \in \mathscr{D}\}.$$

Then the following can be considered as a special case of Theorem 4.2:

Corollary 5.1 $(\mathscr{C}_{\circ\mathrm{sa}})^{\circ\mathrm{sa}} = \mathscr{C}$ *and* $(\mathscr{D}^{\circ\mathrm{sa}})_{\circ\mathrm{sa}} = \mathscr{D}$ *for all mapping cones* $\mathscr{C} \subseteq \mathrm{CB}_{\mathrm{sa}}(V,\mathrm{B}(\mathscr{L}))$ *(which are weak* closed by definition) and all operator cones in* $\mathscr{D} \subseteq \mathrm{CB}_{\mathrm{sa}}(\mathrm{T}(\mathscr{L}),V)$ *that are closed in the topology determined by* $V^\sharp\hat{\otimes}\mathrm{T}(\mathscr{L})$.

In Corollary 5.1 the duality between $\mathrm{CB}(\mathrm{T}(\mathscr{L}),V)$ and $V^\sharp\hat{\otimes}\mathrm{T}(\mathscr{L})$ is of course given by $\langle\varphi, v^\sharp\otimes t\rangle = \langle v^\sharp, \varphi(t)\rangle$. Note also that $\mathscr{C}\subseteq \mathrm{CB}_{\mathrm{sa}}(V,\mathrm{B}(\mathscr{L}))$ may be any weak* closed cone such that $a^*\mathscr{C}a\subseteq\mathscr{C}$ for all $a\in\mathrm{B}(\mathscr{L})$, where V is any operator system.

6 Positivity in operator-projective tensor products

Let τ be an anti-automorphism of $\mathrm{B}(\mathscr{K})$. The pairing

$$\langle b,t\rangle_\tau := \mathrm{Tr}(\tau(b)t) \;\; (b\in\mathrm{B}(\mathscr{K}),\, t\in\mathrm{T}(\mathscr{K})), \tag{6.1}$$

makes $\mathrm{T}(\mathscr{K})$ a predual of $\mathrm{B}(\mathscr{K})$. Since τ is the composition of an automorphism and the transposition, τ is weak* continuous, hence $\tau_\sharp : \mathrm{T}(\mathscr{K})\to\mathrm{T}(\mathscr{K})$ exists.

The reason of using the pairing (6.1), instead of just $(b,t)\mapsto\mathrm{Tr}(bt)$, is in the well-known correspondence between maps in $\mathrm{CP}(\mathrm{B}(\mathscr{H}),\mathrm{B}(\mathscr{K}))$ and positive functionals on $\mathrm{B}(\mathscr{H})\hat{\otimes}\mathrm{T}(\mathscr{K})$. Namely, an element $w\in\mathrm{B}(\mathscr{H})\hat{\otimes}\mathrm{T}(\mathscr{K})$ is called positive if and only if it is positive in the C*-algebra $\mathrm{B}(\mathscr{H})\otimes\mathrm{B}(\mathscr{K})$ and it is well-known (see [26, 4.2.7] in the case τ is a transposition) that a map $\varphi\in\mathrm{CB}(\mathrm{B}(\mathscr{H}),\mathrm{B}(\mathscr{K}))$ is completely positive if and only if the *dual functional* $\tilde{\varphi}\in(\mathrm{B}(\mathscr{H})\hat{\otimes}\mathrm{T}(\mathscr{K}))^\sharp$, defined by

$$\tilde{\varphi}(x\otimes t) := \langle\tau(\varphi(x)),t\rangle = \mathrm{Tr}(\tau(\varphi(x))t) \;\; (x\in\mathrm{B}(\mathscr{H}),\, t\in\mathrm{T}(\mathscr{K}))$$

is positive. (In general W*-algebras it seems more natural to use anti-automorphisms, when they exist [4], instead of transpositions.) This will suggest how to define positive elements in tensor products of the form $X_A\hat{\otimes}_A\mathscr{R}_\sharp$, where X is an operator A-system and $\mathscr{R}_\sharp$ is the predual of a W*-algebra $\mathscr{R}$ such that $\mathscr{R}$ contains the image of A under a $*$-homomorphism π, so that $\mathscr{R}$ is an operator A-bimodule by $axb := \pi(a)x\pi(b)$ $(x\in\mathscr{R},\, a,b\in A)$. Thus $\mathscr{R}_\sharp$ is a Banach A-bimodule by

$$\langle x, a\omega b\rangle := \langle bxa,\omega\rangle \;\; (a,b\in A,\; x\in\mathscr{R},\; \omega\in\mathscr{R}_\sharp).$$

We would like to associate to each map $\varphi\in\mathrm{CB}_A(X,\mathscr{R})_A$ a linear functional $\tilde{\varphi}$ on $X_A\hat{\otimes}_A\mathscr{R}_\sharp$ by

$$\tilde{\varphi}(x_A\otimes_A\omega) = \langle\tau(\varphi(x)),\omega\rangle \;\; (x\in X,\; \omega\in\mathscr{R}_\sharp), \tag{6.2}$$

where τ is a fixed anti-automorphism of $\mathscr{R}$. For $\tilde{\varphi}$ to be well-defined, we must modify the A-bimodule multiplications in $\mathscr{R}_\sharp$ so that (denoting by $\circ$ the new multiplication)

$$\langle\tau(\varphi(axb)),\omega\rangle = \tilde{\varphi}(axb_A\otimes_A\omega) = \tilde{\varphi}(x_A\otimes_A b\circ\omega\circ a) = \langle\tau(\varphi(x)), b\circ\omega\circ a\rangle$$

for all $a,b\in A$, $x\in X$ and $\omega\in\mathscr{R}_\sharp$. This means that $\langle\tau(\pi(a)\varphi(x)\pi(b)),\omega\rangle = \langle\tau(\varphi(x)), b\circ\omega\circ a\rangle$, which can also be written as

$$\langle\tau(\varphi(x)), \tau(\pi(a))\omega\tau(\pi(b))\rangle = \langle\tau(\varphi(x)), b\circ\omega\circ a\rangle.$$

Thus we introduce to $\mathscr{R}_\sharp$ the new A-bimodule multiplications by

$$a\circ\omega := \omega\tau(\pi(a)) \;\text{ and }\; \omega\circ a := \tau(\pi(a))\omega \;\; (\omega\in\mathscr{R}_\sharp,\, a\in A). \tag{6.3}$$

$\mathscr{R}_\sharp$ equipped with the A-bimodule operations (6.3) is denoted by $\mathscr{R}_\sharp^\tau$.

$\mathscr{R}_\sharp^\tau$ is regarded as a predual of $\mathscr{R}$ through the pairing

$$\langle x,\omega\rangle_\tau := \langle\tau(x),\omega\rangle \tag{6.4}$$

and as such carries the operator space structure, where for each $n \in \mathbb{N}$ the norm on $\mathrm{M}_n(\mathscr{R}_\sharp^\tau)$ is defined by

$$\|[\omega_{i,j}]\|_\tau = \sup\{\|[\langle \tau(x_{k,l}), \omega_{i,j}\rangle]\| : [x_{k,l}] \in \mathrm{M}_m(\mathscr{R}),\ \|[x_{k,l}]\| = 1,\ m \in \mathbb{N}\}.$$

The usual involution on $\mathscr{R}_\sharp$ is also an involution on $\mathscr{R}_\sharp^\tau$ and has the property

$$(a \circ \omega \circ b)^* = b^* \circ \omega^* \circ a^* \quad (a, b \in A,\ t \in \mathscr{R}_\sharp^\tau).$$

Definition 6.1 Given an operator A-system X, an element $w \in X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau$ is *positive* ($w \geq 0$) if and only if $\tilde{\varphi}(w) \geq 0$ for all $\varphi \in \mathrm{CP}_A(X, \mathscr{R})$, where $\tilde{\varphi} \in (X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau)^\sharp$ is determined by (6.2). Let $(X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau)_+$ denote the cone of all positive elements in $X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau$.

Under the isomorphism $(X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau)^\sharp \cong \mathrm{CB}_A(X, \mathscr{R})_A$ (where the duality between $\mathscr{R}_\sharp^\tau$ and $\mathscr{R}$ is given by (6.4)) $\tilde{\varphi}$ corresponds to φ, so $\tilde{\varphi}$ is bounded.

There is a natural *involution* on $X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau$, determined by

$$(x_A \hat{\otimes}_A \omega)^* = x^*{}_A \hat{\otimes}_A \omega^* \quad (x \in X,\ \omega \in \mathscr{R}_\sharp^\tau).$$

A matrix $[\omega_{i,j}] \in \mathrm{M}_n(\mathscr{R}_\sharp^\tau)$ is *positive* if and only if the matrix $[\rho_{i,j}] := [\tau_\sharp(\omega_{i,j})]$ is positive in $\mathrm{M}_n(\mathscr{R}_\sharp)$ (that is, if and only if the map $\mathscr{R} \to \mathrm{M}_n(\mathbb{C})$, $x \mapsto [\rho_{i,j}(x)]$ is positive). This implies the following remark.

Remark 6.1 Let τ and ν be anti-automorphisms of $\mathscr{R}$. A map $\sigma : \mathscr{R}_\sharp^\tau \to \mathscr{R}_\sharp^\nu$ is completely positive if and only if $\nu_\sharp \sigma \tau_\sharp^{-1}$ is a completely positive map on $\mathscr{R}_\sharp$.

Proposition 6.1 *Let τ and ν be anti-automorphisms of $\mathscr{R}$ and let $\alpha = \tau\nu^{-1}$, an automorphism of $\mathscr{R}$. For any operator A-system X let id_X be the identity map on X and denote by Ψ the map $\mathrm{id}_{XA} \hat{\otimes}_A \alpha_\sharp : X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau \to X_A \hat{\otimes}_A \mathscr{R}_\sharp^\nu$. Then:*

(i) Ψ is a completely isometric isomorphism such that $\tilde{\varphi}^\nu(\Psi(w)) = \tilde{\varphi}^\tau(w)$ for each map $\varphi \in \mathrm{CP}_A(X, \mathscr{R})$ and all $w \in X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau$, where $\tilde{\varphi}^\tau$ and $\tilde{\varphi}^\nu$ are the linear functionals on $X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau$ and $X_A \hat{\otimes}_A \mathscr{R}_\sharp^\nu$ (respectively) that correspond to the map φ according to the rule (6.2). In particular

$$\Psi((X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau)_+) = (X_A \hat{\otimes}_A \mathscr{R}_\sharp^\nu)_+.$$

(ii) $(\theta_A \hat{\otimes}_A \sigma)((X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau)_+) \subseteq (Y_A \hat{\otimes}_A \mathscr{R}_\sharp^\nu)_+$ for completely positive A-bimodule maps $\sigma : \mathscr{R}_\sharp^\tau \to \mathscr{R}_\sharp^\nu$ and $\theta : X \to Y$ between any operator A-systems X, Y.

(iii) If $X \subseteq Y$ are operator A-systems and $\mathscr{R}$ is injective, then the natural map

$$X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau \to Y_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau$$

is completely isometric and hence, regarding $X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau$ as a subspace of $Y_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau$, we have $(X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau)_+ = (X_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau) \cap (Y_A \hat{\otimes}_A \mathscr{R}_\sharp^\tau)_+$.

Proof (i) First observe that

$$\alpha_\sharp(c\omega d) = \alpha^{-1}(c)\alpha_\sharp(\omega)\alpha^{-1}(d) \quad (c, d \in \mathscr{R},\ \omega \in \mathscr{R}_\sharp) \tag{6.5}$$

for every automorphism α of $\mathscr{R}$. Indeed, for every $x \in \mathscr{R}$ we compute that

$$\begin{aligned}\langle x, \alpha_\sharp(c\omega d)\rangle =& \langle d\alpha(x)c, \omega\rangle = \langle \alpha(\alpha^{-1}(d)x\alpha^{-1}(c)), \omega\rangle \\ =& \langle \alpha^{-1}(d)x\alpha^{-1}(c), \alpha_\sharp(\omega)\rangle = \langle x, \alpha^{-1}(c)\alpha_\sharp(\omega)\alpha^{-1}(d)\rangle,\end{aligned}$$

from which (6.5) follows. Using (6.5) with $\alpha = \tau\nu^{-1}$, it is straightforward to verify that $\alpha_\sharp : \mathscr{R}_\sharp^\tau \to \mathscr{R}_\sharp^\nu$ is a homomorphism of A-bimodules. It can be verified that $\alpha_\sharp$ is also a completely isometric isomorphism, hence so is $\Psi = \mathrm{id}_X{}_A\hat{\otimes}_A\alpha_\sharp$. Further, for any $\varphi \in \mathrm{CP}_A(X,\mathscr{R})$, $x \in X$ and $\omega \in \mathscr{R}_\sharp$ we have

$$\begin{aligned}\tilde{\varphi}^\nu(\Psi(x_A\hat{\otimes}_A\omega) =&\tilde{\varphi}^\nu(x_A\hat{\otimes}_A\alpha_\sharp(\omega)) = \langle \nu(\varphi(x)), \alpha_\sharp(\omega)\rangle\\ =&\langle \alpha(\nu(\varphi(x))), \omega\rangle = \langle \tau(\varphi(x)), \omega\rangle = \tilde{\varphi}^\tau(x_A\hat{\otimes}_A\omega),\end{aligned}$$

hence by linearity and continuity $\tilde{\varphi}^\nu(\Psi(w)) = \tilde{\varphi}^\tau(w)$ for all $w \in X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau$.

(ii) Let $\rho = \nu_\sharp\sigma\tau_\sharp^{-1}$, so that ρ is a c.p. map on $\mathscr{R}_\sharp$ by Remark 6.1. For any $\varphi \in \mathrm{CP}_A(Y,\mathscr{R})$ and $x \in X$, $\omega \in \mathscr{R}_\sharp^\tau$ we have for $w := x_A\hat{\otimes}_A\omega$ that

$$\begin{aligned}\tilde{\varphi}^\nu((\theta_A\hat{\otimes}_A\sigma)(w)) =&\langle \nu(\varphi(\theta(x))), \sigma(\omega)\rangle = \langle (\varphi\theta)(x), \nu_\sharp((\nu_\sharp^{-1}\rho\tau_\sharp)(\omega))\rangle\\ =&\langle \tau((\rho^\sharp\varphi\theta)(x)), \omega\rangle = (\widetilde{(\rho^\sharp\varphi\theta)}^\tau(w),\end{aligned}$$

hence by linearity and continuity

$$\tilde{\varphi}^\nu((\theta_A\hat{\otimes}_A\sigma)(w)) = \widetilde{\rho^\sharp\varphi\theta}^\tau(w)$$

for all $w \in X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau$. If w is positive, so is by definition also $\widetilde{\rho^\sharp\varphi\theta}^\tau(w)$ since $\rho^\sharp\varphi\theta \in \mathrm{CP}_A(X,\mathscr{R})$, hence by the last equality $(\theta_A\hat{\otimes}_A\sigma)(w)$ is positive.

(iii) By injectivity of $\mathscr{R}$ any map in $\mathrm{CB}_A(X,\mathscr{R})_A$ extends to a map in $\mathrm{CB}_A(Y,\mathscr{R})_A$ of the same norm [20, p. 95], so the restriction map $\mathrm{CB}_A(Y,\mathscr{R})_A \to \mathrm{CB}_A(X,\mathscr{R})_A$ is a completely quotient map, hence its pre-adjoint $X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau \to Y_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau$ must be completely isometric. The last identity in (iii) follows now directly from the definition of the cones $(X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau)_+$ and $(Y_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau)_+$ since any map in $\mathrm{CP}_A(X,\mathscr{R})$ can be extended to a map in $\mathrm{CP}_A(Y,\mathscr{R})$. □

The operator space structure on $\mathscr{R}_\sharp^\tau$ is independent of the choice of τ since for any other anti-automorphism ν of $\mathscr{R}$ the map $\alpha = \tau\nu^{-1}$ is an automorphism, hence completely isometric. When $\pi(A)$ is contained in the center $\mathscr{Z}$ of $\mathscr{R}$ and $\tau(z) = z$ for each $z \in \mathscr{Z}$ (for example, if $A = \mathbb{C}$), then the bimodule operations (6.3) are the usual ones on $\mathscr{R}_\sharp$, hence $\mathscr{R}_\sharp^\tau$ is independent of the choice of τ and so $X_A\hat{\otimes}_A\mathscr{R}_\sharp^\nu = X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau$. Further, one can verify that $\tilde{\varphi}^\tau(\Psi(w)) = (\tau^{-1}\alpha\tau\varphi)^{\sim\tau}(w)$ for all $w \in X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau$ and $\varphi \in \mathrm{CP}_A(X,\mathscr{R})$. Since any automorphism (in particular $\tau^{-1}\alpha\tau$) is a c.p. map, it follows from the definition of $(X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau)_+$ that the map $\Psi = \mathrm{id}_X{}_A\hat{\otimes}_A\alpha_\sharp$ preserves the set $(X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau)_+$. On the other hand, by Proposition 6.1 $\Psi((X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau)_+) = (X_A\hat{\otimes}_A\mathscr{R}_\sharp^\nu)_+$, hence $(X_A\hat{\otimes}_A\mathscr{R}_\sharp^\nu)_+ = (X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau)_+$. This proves the following corollary.

Corollary 6.2 *If $\pi(A) \subseteq \mathscr{Z}$, then for all anti-automorphisms τ of $\mathscr{R}$ satisfying $\tau|\mathscr{Z} = \mathrm{id}_\mathscr{Z}$ the space $X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau$ and the set $(X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau)_+$ are independent of τ.*

Given a homomorphism $\pi_A : A \to R$ of C*-algebras and an anti-automorphism τ of R, regarding $R^\sharp$ as the pre-dual $(R^{\sharp\sharp})_\sharp$ of the universal W*-envelope $R^{\sharp\sharp}$ of R, we see that $R^\sharp$ inherits from $(R^{\sharp\sharp})_\sharp^{\tau^{\sharp\sharp}}$ an operator A-bimodule structure. $R^\sharp$ with this new structure is denoted by $R_\tau^\sharp$.

If $\mathscr{R}_\sharp$ is the dual of a not necessarily unital C*-algebra B (for example, if $\mathscr{R} = \mathrm{B}(\mathscr{H})$), then the adjoint map of the natural embedding $\iota : B \to B^{\sharp\sharp} = \mathscr{R}$ is a projection $\iota^\sharp : \mathscr{R}^\sharp = B^{\sharp\sharp\sharp} \to B^\sharp = \mathscr{R}_\sharp$. This implies that for any homomorphism $\pi : A \to B$ of C*-algebras (making B an operator A-bimodule) and any operator A-bimodule X the natural map $\Theta : X_A\hat{\otimes}_A\mathscr{R}_\sharp \to X_A\hat{\otimes}_A\mathscr{R}^\sharp$ is completely isometric.

Proposition 6.3 *If $\mathscr{R}_\sharp = B^\sharp$ for a not necessarily unital C^*-algebra B which is a bimodule over a C^*-algebra A, X is an operator A-system, σ is an anti-automorphism of B and $\tau = \sigma^{\sharp\sharp}$ (an automorphism of $\mathscr{R}$), then an element $w \in X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau$ is positive if and only if $\Theta(w)$ is positive in $X_A\hat{\otimes}_A(\mathscr{R}^\sharp)^{\tau^{\sharp\sharp}}$, where $\Theta : X_A\hat{\otimes}_A\mathscr{R}_\sharp \to X_A\hat{\otimes}_A\mathscr{R}^\sharp$ is the natural map.*

Proof By definition w is positive if and only if $\tilde{\varphi}^\tau(w) \geq 0$ for all $\varphi \in \mathrm{CP}_A(X,\mathscr{R})$, while $\Theta(w)$ is positive if and only if $\tilde{\psi}^{\tau^{\sharp\sharp}}(\Theta(w)) \geq 0$ for all $\psi \in \mathrm{CP}_A(X,\mathscr{R}^{\sharp\sharp})$. Let $\kappa : \mathscr{R}_\sharp \to \mathscr{R}^\sharp$ be the canonical map (regarded as inclusion) and $q = \kappa^\sharp$, a projection from $\mathscr{R}^{\sharp\sharp}$ onto $\mathscr{R}$. On elements of the form $x_A\otimes_A\omega$ ($x \in X$, $\omega \in \mathscr{R}_\sharp^\tau$) we have

$$\begin{aligned}\tilde{\psi}^{\tau^{\sharp\sharp}}(\Theta(x_A\otimes_A\omega)) =& \langle \tau^{\sharp\sharp}(\psi(x)),\omega\rangle = \langle \psi(x),\tau^\sharp(\omega)\rangle \\ =& \langle \psi(x),\kappa(\tau^\sharp(\omega))\rangle = \langle \tau q\psi(x),\omega\rangle = \widetilde{q\psi}^\tau(x_A\otimes_A\omega).\end{aligned}$$

By linearity and continuity the identity $\tilde{\psi}^{\tau^{\sharp\sharp}}(\Theta(w)) = \widetilde{q\psi}^\tau(w)$ holds for all $w \in X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau$. Since $q\psi \in \mathrm{CP}_A(X,\mathscr{R})$ for each $\psi \in \mathrm{CP}_A(X,\mathscr{R}^{\sharp\sharp})$, and since each $\varphi \in \mathrm{CP}_A(X,\mathscr{R})$ may be regarded as an element of $\mathrm{CP}_A(X,\mathscr{R}^{\sharp\sharp})$, it follow that w is positive if and only if $\Theta(w)$ is positive. □

7 $\mathscr{C}$-positive maps

Motivated by the work of Størmer [24], [26] we state the following definition.

Definition 7.1 Let $\mathscr{R}$ be a W*-algebra, $\pi : A \to \mathscr{R}$ a homomorphism of C*-algebras (making $\mathscr{R}$ an A-bimodule) and τ an anti-automorphism of $\mathscr{R}$ such that $\tau(\pi(A)) = \pi(A)$. For a mapping cone $\mathscr{C} \subseteq \mathrm{CB}_A(\mathscr{R})_A$ consisting of self-adjoint maps φ (that is, $\varphi(x^*) = \varphi(x)^*$) and an operator A-system X define

$$P_{\mathscr{C}}(X) = \{w \in X_A\hat{\otimes}_A\mathscr{R}_\sharp^\tau :\ w = w^*,\ (\mathrm{id}_{XA}\hat{\otimes}_A\theta^\sharp)(w) \in (X_A\hat{\otimes}_A\mathscr{R}_\tau^\sharp)_+\ \ \forall\theta \in \mathscr{C}\}.$$

A map $\psi = \psi^* \in \mathrm{CB}_A(X,\mathscr{R})_A$ is called *$\mathscr{C}$-positive* if

$$\tilde{\psi}^\tau(P_{\mathscr{C}}(X)) \subseteq [0,\infty).$$

The set of all $\mathscr{C}$-positive maps is denoted by $P_{\mathscr{C}}(X,\mathscr{R})$.

Lemma 7.1 *$\tau\theta\tau^{-1} \in \mathrm{CB}_A(\mathscr{R})_A$ for all $\theta \in \mathrm{CB}_A(\mathscr{R})_A$ and $\|\tau\theta\tau^{-1}\|_{\mathrm{cb}} = \|\theta\|_{\mathrm{cb}}$.*

It is routine to verify that $\tau\theta\tau^{-1}$ is an A-bimodule map. The verification of the norm equality in Lemma 7.1 is based on the fact that the map $\alpha : \mathscr{R} \to \mathscr{R}$, $\alpha(x) = \tau(x)^*$, is a $*$-isomorphism of $\mathscr{R}$ onto the conjugate algebra of $\mathscr{R}$, hence completely isometric, as well as on the fact that the involution is isometric. We will omit the details of this verification.

Theorem 7.1 *In the situation as in Definition 7.1 assume that $\mathscr{R}_\sharp = B^\sharp$ for a not necessarily unital C^*-algebra B, $\tau = \tau_B^{\sharp\sharp}$ for an anti-automorphism τ_B of B, $\pi = \iota\pi_A$ for a $*$-homomorphism $\pi_A : A \to B$, where $\iota : B \to B^{\sharp\sharp} = \mathscr{R}$ is the canonical map, $\tau\mathscr{C}\tau^{-1} = \mathscr{C}$ and $\tau(\pi(A)) = \pi(A)$. Then $P_{\mathscr{C}}(X,\mathscr{R})$ is the smallest mapping cone $\tilde{\mathscr{C}}$ in $\mathrm{CB}_A(X,\mathscr{R})_A$ that contains all maps $\theta\varphi$, where $\theta \in \mathscr{C}$ and $\varphi \in \mathrm{CP}_A(X,\mathscr{R})$. (By definition $\tilde{\mathscr{C}}$ is the weak* closure of all finite sums of maps of the form $\theta\varphi$).*

Proof Let us first verify the inclusion $\tilde{\mathscr{C}} \subseteq P_{\mathscr{C}}(X,\mathscr{R})$, in other words, that $\theta\varphi \in P_{\mathscr{C}}(X,\mathscr{R})$ for all $\theta \in \mathscr{C}$ and $\varphi \in \mathrm{CP}_A(X,\mathscr{R})$. Since $(\theta\varphi)^* = \theta^*\varphi^* = \theta\varphi$, we need only to show that $\widetilde{\theta\varphi}^{\tau}(w) \geq 0$ for all $w \in P_{\mathscr{C}}(X)$. By Lemma 7.1 $\tau\theta\tau^{-1} \in \mathrm{CB}_A(\mathscr{R})_A$, hence $(\tau^{-1})^\sharp\theta^\sharp\tau^\sharp \in \mathrm{CB}_A(\mathscr{R}^\sharp_\tau)_A$, so we have for all $x \in X$ and $\omega \in \mathscr{R}^\tau_\sharp$

$$\begin{aligned}\widetilde{\theta\varphi}^{\tau}(x_A\otimes_A\omega) =& \langle\tau\theta\varphi(x),\omega\rangle = \langle\varphi(x),\theta^\sharp\tau^\sharp(\omega)\rangle \\ =& \langle\tau\varphi(x),(\tau^\sharp)^{-1}\theta^\sharp\tau^\sharp(\omega)\rangle = \tilde{\varphi}^\tau(\mathrm{id}_{XA}\hat{\otimes}_A(\tau^\sharp)^{-1}\theta^\sharp\tau^\sharp)(x_A\otimes_A\omega).\end{aligned}$$

Thus by linearity and continuity

$$\widetilde{\theta\varphi}^{\tau}(w) = \tilde{\varphi}^\tau((\mathrm{id}_{XA}\hat{\otimes}_A(\tau^\sharp)^{-1}\theta^\sharp\tau^\sharp)(w)) \tag{7.1}$$

for all $w \in X_A\hat{\otimes}_A\mathscr{R}^\tau_\sharp$. Since by hypothesis $\tau\theta\tau^{-1} \in \mathscr{C}$ and $w \in P_{\mathscr{C}}(X)$, we have by the definition of $P_{\mathscr{C}}(X)$ that

$$(\mathrm{id}_{XA}\hat{\otimes}_A(\tau\theta\tau^{-1})^\sharp)(w) \in (X_A\hat{\otimes}_A\mathscr{R}^\sharp_\tau)_+$$

for all $w \in P_{\mathscr{C}}(X)$. By the definition of positive elements in $X_A\hat{\otimes}_A\mathscr{R}^\sharp_\tau$ this means that $\tilde{\varphi}^\tau((\mathrm{id}_{XA}\hat{\otimes}_A(\tau\theta\tau^{-1})^\sharp)(w)) \geq 0$ for all $\varphi \in \mathrm{CP}_A(X,\mathscr{R}^{\sharp\sharp})$, hence in particular for all $\varphi \in \mathrm{CP}_A(X,\mathscr{R})$, thus it follows from (7.1) that $\widetilde{\theta\varphi}^{\tau}(w) \geq 0$.

Now, to prove the equality $\tilde{\mathscr{C}} = P_{\mathscr{C}}(X,\mathscr{R})$, assume the contrary, that there exists $\psi \in P_{\mathscr{C}}(X,\mathscr{R}) \setminus \tilde{\mathscr{C}}$. Then by the Hahn-Banach theorem there exists a functional

$$w \in (\mathrm{CB}_A(X,\mathscr{R})_A)_\sharp = X_A\hat{\otimes}_A\mathscr{R}^\tau_\sharp$$

such that

$$\widetilde{\theta\varphi}^{\tau}(\mathrm{Re}\,w) = \mathrm{Re}\,\widetilde{\theta\varphi}^{\tau}(w) \geq 0 \;\; \forall\theta \in \mathscr{C}, \forall\varphi \in \mathrm{CP}_A(X,\mathscr{R}) \tag{7.2}$$

and

$$\psi(\mathrm{Re}\,w) = \mathrm{Re}\,\psi(w) < 0. \tag{7.3}$$

By (7.1) the inequality (7.2) means that $\tilde{\varphi}^\tau((\mathrm{id}_{XA}\hat{\otimes}_A(\tau^\sharp)^{-1}\theta^\sharp\tau^\sharp)(\mathrm{Re}\,w)) \geq 0$; since this holds for all $\varphi \in \mathrm{CP}_A(X,\mathscr{R})$, it follows from Proposition 6.3 that

$$(\mathrm{id}_{XA}\hat{\otimes}_A(\tau^\sharp)^{-1}\theta^\sharp\tau^\sharp)(\mathrm{Re}\,w) \in (X_A\hat{\otimes}_A\mathscr{R}^\sharp_\tau)_+.$$

Since this holds for all $\theta \in \mathscr{C}$ and $\tau\mathscr{C}\tau^{-1} = \mathscr{C}$, we conclude that $\mathrm{Re}\,w \in P_{\mathscr{C}}(X)$, hence $\psi(\mathrm{Re}\,w) \geq 0$ since $\psi \in P_{\mathscr{C}}(X,\mathscr{R})$. But this contradicts (7.3). □

The following is a partial generalization of the Størmer extension theorem [24], [29]. (In the case $A = \mathbb{C}$ it covers the symmetric mapping cones of [26], but not the general ones since our definition of $\mathscr{C}$-positivity is slightly different from [26].)

Theorem 7.2 *In the same situation as in Theorem 7.1 assume that $\mathscr{C}$ consists only of positive maps and that $\mathscr{R}$ is injective. Let Y be an operator A-system containing X. Then each $\psi \in P_{\mathscr{C}}(X,\mathscr{R})$ can be extended to a map in $P_{\mathscr{C}}(Y,\mathscr{R})$.*

Proof By Theorem 7.1 there is a net of maps ψ_k of the form $\psi_k = \sum_j \theta_{k,j}\varphi_{k,j}$ converging to ψ in the weak*, hence also in the point weak* topology. Thus in particular $\psi_k(1) \to \psi(1)$, so, passing to a subnet, we may assume that the net $(\psi_k(1))$ is bounded. Since $\mathscr{R}$ is assumed to be injective, each $\varphi_{k,j} \in \mathrm{CP}_A(X,\mathscr{R})$ can be extended to a map $\eta_{k,j} \in \mathrm{CP}_A(Y,\mathscr{R})$ (see [20], [31]). Then $\eta_k := \sum_j \theta_{k,j}\eta_{k,j}$ extends ψ_k and, since η_k is a positive map (because all $\eta_{k,j}$ are completely positive and $\theta_{k,j}$ are positive by our hypothesis about $\mathscr{C}$), we have that $\|\eta_k\| \leq 2\|\eta_k(1)\| = 2\|\psi_k(1)\|$, hence the net (η_k) is bounded. Clearly each weak* limit point η of the net (η_k) is an extension of ψ, and $\eta \in P_{\mathscr{C}}(Y,\mathscr{R})$ by Theorem 7.1. □

8 When is $\mathrm{CP}_A(X,R)$ weak* dense in $\mathrm{CP}_A(X,R^{\sharp\sharp})$?

Positive cone in $X_A\hat{\otimes}_A R^{\sharp}_{\tau}$ is defined in terms of maps in $\mathrm{CP}_A(X,R^{\sharp\sharp})$. Since $R^{\sharp\sharp}$ is in general very large, it is natural to ask: does the condition that $\tilde{\varphi}(w)\geq 0$ for all $\varphi\in\mathrm{CP}_A(X,R)$ already guarantee that an element $w\in X_A\hat{\otimes}_A\mathscr{R}^{\sharp}_{\tau}$ is positive? It follows by an application of the Hahn-Banach separation theorem that the answer is positive if and only if the set $\mathrm{CP}_A(X,R)$ is weak* dense in $\mathrm{CP}_A(X,R^{\sharp\sharp})$. (Here we regard R as a C*-subalgebra of $R^{\sharp\sharp}$, through the canonical map $\iota:R\to R^{\sharp\sharp}$, and consequently $\mathrm{CB}_A(X,R)_A$ as a subspace in $\mathrm{CB}_A(X,R^{\sharp\sharp})_A$ by identifying each $\varphi\in\mathrm{CB}_A(X,R)_A$ with $\iota\varphi$.) This density question is interesting even in the case $A=\mathbb{C}$. Namely arguments of Kirchberg [10, 2.5.1] and Ozawa [19, 2.8] show, for a separable X, that the set $\mathrm{UCP}(X,R)$ (unital c.p. maps) is weak* dense in $\mathrm{UCP}(X,R^{\sharp\sharp})$ for all R if and only if X has the *lifting property* in the sense that for each C*-algebra R and ideal J in R every u.c.p. map $\varphi:X\to R/J$ can be lifted to a u.c.p. map $\theta:X\to R$, so that $q\theta=\varphi$, where $q:R\to R/J$ is the quotient map. (For example, X may be a separable nuclear C*-algebra [2, C3].) Restricting to W*-algebras, we can dispense with the separability assumption on X:

Proposition 8.1 *Let X be a (not necessarily separable) operator system. If* $\mathrm{UCP}(X,\mathscr{R})$ *is weak* dense as a subset of* $\mathrm{UCP}(X,\mathscr{R}^{\sharp\sharp})$ *for all W*-algebras* $\mathscr{R}$*, then X has the lifting property for all von Neumann algebras* $\mathscr{R}$ *and norm-closed ideals in* $\mathscr{R}$.

Conversely, if A is finite-dimensional, X is an operator A-system which has the lifting property as an operator system, then the set $\mathrm{UCP}_A(X,R)$ ($\mathrm{CP}_A(X,R)$) *is weak* dense in* $\mathrm{UCP}_A(X,R^{\sharp\sharp})$ *(in* $\mathrm{CP}_A(X,R^{\sharp\sharp})$*, respectively) for every C*-algebra R which contains A as a C*-subalgebra.*

Proof Let J be a norm closed two-sided ideal in $\mathscr{R}$, $q:\mathscr{R}\to\mathscr{R}/J$ the quotient map and $\varphi:X\to\mathscr{R}/J$ a unital c.p. map. Then $q^{\sharp\sharp}:A^{\sharp\sharp}\to(\mathscr{R}/J)^{\sharp\sharp}$ splits (since $J^{\sharp\sharp}$ is of the form $p\mathscr{R}^{\sharp\sharp}$ for a central projection $p\in\mathscr{R}^{\sharp\sharp}$), hence there exists a complete contraction $\theta:X\to\mathscr{R}^{\sharp\sharp}$ such that $q^{\sharp\sharp}\theta=\iota_{\mathscr{R}/J}\varphi$, where $\iota_{\mathscr{R}/J}:\mathscr{R}/J\to(\mathscr{R}/J)^{\sharp\sharp}$ is the canonical map. If $\mathrm{UCP}(X,\mathscr{R})$ is weak* dense in $\mathrm{UCP}(X,\mathscr{R}^{\sharp\sharp})$, there exists a net of u.c.p. maps $\sigma_k:X\to\mathscr{R}$ such that the maps $\iota_{\mathscr{R}}\sigma_k$ weak* converge to θ, where $\iota_{\mathscr{R}}:\mathscr{R}\to\mathscr{R}^{\sharp\sharp}$ is the canonical map. Let $\overline{\iota_{\mathscr{R}}^{-1}}:\mathscr{R}^{\sharp\sharp}\to\mathscr{R}$ be the weak* continuous extension of $(\iota_{\mathscr{R}})^{-1}$ [9, Section 10.1]. Then $\overline{\iota_{\mathscr{R}}^{-1}}\theta=$ p.w.$^*\lim_k\overline{\iota_{\mathscr{R}}^{-1}}\iota_{\mathscr{R}}\sigma_k=$ p.w.$^*\lim_k\sigma_k$. Since the maps $\overline{\iota_{\mathscr{R}}^{-1}}$ and σ_k all have their ranges in $\mathscr{R}$ and the restriction of the weak* topology of $\mathscr{R}^{\sharp\sharp}$ to $\mathscr{R}$ is just the weak topology, we can replace the maps σ_k by suitable convex combinations (denoted again by σ_k) to achieve the point-norm (p.n.) convergence, thus $\overline{\iota_{\mathscr{R}}^{-1}}\theta=$ p.n.$\lim_k\sigma_k$. Note that $q^{\sharp\sharp}\iota_{\mathscr{R}}=\iota_{\mathscr{R}/J}q$. Using also that $q^{\sharp\sharp}\theta=\iota_{\mathscr{R}/J}\varphi$ and setting $\sigma=\overline{\iota_{\mathscr{R}}^{-1}}\theta$, we have

$$\begin{aligned}\iota_{\mathscr{R}/J}q\sigma=&\iota_{\mathscr{R}/J}q\overline{\iota_{\mathscr{R}}^{-1}}\theta=\iota_{\mathscr{R}/J}q(\text{p.n.}\lim_k\sigma_k)=\text{p.n.}\lim_k\iota_{\mathscr{R}/J}q\sigma_k\\ =&\text{p.n.}\lim_k q^{\sharp\sharp}\iota_{\mathscr{R}}\sigma_k=q^{\sharp\sharp}(\text{p.w.}^*\lim_k\iota_{\mathscr{R}}\sigma_k)=q^{\sharp\sharp}\theta=\iota_{\mathscr{R}/J}\varphi.\end{aligned}$$

Since the map $\iota_{\mathscr{R}/J}$ is injective, this implies that $q\sigma=\varphi$, that is, σ is a lift of φ.

For the converse, note that by the lifting property of X each map $\psi\in\mathrm{UCP}(X,R^{\sharp\sharp})$ can be weak* approximated by maps θ_k in $\mathrm{UCP}(X,R)$ (see [19, 2.8]). Assume now that $\psi\in\mathrm{UCP}_A(X,R^{\sharp\sharp})$. Since A is finite-dimensional, it is linearly spanned by a finite group G of unitaries [9, p. 526], and the averages

$$\varphi_k:X\to R,\ \ \varphi_k(x):=\frac{1}{|G|}\sum_{u\in G}u^*\theta_k(ux)$$

are completely contractive unital (hence c.p.) A-bimodule maps (as selfadjoint left A-module maps) that approximate ψ. If $\varphi \in \mathrm{CP}_A(X,R^{\sharp\sharp})$, then by an argument from [6, 5.1.6] we can write $\varphi = a\psi a$, where $\psi \in \mathrm{UCP}_A(X,R^{\sharp\sharp})$ and $a = \sqrt{\varphi(1)}$ is in the relative commutant $C_A(R^{\sharp\sharp})$ of A in R. An averaging over a finite group of unitaries that span A shows that $C_A(R)$ is strong* dense in $C_A(R^{\sharp\sharp})$. Strong* approximating a with a bounded net of elements of $C_A(R)$ and weak* approximating ψ by maps in $\mathrm{UCP}_A(X,R)$ we get the desired approximation of φ. □

For which R is $\mathrm{CP}(X,R)$ weak* dense in $\mathrm{CP}(X,R^{\sharp\sharp})$ for all operator systems X? At least for W*-algebras the answer is simple:

Proposition 8.2 *Let $\mathscr{R}$ be a W*-algebra. The set $\mathrm{CP}(X,\mathscr{R})$ is weak* dense in $\mathrm{CP}(X,\mathscr{R}^{\sharp\sharp})$ for all operator systems X if and only if $\mathscr{R}$ is of the form*

$$\mathscr{R} = \overline{\oplus}_k \mathrm{M}_{n_k}(\mathscr{Z}_k), \text{ where } \mathscr{Z}_k \text{ are abelian}, \; n_k \in \mathbb{N} \text{ and } \sup_k n_k < \infty. \tag{8.1}$$

Proof If $\mathscr{R}$ is not of the form (8.1), then $\mathscr{R}$ contains a copy of an algebra of the form $\overline{\oplus_k}\mathrm{M}_{n_k}(\mathscr{Z}_k)$, where $\mathscr{Z}_k$ are abelian and $\sup_k n_k = \infty$ (this follows from the type decomposition and the halving lemma), which contains a completely isometric copy (as an operator system) of $\mathrm{B}(\ell^2)$ (as can be seen by considering suitable compressions). Hence $\mathscr{R}$ is not locally reflexive [2], [22], [6], which means that there exist a finite-dimensional operator system $X \subseteq \mathscr{R}^{\sharp\sharp}$ such that the inclusion $\psi : X \to \mathscr{R}^{\sharp\sharp}$ can not be weak* approximated by complete contractions from $\mathrm{CP}(X,\mathscr{R})$ [2, 9.1.2]. By a well-known argument [2, p. 35] this implies that ψ can not be approximated by maps in $\mathrm{CP}(X,\mathscr{R})$.

Conversely, if $\mathscr{R}$ is of the form (8.1), then $\mathscr{R}$ is injective and nuclear, so by [2, 9.4.1] locally reflexive. Therefore, given an operator system X and $\psi \in \mathrm{UCP}(X,\mathscr{R}^{\sharp\sharp})$, for each finite-dimensional operator system $V \subseteq X$ the map $\psi|V$ can be weak* approximated by maps $\varphi_k \in \mathrm{UCP}(V,\mathscr{R})$. Each such φ_k can be extended to a map $\psi_k \in \mathrm{UCP}(X,\mathscr{R})$. Since this holds for every finite-dimensional $V \subseteq X$, it follows easily that ψ is in the weak* closure of $\mathrm{UCP}(X,\mathscr{R})$. For a non-unital map $\psi \in \mathrm{CP}(X,\mathscr{R}^{\sharp\sharp})$, write ψ as $a\psi_1 a$, where $a \in \mathscr{R}^{\sharp\sharp}$ and ψ_1 is unital (see the proof in [6, 5.1.6]), approximate ψ_1 and approximate a by elements from $\mathscr{R}$ in the strong* topology. □

When $X = A$, any A-bimodule map φ from X to R (or to $R^{\sharp\sharp}$) is determined by the element $\varphi(1)$, which is in the *centralizer* $C_A(R) := \{x \in R : xa = ax \;\forall a \in A\}$ (in $C_A(R^{\sharp\sharp})$, respectively). Thus $\mathrm{CP}_A(A,R) \cong C_A(\mathscr{R})_+$ and $\mathrm{CP}_A(A,R^{\sharp\sharp}) \cong C_A(R^{\sharp\sharp})_+$. However, $C_A(R)$ is usually not weak* dense in $C_A(R^{\sharp\sharp})$. The following example shows that even in an entirely commutative W^*-context (where there is no problem with the density of centralizers) $\mathrm{CP}_A(X,\mathscr{R})$ can fail to be weak* dense in $\mathrm{CP}_A(X,\mathscr{R}^{\sharp\sharp})$.

Example 8.1 Let $A = \mathscr{R} = \mathscr{Z}$ be an abelian W*-algebra and $X = \mathscr{Z}^{\sharp\sharp}$. Let $\pi_{\mathrm{u}} : \mathscr{Z} \to \mathrm{B}(\mathscr{H}_{\mathrm{u}})$ be the universal representation of $\mathscr{Z}$. Each character χ on $\mathscr{Z}$ is a direct summand in π_{u}, acting on a one-dimensional subspace $\mathbb{C}\xi$ of $\mathscr{H}_{\mathrm{u}}$. The corresponding projection $e : \mathscr{H}_{\mathrm{u}} \to \mathbb{C}\xi$ is minimal in $\mathrm{B}_{\mathscr{Z}}(\mathscr{H}_{\mathrm{u}})$ and can not be equivalent to any sub-projection of $1-e$ in $\mathrm{B}_{\mathscr{Z}}(\mathscr{H}_{\mathrm{u}})$ (since χ is not equivalent to any other representation of $\mathscr{Z}$). Thus the central carriers C_e and C_{1-e} of e and $1-e$ must be orthogonal, which means that $e = C_e$, hence e is in the center $\mathscr{Z}^{\sharp\sharp}$ of $\mathrm{B}_{\mathscr{Z}}(\mathscr{H}_{\mathrm{u}})$ ($= \mathrm{B}_{\mathscr{Z}^{\sharp\sharp}}(\mathscr{H}_{\mathrm{u}})$). We claim that for each $\varphi \in \mathrm{CP}_{\mathscr{Z}}(\mathscr{Z}^{\sharp\sharp}, \mathscr{Z})$, $\varphi(e)$ must be of the form $\varphi(e) = \sum_{n=1}^{\infty} \lambda_n q_n$, where q_n are minimal projections in $\mathscr{Z}$ and $\lambda_n \in \mathbb{R}_+$ satisfy $\lim_{n\to\infty} \lambda_n = 0$. To show this, assume that

$$\delta \in \sigma(\varphi(e)) \text{ (the spectrum of } \varphi(e)) \text{ and } \delta > 0.$$

For each $n \in \mathbb{N}$ let p_n and $p = p_{\{\delta\}}$ be the spectral projections of $\varphi(e)$ in $\mathscr{Z}$ corresponding to the interval $[\delta\frac{2^n-1}{2^n}, \delta\frac{2^{n+1}-1}{2^{n+1}})$ and to the singleton $\{\delta\}$, respectively. If $p \neq 0$ and p is not a minimal projection in $\mathscr{Z}$, we can write $p = f_1 + f_2$, where f_1, f_2 are non-zero projections in $\mathscr{Z}$. If $ef_j = 0$, then $\varphi(e)f_j = \varphi(ef_j) = 0$, in contradiction with $\varphi(e)f_j = \delta f_j$. Thus, $ef_j \neq 0$, and by minimality of e we have $e \leq f_j$, so $e \leq f_1 \wedge f_2 = 0$, a contradiction. Thus $p = p_{\{\delta\}}$ must be a minimal projection in $\mathscr{Z}$ (or 0). Assume now (to reach a contradiction) that δ is not an isolated point of $\sigma(\varphi(e))$. Then, since $\sum_{n=m}^{\infty} p_n$ is the spectral projection of $\varphi(e)$ corresponding to the interval $[\delta\frac{2^m-1}{2^m}, \delta)$, which is non-zero for each $m \in \mathbb{N}$ because δ is not isolated in $\sigma(\varphi(e))$, it follows $p_n \neq 0$ for infinitely many n. If $ep_n = 0$ for some such n, then

$$0 \leq \delta\frac{2^n-1}{2^n} p_n \leq \varphi(e)p_n = \varphi(ep_n) = 0$$

would imply that $p_n = 0$, a contradiction. Thus, by the minimality of e it follows that $e \leq p_n$ for each such n. But taking two different such n_1 and n_2 it follows that $e \leq p_{n_1}p_{n_2} = 0$, a contradiction. Thus all nonzero points δ in $\sigma(\varphi(e))$ are isolated, hence $\sigma(\varphi(e))$ must be a sequence (λ_n) converging to 0 (together with 0) and the corresponding spectral projections q_n must be minimal in $\mathscr{Z}$. If $\mathscr{Z}$ has no minimal projections, then this implies that $\varphi(e) = 0$, hence in this case $\mathrm{id}_{\mathscr{Z}^{\sharp\sharp}}$ is not in the point-weak* closure of $\mathrm{CP}_{\mathscr{Z}}(\mathscr{Z}^{\sharp\sharp}, \mathscr{Z})$. However, even in the discrete case $\mathscr{Z} = \ell^\infty$, any character $\chi \in \mathscr{Z}^\sharp$, such that χ is the evaluation at a point in $\beta(\mathbb{N}) \setminus \mathbb{N}$, annihilates all minimal projections in ℓ^∞, hence also all elements of the form $\varphi(e) = \sum_{n=1}^{\infty} \lambda_n q_n$, where $\lim_{n\to\infty} \lambda_n = 0$ and q_n are minimal projections in $\mathscr{Z}$. Thus again $\mathrm{id}_{\mathscr{Z}^{\sharp\sharp}}$ can not be in the point-weak* closure of $\mathrm{CP}_{\mathscr{Z}}(\mathscr{Z}^{\sharp\sharp}, \mathscr{Z})$.

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