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WHICH STATES CAN BE REACHED FROM A GIVEN STATE BY UNITAL COMPLETELY POSITIVE MAPS?

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Abstract For a state ω on a C*-algebra A, we characterize all states ρ in the weak* closure of the set of all states of the form $\omega \circ \varphi$, where φ is a map on A of the form $\varphi(x) = \sum_{i=1}^{n} a_i^* x a_i$, $\sum_{i=1}^{n} a_i^* a_i = 1$ ($a_i \in A$, $n \in \mathbb{N}$). These are precisely the states ρ that satisfy $\|\rho|J\| \leq \|\omega|J\|$ for each ideal J of A. The corresponding question for normal states on a von Neumann algebra $\mathcal{R}$ (with the weak* closure replaced by the norm closure) is also considered. All normal states of the form $\omega \circ \psi$, where ψ is a quantum channel on $\mathcal{R}$ (that is, a map of the form $\psi(x) = \sum_j a_j^* x a_j$, where $a_j \in \mathcal{R}$ are such that the sum $\sum_j a_j^* a_j$ converge to 1 in the weak operator topology) are characterized. A variant of this topic for hermitian functionals instead of states is investigated. Maximally mixed states are shown to vanish on the strong radical of a C*-algebra and for properly infinite von Neumann algebras the converse also holds.

Keywords: state; completely positive map; C*-algebra; von Neumann algebra; quantum channel

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1. Introduction

For two states ω and ρ on a C*-algebra $\mathcal{R}$, ρ is regarded to be unitarily more mixed than ω if ρ is contained in the weak* closure of the convex hull of the unitary orbit of ω. In [1, 2, 29], Alberti, Uhlmann and Wehrl studied the notion of maximally unitarily mixed states on von Neumann algebras and such states were characterized by Alberti in [1]. Recently, this topic has been revitalized in the broader context of C*-algebras by Archbold *et al.* [4], who proved among other things that the weak*closure of the set of maximally unitarily mixed states on a C*-algebra A is equal to the weak* closure of the convex hull of tracial states and states that factor through simple traceless quotients of A. However, the evolution of open quantum systems is not always unitary, but is described by more general completely positive (trace preserving) maps of the form $\omega \mapsto \sum a_i \omega a_i^*$, say on the predual of $\mathrm{B}(\mathcal{H})$, so it seems worthwhile to study also a less restrictive notion of when one state is more mixed than the other. The dual of such a map is a unital completely positive map of the form

$$x \mapsto \sum a_i^* x a_i, \quad \sum a_i^* a_i = 1 \tag{1.1}$$

on $\mathcal{R} = \mathrm{B}(\mathcal{H})$. Let $\mathrm{E}(A)$ be the set of all unital completely positive maps on A of the form (1.1), where $a_i \in A$ and the sums have only finitely many terms. A natural question in this context is, when a state ρ on a C*-algebra A (or a normal state on a von Neumann algebra $\mathcal{R}$) is in the weak* closure (or the norm closure) of the set $\omega \circ \mathrm{E}(A)$ of all states of the form $\omega \circ \psi$, where ω is a fixed state (perhaps normal in the case of von Neumann algebras) and ψ runs over the set $\mathrm{E}(A)$. In § 2, we show for normal states on a von Neumann algebra $\mathcal{R}$ that ρ is in the norm closure of $\omega \circ \mathrm{E}(\mathcal{R})$ if and only if ρ and ω agree on the centre of $\mathcal{R}$. We also study the same topic for hermitian normal functionals on $\mathcal{R}$ and provide an explicit normal mapping ψ in the point-weak* closure of $\mathrm{E}(\mathcal{R})$ such that $\rho = \omega \circ \psi$. In the special case of $\mathcal{R} = \mathrm{B}(\mathcal{H})$, hermitian normal functionals are just hermitian trace class operators and maps mapping one such operator to another have been constructed by Hsu *et al.* in [19] and by Li and Du in [21], but they do not study the question if such maps are in the closure of $\mathrm{E}(\mathrm{B}(\mathcal{H}))$.

For a normal state ω on a von Neumann algebra $\mathcal{R} \subseteq \mathrm{B}(\mathcal{H})$ and a map ϕ of the form (1.1), where $a_i \in \mathcal{R}$ and the sums may have infinitely many terms (that is, ϕ is a quantum channel) any state of the form $\rho = \omega \circ \phi$ has the following property: if $\tilde{\omega}$ is a normal state on $\mathrm{B}(\mathcal{H})$ that extends ω, then there is a normal state $\tilde{\rho}$ on $\mathrm{B}(\mathcal{H})$ that extends ρ such that $\tilde{\rho}$ and $\tilde{\omega}$ coincide on the commutant $\mathcal{R}'$ of $\mathcal{R}$ (namely, $\tilde{\rho} = \tilde{\omega} \circ \tilde{\phi}$, where $\tilde{\phi}$ is the map on $\mathrm{B}(\mathcal{H})$ given by the same formula as ϕ on $\mathcal{R}$). This property holds in any faithful normal representation of $\mathcal{R}$ on a Hilbert space $\mathcal{H}$. In § 2, we will see that this property characterizes states of the form $\omega \circ \phi$, where ϕ runs over quantum channels on $\mathcal{R}$.

Then, in § 3, we study the analogous topic for hermitian functionals ρ, ω on a unital C*-algebra A. If A has Hausdorff primitive spectrum, Theorem 3.1 shows that ρ is in the weak* closure of $\omega \circ \mathrm{E}(A)$ if and only if ω and ρ agree on the centre of A and $\|c\rho\| \leq \|c\omega\|$ for each positive element c in the centre of A. If the primitive spectrum of A is not Hausdorff, this characterization is not true any more, but an alternative one is given in Theorem 3.7.

For two states ω and ρ on a C*-algebra A, ρ is regarded here to be *more mixed than* ω. if ρ is contained in the weak* closure $\overline{\omega \circ \mathrm{E}(A)}$ of the set $\omega \circ \mathrm{E}(A) := \{\omega \circ \psi : \psi \in \mathrm{E}(A)\}$. Then, ω is called *maximally mixed* if for each state ρ on A the condition that $\rho \in \overline{\omega \circ \mathrm{E}(A)}$ implies that $\omega \in \overline{\rho \circ \mathrm{E}(A)}$; in other words, $\overline{\omega \circ \mathrm{E}(A)}$ is minimal among weak* closed $\mathrm{E}(A)$-invariant subsets of the set $S(A)$ of all states on A. This is a coarser relation than the one considered in the references mentioned above, where instead of $\mathrm{E}(A)$, only convex combinations of unitary similarities are considered. In § 4, we show that each maximally mixed state on a unital C*-algebra A must annihilate the strong radical J_A of A (= the intersection of all two-sided maximal ideals of A) and, if A is a properly infinite von Neumann algebra, the converse is also true. Furthermore, the set $S_m(A)$ of all maximally mixed states contains all states that annihilate some intersection of finitely many maximal ideals of A and is therefore weak* dense in $S(A/J_A)$. These results are analogous to those of [4] and [1] for unitarily maximally mixed states. For C*-algebras with the Dixmier property, the authors of [4] provided a more precise determination of maximally unitarily mixed states than for general C*-algebras. In our present context, the role of C*-algebras

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with the Dixmier property can be played by weakly central C*-algebras. For a weakly central C*-algebra A, we show that the set $S_m(A)$ is weak* closed (and hence equal to the set of all states that annihilate J_A) if and only if each primitive ideal of A which contains J_A is maximal. States in $S_m(\mathcal{R})$ for a general von Neumann algebra $\mathcal{R}$ are also characterized.

Throughout the paper, an ideal means a norm closed two-sided ideal and all C-algebras are assumed to be unital unless explicitly stated otherwise.*

2. The case of normal states on a von Neumann algebra

We denote by $A^\sharp$ the dual of a Banach case A. In what follows A will usually be a C*-algebra. Throughout this article, $\mathcal{R}$ is a von Neumann algebra, $\mathcal{R}_\sharp$ its predual (that is, the space of all weak* continuous linear functionals on $\mathcal{R}$) and $\mathcal{Z}$ the centre of R. Basic facts concerning von Neumann algebras, that will be used here without explicitly mentioning a reference, can be found in [20, 28].

We will need a preliminary result of independent interest, which in the special case (when, in the notation of Theorem 2.1, $\mathcal{A} = \mathcal{R}$ and $\mathcal{R}$ is a factor or has a separable predual, and positivity was not considered), has been proved by Chatterjee and Smith [9]. We would like to avoid the separability assumption. In its proof, we will use the notion of the minimal C*-tensor product over $\mathcal{Z}$ of two C*-algebras A and B both containing an abelian W*-algebra $\mathcal{Z}$ in their centres. This product $A \otimes_{\mathcal{Z}} B$ [6, 13, 22], can be defined as the closure of the image of the algebraic tensor product $A \odot_{\mathcal{Z}} B$ in $\oplus_{t\in\Delta} A(t) \otimes B(t)$, where Δ is the maximal ideal space of $\mathcal{Z}$ and, for each $t \in \Delta$, $A(t)$ denotes the quotient C*-algebra $A/(tA)$, where tA is the closed ideal in A generated by t (and similarly for $B(t)$). (If at least one of the algebras A, B is exact, which will be the case in our application in the proof of Theorem 2.3, $A \otimes_{\mathcal{Z}} B$ coincides with the quotient of $A \otimes B$ by the closed ideal generated by all elements of the form $az \otimes b - a \otimes zb$ ($a \in A$, $b \in B$, $z \in \mathcal{Z}$) [22, 3.12].)

Theorem 2.1. *Let $\mathcal{A}$ be an injective von Neumann subalgebra of a von Neumann algebra $\mathcal{R}$ containing the centre $\mathcal{Z}$ of $\mathcal{R}$. Then, each completely contractive $\mathcal{Z}$-module map $\psi : \mathcal{R} \to \mathcal{A}$ is (as a map into $\mathcal{R}$) in the point-weak* closure of the set consisting of all maps of the form $x \mapsto \sum_{i=1}^{n} a_i^* x b_i$ ($x \in \mathcal{R}$), where $n \in \mathbb{N}$ and $a_i, b_i \in \mathcal{R}$ satisfy $\sum_{i=1}^{n} a_i^* a_i \leq 1$ and $\sum_{i=1}^{n} b_i^* b_i \leq 1$. If in addition ψ is unital, then ψ is in the point-weak* closure of* $\mathrm{E}(\mathcal{R})$.

Proof. Let $\mathcal{H}$ be a Hilbert space such that $\mathcal{R} \subseteq \mathrm{B}(\mathcal{H})$. It follows from [20, 5.5.4] that there is a natural $*$-isomorphism ι from $\mathcal{R}\mathcal{R}'$ (the subalgebra of $\mathrm{B}(\mathcal{H})$ generated by $\mathcal{R} \cup \mathcal{R}'$) onto the algebraic tensor product $\mathcal{R} \odot_{\mathcal{Z}} \mathcal{R}'$, given by $rr' \mapsto r \otimes_{\mathcal{Z}} r'$. By [6, 2.9], the tensor norm on $\mathcal{R} \otimes_{\mathcal{Z}} \mathcal{R}'$ restricted to $\mathcal{R} \odot_{\mathcal{Z}} \mathcal{R}'$ is minimal among all C*-tensor norms on $\mathcal{R} \odot_{\mathcal{Z}} \mathcal{R}'$, hence the $*$-homomorphism ι extends uniquely to the norm closure $\overline{\mathcal{R}\mathcal{R}'}$. Since $\mathcal{A}$ is injective and commutes with $\mathcal{R}'$ the multiplication $\mu_0 : \mathcal{A} \otimes \mathcal{R}' \to \overline{\mathcal{A}\mathcal{R}'} \subseteq \mathrm{B}(\mathcal{H})$ is a completely contractive $*$-homomorphism [8, 9.3.3, 3.8.5]. But more is true: by [13, 4.2], the natural map $\mathcal{A} \odot_{\mathcal{Z}} \mathcal{R}' \to \mathcal{A}\mathcal{R}'$ extends (uniquely) to a $*$-isomorphism $\mathcal{A} \otimes_{\mathcal{Z}}$

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$\mathcal{R}' \to \overline{\overline{\mathcal{AR}'}}$. It follows that the composition

$$\mathrm{B}(\mathcal{H}) \supseteq \overline{\overline{\mathcal{RR}'}} \to \mathcal{R}\otimes_{\mathcal{Z}}\mathcal{R}' \overset{\psi\otimes_{\mathcal{Z}}\mathrm{id}}{\longrightarrow} \mathcal{A}\otimes_{\mathcal{Z}}\mathcal{R}' \cong \overline{\overline{\mathcal{AR}'}} \subseteq \mathrm{B}(\mathcal{H})$$

is completely contractive and clearly, it is an $\mathcal{R}'$-bimodule map, hence extending to such a map ϕ on $\mathrm{B}(\mathcal{H})$ by the Wittstock extension theorem (see [30] or [7, 3.6.2]). By [11], ϕ can be approximated in the point-weak* topology by a net of elementary complete contractions of the form

$$x \mapsto \sum_i a_i^*(k)xb_i(k) = a(k)^*xb(k) \quad (x \in \mathrm{B}(\mathcal{H})) \tag{2.1}$$

where $a(k) = (a_1(k), \ldots, a_n(k))^T$ and $b(k) = (b_1(k), \ldots, b_n(k))^T$ are columns with the entries $a_i(k), b_i(k) \in \mathcal{R}$ and

$$a^*(k)a(k) = \sum_i a_i^*(k)a_i(k) \le 1, \quad b^*(k)b(k) = \sum_i b_i^*(k)b_i(k) \le 1. \tag{2.2}$$

Thus, ψ $(=\phi|\mathcal{R})$ can also be approximated by such maps.

Assume now in addition that ψ is unital and consider a point-weak* approximation of ψ of the form (2.1), (2.2). Since

$$\begin{aligned} 0 &\le (b(k)-a(k))^*(b(k)-a(k)) = b(k)^*b(k) + a(k)^*a(k) - a(k)^*b(k) - b(k)^*a(k) \\ &\le 2 - a(k)^*b(k) - b(k)^*a(k) \to 2 - 2\psi(1) = 0, \end{aligned}$$

it follows that $b(k) - a(k)$ tends to 0 in the strong operator topology. Hence, ψ can be approximated by maps of the form $x \mapsto a(k)^*xa(k)$ in the point-weak* operator topology. To see this, write

$$\psi(x) - a(k)^*xa(k) = (\psi(x) - a(k)^*xb(k)) + (a(k)^*x(b(k) - a(k)))$$

and note that $\|a(k)^*x(b(k)-a(k))\xi\| \le \|x\|\|(b(k)-a(k))\xi\|$ for each vector $\xi \in \mathcal{H}$. Finally, as $a(k)^*a(k)$ tends to $\psi(1) = 1$ in the strong operator topology, ψ can be approximated by maps of the form

$$x \mapsto a(k)^*xa(k) + \sqrt{1 - a(k)^*a(k)}\,x\sqrt{1 - a(k)^*a(k)},$$

that is, by unital completely positive elementary maps. □

Lemma 2.2. *Let ω and ρ be hermitian functionals on a C^*-algebra A such that $\rho|Z = \omega|Z$ and $\|c\rho\| \le \|c\omega\|$ for all $c \in Z_+$, where Z is the centre of A. Then, $\rho_+|Z \le \omega_+|Z$ and $\rho_-|Z \le \omega_-|Z$.*

Thus, if Z is a von Neumann algebra, ω and ρ are normal and p^+ and p^- are the support projections of $\omega_+|Z$ and $\omega_-|Z$, then there exist elements c_+ and c_- in Z such that $0 \le c_+ \le p^+$, $0 \le c_- \le p^-$,

$$\rho_+|Z = c_+\omega_+|Z, \quad \rho_-|Z = c_-\omega_-|Z, \ \text{and}\ (p^+ - c_+)\omega_+|Z = (p^- - c_-)\omega_-|Z.$$

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Proof. For each $c \in Z_+$ and $\theta \in (A^\sharp)_+$, we have that $\|c\theta\| = (c\theta)(1) = \theta(c)$ and it is also well-known that for each hermitian functional σ the equality $\|\sigma\| = \sigma_+(1) + \sigma_-(1) = \|\sigma_+\| + \|\sigma_-\|$ holds, hence

$$\rho_+(c) + \rho_-(c) = \|c\rho\| \leq \|c\omega\| = \omega_+(c) + \omega_-(c),$$
$$\rho_+(c) - \rho_-(c) = \rho(c) = \omega(c) = \omega_+(c) - \omega_-(c).$$

Adding and subtracting these two relations, we find that $\rho_+(c) \leq \omega_+(c)$ and $\rho_-(c) \leq \omega_-(c)$ for all $c \in Z_+$. If Z, ω, ρ, p^+ and p^- are as in the second part of the lemma, we may regard Z as $L^\infty(\mu)$ for some positive measure μ and then the existence of elements c_+ and c_- in Z satisfying $0 \leq c_+ \leq p_+$, $0 \leq c_- \leq p^-$ and $\rho_+|Z = c_+\omega_+|Z$, $\rho_-|Z = c_-\omega_-|Z$ follows easily, so we will verify here only the last equality in the lemma. The condition $\rho|Z = \omega|Z$ can be written as $(c_+\omega_+ - c_-\omega_-)|Z = (\omega_+ - \omega_-)|Z$, hence $(1-c_+)\omega_+|Z = (1-c_-)\omega_-|Z$. But $\omega_+ = p^+\omega_+$ and $\omega_- = p^-\omega_-$, since p^+ and p^- are the support projections of $\omega_+|Z$ and $\omega_-|Z$, hence the required equality follows. □

By [17] or [27], each positive functional ω on $\mathcal{R}$, such that $\omega|\mathcal{Z}$ is weak* continuous, can be uniquely expressed as

$$\omega = (\omega|\mathcal{Z}) \circ \omega_{\mathcal{Z}}, \tag{2.3}$$

where $\omega_{\mathcal{Z}}$ is a (completely) positive $\mathcal{Z}$-module map from $\mathcal{R}$ to $\mathcal{Z}$ such that $\omega_{\mathcal{Z}}(1)$ is the support projection $q \in \mathcal{Z}$ of $\omega|\mathcal{Z}$. If ω is weak* continuous, then so is also $\omega_{\mathcal{Z}}$. *Observe that the support projections of ω and $\omega_{\mathcal{Z}}$ coincide, if ω is normal.* (Indeed, for each projection $e \in \mathcal{R}$, we have $0 \leq \omega_{\mathcal{Z}}(e) \leq \omega_{\mathcal{Z}}(1) = q$, hence $\omega(e) = (\omega|\mathcal{Z})(\omega_{\mathcal{Z}}(e)) = 0$ if and only if $\omega_{\mathcal{Z}}(e) = 0$ since q is the support projection of $\omega|\mathcal{Z}$.)

Theorem 2.3. *Let ω, ρ be normal hermitian functionals on $\mathcal{R}$. There exists a normal unital completely positive map $\psi : \mathcal{R} \to \mathcal{R}$ in the point-weak* closure of* $\mathrm{E}(\mathcal{R})$ *satisfying $\psi(1) = 1$ and $\psi_\sharp(\omega) = \rho$ if and only if*

$$\rho|\mathcal{Z} = \omega|\mathcal{Z} \text{ and } \|c\rho\| \leq \|c\omega\| \quad \forall c \in \mathcal{Z}_+. \tag{2.4}$$

Under this condition, ρ is in the norm closure of $\omega \circ \mathrm{E}(\mathcal{R})$.

Proof. Since maps in $\mathrm{E}(\mathcal{R})$ are unital and completely positive, they are also completely contractive. Each map in $\mathrm{E}(\mathcal{R})$ is of the form $\psi(x) = \sum_{i=1}^n a_i^* x a_i$, where $a_i \in \mathcal{R}$ and $\sum_{i=1}^n a_i^* a_i = 1$, hence weak* continuous and the corresponding map $\psi_\sharp$ on the predual $\mathcal{R}_\sharp$ of $\mathcal{R}$ is given by $\psi_\sharp(\omega) = \sum_{i=1}^n a_i \omega a_i^*$ and is a $\mathcal{Z}$-module map with $\|\psi_\sharp\| = \|\psi\| = 1$. Hence $\|c\psi_\sharp(\omega)\| = \|\psi_\sharp(c\omega)\| \leq \|c\omega\|$ for each $c \in \mathcal{Z}$. This means that the inequality $\|(c\omega) \circ \psi\| \leq \|c\omega\|$ holds for all $\psi \in \mathrm{E}(\mathcal{R})$, hence also for all ψ in the point-weak* closure of $\mathrm{E}(\mathcal{R})$ since $c\omega$ is weak* continuous. If ψ is a weak* continuous such map and $\rho = \psi_\sharp(\omega)$, then $\|c\rho\| = \|(c\omega) \circ \psi\| \leq \|c\omega\|$. Furthermore, since $\psi|\mathcal{Z} = \mathrm{id}$ for each such map ψ, it follows that $\rho|\mathcal{Z} = \omega|\mathcal{Z}$ for each $\rho \in \overline{\omega \circ \mathrm{E}(\mathcal{R})}$.

Assume now that the condition (2.4) holds. Decompose each of the functionals ω_+, ω_-, ρ_+, ρ_- as described in (2.3), so that

$$\omega = (\omega_+|\mathcal{Z}) \circ \omega_{\mathcal{Z}}^+ - (\omega_-|\mathcal{Z}) \circ \omega_{\mathcal{Z}}^- \text{ and } \rho = (\rho_+|\mathcal{Z}) \circ \rho_{\mathcal{Z}}^+ - (\rho_-|\mathcal{Z}) \circ \rho_{\mathcal{Z}}^-,$$

where $\rho_{\mathcal{Z}}^+$, $\rho_{\mathcal{Z}}^-$, $\omega_{\mathcal{Z}}^+$, $\omega_{\mathcal{Z}}^-$ are $\mathcal{Z}$-module homomorphisms from $\mathcal{R}$ to $\mathcal{Z}$ such that $p^+ := \omega_{\mathcal{Z}}^+(1)$ and $p^- := \omega_{\mathcal{Z}}^-(1)$ are the support projections of $\omega_+|\mathcal{Z}$ and $\omega_-|\mathcal{Z}$. Let p_+ and p_- be the support projections of ω_+ and ω_-. Observe that $p_+ \leq p^+$ and $p_- \leq p^-$. (Namely, $\omega_+(1-p^+) = (\omega_+|\mathcal{Z})(1-p^+) = 0$ implies that $1-p^+ \leq 1-p_+$, hence $p_+ \leq p^+$.) By Lemma 2.2, there exists c_+, $c_- \in \mathcal{Z}$ such that $0 \leq c_+ \leq p^+$, $0 \leq c_- \leq p^-$,

$$\rho_+|\mathcal{Z} = c_+\omega_+|\mathcal{Z},\ \rho_-|\mathcal{Z} = c_-\omega_-|\mathcal{Z} \text{ and } (p^+ - c_+)\omega_+|\mathcal{Z} = (p^- - c_-)\omega_-|\mathcal{Z}. \tag{2.5}$$

When we first tried to find a map ψ satisfying the requirements of the theorem to be of the form $\psi = a\rho^+_{\mathcal{Z}} + b\rho^-_{\mathcal{Z}}$, where a, $b \in \mathcal{R}_+$, we found that it is not always possible to simultaneously satisfy the conditions $\psi(1) = 1$ and $\omega \circ \psi = \rho$ by maps of such a form. But after several attempts we arrived to the following map:

$$\psi = c_+p_+\rho^+_{\mathcal{Z}} + (1 - c_+p_+)(\rho^-_{\mathcal{Z}} + (1 - p^-)\theta). \tag{2.6}$$

Here, θ is any fixed normal positive unital $\mathcal{Z}$-module map from $\mathcal{R}$ to $\mathcal{Z}$. (Such a map exists even on $\mathcal{Z}' \supseteq \mathcal{R}$ since $\mathcal{Z}'$ is of type I, hence isomorphic to a direct sum of matrix algebras of the form $\mathrm{M}_n(\mathcal{Z})$, where n can be infinite.) This map ψ is positive, weak* continuous, $\mathcal{Z}$-module map, with the range contained in the commutative C*-algebra generated by $\mathcal{Z} \cup \{p_+\}$, hence completely positive. We can immediately verify that ψ is also unital:

$$\psi(1) = c_+p_+\rho^+_{\mathcal{Z}}(1) + (1 - c_+p_+)(\rho^-_{\mathcal{Z}}(1) + 1 - p^-) = c_+p_+ + (1 - c_+p_+)(p^- + 1 - p^-) = 1.$$

Now, we are going to compute

$$\psi_\sharp(\omega) = \omega \circ \psi = (\omega_+|\mathcal{Z}) \circ \omega^+_{\mathcal{Z}} \circ \psi - (\omega_-|\mathcal{Z}) \circ \omega^-_{\mathcal{Z}} \circ \psi. \tag{2.7}$$

For this, first observe that if $f, g : \mathcal{R} \to \mathcal{Z}$ are $\mathcal{Z}$-module maps and $a \in \mathcal{R}$, then $(f \circ (ag))(x) = f(ag(x)) = f(a)g(x) = (f(a)g)(x)$, that is, $f \circ (ag) = f(a)g$. Note also that $p^+\rho^+_{\mathcal{Z}} = \rho^+_{\mathcal{Z}}$ and $p^-\rho^-_{\mathcal{Z}} = \rho^-_{\mathcal{Z}}$ since Lemma 2.2 implies that the support projection of $\rho_+|\mathcal{Z}$ is dominated by the support projection of $\omega_+|\mathcal{Z}$ and similarly for $\rho_-|\mathcal{Z}$ and $\omega_-|\mathcal{Z}$. From the definition (2.6) of ψ and using that $\omega^+_{\mathcal{Z}}$ and $\omega^-_{\mathcal{Z}}$ are $\mathcal{Z}$-module maps with ranges contained in $\mathcal{Z}$ and mutually orthogonal support projections p_+ and p_- (which are just the support projections of ω_+ and ω_-, respectively), we now compute

$$\begin{aligned}\omega^+_{\mathcal{Z}} \circ \psi &= \omega^+_{\mathcal{Z}}(c_+p_+)\rho^+_{\mathcal{Z}} + \omega^+_{\mathcal{Z}}(1 - c_+p_+)(\rho^-_{\mathcal{Z}} + (1 - p^-)\theta) \\ &= c_+\rho^+_{\mathcal{Z}} + (p^+ - c_+)(\rho^-_{\mathcal{Z}} + (1 - p^-)\theta)\end{aligned} \tag{2.8}$$

and similarly

$$\omega^-_{\mathcal{Z}} \circ \psi = \omega^-_{\mathcal{Z}}(1 - c_+p_+)\rho^-_{\mathcal{Z}} = p^-\rho^-_{\mathcal{Z}} = \rho^-_{\mathcal{Z}}. \tag{2.9}$$

From (2.7), (2.8) and (2.9) we have, using also (2.5) and (2.6),

$$\begin{aligned}\omega \circ \psi &= (\omega_+|\mathcal{Z}) \circ [c_+\rho^+_{\mathcal{Z}} + (p^+ - c_+)(\rho^-_{\mathcal{Z}} + (1 - p^-)\theta)] - (\omega_-|\mathcal{Z}) \circ \rho^-_{\mathcal{Z}} \\ &= (c_+\omega_+|\mathcal{Z}) \circ \rho^+_{\mathcal{Z}} + [(p^+ - c_+)\omega_+|\mathcal{Z}] \circ \rho^-_{\mathcal{Z}} + [(p^+ - c_+)(1 - p^-)\omega_+|\mathcal{Z}] \circ \theta \\ &\quad - (\omega_-|\mathcal{Z}) \circ \rho^-_{\mathcal{Z}} \\ &= (\rho_+|\mathcal{Z}) \circ \rho^+_{\mathcal{Z}} + [(p^- - c_-)\omega_-|\mathcal{Z}] \circ \rho^-_{\mathcal{Z}} + [(1 - p^-)(p^- - c_-)p^-\omega_-|\mathcal{Z}] \circ \theta \\ &\quad - (p^-\omega_-|\mathcal{Z}) \circ \rho^-_{\mathcal{Z}} \\ &= \rho_+ - (c_-\omega_-|\mathcal{Z}) \circ \rho^-_{\mathcal{Z}} = \rho_+ - (\rho_-|\mathcal{Z}) \circ \rho^-_{\mathcal{Z}} = \rho_+ - \rho_- = \rho.\end{aligned}$$

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It follows from Theorem 2.1 that ψ is in the point-weak* closure of $\mathrm{E}(\mathcal{R})$. Thus, $\rho = \omega \circ \psi$ is in the weak closure of the convex set $\omega \circ \mathrm{E}(\mathcal{R})$, which is the same as the norm closure by the Hahn–Banach theorem and the fact that $\mathcal{R}$ is the dual of $\mathcal{R}_\sharp$. □

When ω and ρ are states, Theorem 2.3 simplifies to the following corollary:

Corollary 2.4. *Let ω and ρ be normal states on $\mathcal{R}$. There exists a normal unital completely positive map ψ in the point-weak* closure of $\mathrm{E}(\mathcal{R})$ satisfying $\psi_\sharp(\omega) = \rho$ if and only if $\rho|\mathcal{Z} = \omega|\mathcal{Z}$. This condition is satisfied if and only if $\|c\rho\| \leq \|c\omega\|$ for all $c \in \mathcal{Z}_+$.*

Proof. By Theorem 2.3, we only need to verify that the condition $\rho|\mathcal{Z} = \omega|\mathcal{Z}$ implies that $\|c\rho\| \leq \|c\omega\|$ for all $c \in \mathcal{Z}_+$ and conversely. Since ω and ρ are positive, we have $\|c\rho\| = (c\rho)(1) = \rho(c)$ and $\omega(c) = \|c\omega\|$ for all $c \in \mathcal{Z}_+$. If $\|c\rho\| \leq \|c\omega\|$ for all $c \in \mathcal{Z}_+$, then $\rho(c) \leq \omega(c)$. Applying this to $1 - c$ instead of c, where $0 \leq c \leq 1$, it follows that $\rho(c) = \omega(c)$ for all such c. But such elements span $\mathcal{Z}$, hence it follows that $\rho|\mathcal{Z} = \omega|\mathcal{Z}$ if and only if $\|c\rho\| \leq \|c\omega\|$ for all $c \in \mathcal{Z}_+$. □

It is well known that on $\mathcal{R} = \mathrm{B}(\mathcal{H})$, all normal completely positive unital maps are of the form

$$\phi(x) = \sum_{j \in \mathbb{J}} a_j^* x a_j \quad (x \in \mathbb{R}), \tag{2.10}$$

where $\mathbb{J}$ is some set of indexes and $a_j \in \mathcal{R}$ are such that $\sum_{j \in \mathbb{J}} a_j^* a_j = 1$ with the convergence in the strong operator topology. Maps on $\mathrm{B}(\mathcal{H})$ of the form (2.10) are called *quantum channels* and we will use the same name for maps of such a form on a general von Neumann algebra $\mathcal{R}$. It is well known that on a general von Neumann algebra, not all unital normal completely positive maps are of the form (2.10), so we still have to answer the following question: If ω and ρ are normal states on a von Neumann algebra $\mathcal{R}$, when does there exist a quantum channel ϕ on $\mathcal{R}$ such that $\omega \circ \phi = \rho$?

Theorem 2.5. *For normal states ω and ρ on $\mathcal{R}$, the following statements are equivalent:*

(i) *There exists a quantum channel ϕ on $\mathcal{R}$ such that $\omega \circ \phi = \rho$.*

(ii) *For every faithful normal representation π of $\mathcal{R}$ on a Hilbert space $\mathcal{H}_\pi$ and any normal state $\tilde{\omega}$ on $\mathrm{B}(\mathcal{H}_\pi)$ that extends $\omega \circ \pi^{-1}$, there exists a normal state $\tilde{\rho}$ on $\mathrm{B}(\mathcal{H}_\pi)$ that extends $\rho \circ \pi^{-1}$ such that $\tilde{\omega}|\pi(\mathcal{R})' = \tilde{\rho}|\pi(\mathcal{R})'$.*

(iii) *For some faithful normal representation of $\mathcal{R}$ on a Hilbert space $\mathcal{H}$, such that ω is the restriction to $\mathcal{R}$ of a vector state $\tilde{\omega}$ on $\mathrm{B}(\mathcal{H})$, there exists a normal state $\tilde{\rho}$ on $\mathrm{B}(\mathcal{H})$ such that $\tilde{\rho}|\mathcal{R} = \rho$ and $\tilde{\rho}|\mathcal{R}' = \tilde{\omega}|\mathcal{R}'$.*

(iv) *Let π_ω be the GNS representation of $\mathcal{R}$ engendered by ω on a Hilbert space $\mathcal{H}_\omega$ and let ξ_ω be the corresponding cyclic vector. The state ρ annihilates the kernel of π_ω and there exists a normal state $\tilde{\rho}$ on $\mathrm{B}(\mathcal{H}_\omega)$ such that $\tilde{\rho}|\pi_\omega(\mathcal{R})$ is the state induced by ρ on $\pi_\omega(\mathcal{R}) \cong \mathcal{R}/\ker \pi_\omega$ and $\tilde{\rho}|\pi_\omega(\mathcal{R})' = \tilde{\omega}|\pi_\omega(\mathcal{R})'$, where $\tilde{\omega}$ is the vector state $x \mapsto \langle x\xi_\omega, \xi_\omega \rangle$ on $\mathrm{B}(\mathcal{H}_\omega)$.*

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Proof. (i)⇒(ii) If $\rho = \omega \circ \phi$, where ϕ is of the form (2.10), then let $\tilde{\omega}$ be any state on $B(\mathcal{H}_\pi)$ extending $\omega \circ \pi^{-1}$, let $\tilde{\phi}$ be the map on $B(\mathcal{H}_\pi)$ defined by $\tilde{\phi}(x) = \sum_{j\in\mathbb{J}} \pi(a_j^*)x\pi(a_j)$ and set $\tilde{\rho} = \tilde{\omega} \circ \tilde{\phi}$. Then, $\tilde{\phi}(x) = x$ for each $x \in \pi(\mathcal{R})'$, hence $\tilde{\rho}|\pi(\mathcal{R})' = \tilde{\omega}|\pi(\mathcal{R})'$. Moreover, $\tilde{\rho}$ extends $\rho \circ \pi^{-1}$.

(ii)⇒(iii) Take for π a faithful normal representation on a Hilbert space $\mathcal{H}$ such that ω is the restriction of a vector state $\tilde{\omega}$ on $B(\mathcal{H})$. (For example, $\mathcal{R}$ may be in the standard form [28, Chapter IX] so that all normal states on $\mathcal{R}$ and $\mathcal{R}'$ are vector states.) For simplicity of notation, we may assume that $\mathcal{R} \subseteq B(\mathcal{H})$, that is, $\pi = \mathrm{id}$. Then, with $\tilde{\rho}$ as in (ii), we have $\tilde{\rho}|\mathcal{R} = \rho$ and $\tilde{\rho}|\mathcal{R}' = \tilde{\omega}|\mathcal{R}'$.

(iii)⇒(i) Assume that $\mathcal{R}$ is represented faithfully on a Hilbert space $\mathcal{H}$ such that ω is the restriction of a vector state $\tilde{\omega}$ on $B(\mathcal{H})$ and that $\tilde{\rho}$ is a normal state on $B(\mathcal{H})$ such that $\tilde{\rho}|\mathcal{R} = \rho$ and $\tilde{\rho}|\mathcal{R}' = \tilde{\omega}|\mathcal{R}'$. Let $\xi \in \mathcal{H}$ be such that $\tilde{\omega}(x) = \langle x\xi, \xi\rangle$ $(x \in B(\mathcal{H}))$. As a normal state, $\tilde{\rho}$ is of the form

$$\tilde{\rho}(x) = \langle x^{(\infty)}\eta, \eta\rangle \quad (x \in B(\mathcal{H})),$$

where $x^{(\infty)}$ denotes the direct sum of countably many copies of x acting on the direct sum $\mathcal{H}^\infty$ of countably many copies of $\mathcal{H}$ and $\eta \in \mathcal{H}^\infty$. Now, from $\tilde{\omega}(x) = \tilde{\rho}(x)$ for all $x \in \mathcal{R}'$, we have

$$\langle x\xi, \xi\rangle = \langle x^{(\infty)}\eta, \eta\rangle \quad (x \in (\mathcal{R}')).$$

Replacing x by x^*x, it follows that there exists an isometry $u : [\mathcal{R}'\xi] \to [(\mathcal{R}')^{(\infty)}\eta]$ such that $u\xi = \eta$ and $uy = y^{(\infty)}u$ for all $y \in \mathcal{R}'$. This u can be extended to a partial isometry from $\mathcal{H}$ into $\mathcal{H}^\infty$, denoted again by u, by declaring it to be 0 on the orthogonal complement of $[\mathcal{R}'\xi]$ in $\mathcal{H}$. Then, u intertwines the identity representation id of $\mathcal{R}'$ and the representation id^∞ and, is therefore, a column (u_j), where $u_j \in \mathcal{R}$. For $r \in \mathcal{R}$, we have

$$\rho(r) = \tilde{\rho}(r) = \langle r^{(\infty)}\eta, \eta\rangle = \langle r^{(\infty)}u\xi, u\xi\rangle = \langle u^*r^{(\infty)}u\xi, \xi\rangle = \omega(u^*r^{(\infty)}u).$$

Thus, $\rho = \omega \circ \psi$, where ψ is a map on $\mathcal{R}$, defined by $\psi(r) = u^*r^{(\infty)}u = \sum_j u_j^* r u_j$. This map ψ is not necessarily unital, but from

$$\omega(1) = 1 = \rho(1) = \omega(\psi(1)) = \omega\left(\sum_j u_j^*u_j\right) = \omega(u^*u) \text{ and } u^*u \le 1$$

we infer that $1 - u^*u \le 1 - p$, where p is the support projection of ω. Hence, $p \le u^*u$ and we may replace ψ by the unital map ϕ defined by $\phi(r) = p\psi(r)p + (1-p)r(1-p)$, which satisfies $\omega \circ \phi = \omega \circ \psi = \rho$ and has the required form:

$$\psi(r) = \sum_j pu_j^*ru_jp + p^\perp rp^\perp, \quad \text{where } \sum_j pu_j^*u_jp + p^\perp p^\perp = pu^*up + p^\perp = 1.$$

The equivalence (i)⇔(iv) is proved by similar arguments and we will omit the details, just note that $\mathcal{R} \cong \pi_\omega(\mathcal{R}) \oplus \ker \pi_\omega$. □

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3. The case of C*-algebras

In a general C*-algebra A, there are usually not enough module homomorphisms of A into its centre Z and even if $Z = \mathbb{C}1$, there can be many ideals in A. Functionals on A usually do not preserve ideals, hence can not be approximated by elementary operators. Therefore, we will use for general C*-algebras a different approach from that in the previous section, not trying to construct an explicit map sending one state to another. For C*-algebras with Hausdorff primitive spectrum, the situation nevertheless resembles the one for von Neumann algebras.

Theorem 3.1. *Let ω, ρ be hermitian linear functionals on a C*-algebra A with Hausdorff primitive spectrum $\check{A}$ and centre Z. Then, ρ is in the weak* closure $\overline{\omega \circ \mathrm{E}(A)}$ of the set $\omega \circ \mathrm{E}(A)$ if and only if the following condition is satisfied: (A) $\rho|Z = \omega|Z$ and $\|c\rho\| \leq \|c\omega\|$ for each $c \in Z_+$.*

Proof. To prove the non-trivial direction of the theorem, suppose that the condition (A) is satisfied, but that $\rho \notin \overline{\omega \circ \mathrm{E}(A)}$. Then, by the Hahn–Banach theorem, there exist $h \in A_h$ and α, $\delta \in \mathbb{R}$, $\delta > 0$, such that

$$\omega(\psi(h)) \leq \alpha \ \forall \psi \in \mathrm{E}(A) \text{ and } \rho(h) \geq \alpha + \delta. \tag{3.1}$$

Since $\rho|Z = \omega|Z$, in particular $\rho(1) = \omega(1)$, we may replace h by $h + \gamma 1$ (and α with $\alpha + \gamma\omega(1)$) for a sufficiently large $\gamma \in \mathbb{R}$ and thus assume that h is positive in invertible. Given $\varepsilon > 0$, let $a \in A_h$ be such that

$$-1 \leq a \leq 1 \text{ and } \omega_+(a) - \omega_-(a) = \omega(a) > \|\omega\| - \varepsilon = \omega_+(1) + \omega_-(1) - \varepsilon.$$

By a well-known argument, which we now recall, this implies the relations (3.2). Namely, from the above, we have $\omega_+(1-a) < -\omega_-(1+a) + \varepsilon$ and (since $1 - a \geq 0$ and $1 + a \geq 0$) this implies that $\omega_+(1-a) < \varepsilon$ and $\omega_-(1+a) \leq \varepsilon$. Thus $\omega_+(a_+) \geq \omega_+(a) > \omega_+(1) - \varepsilon$ and $\omega_-(a_+) = \omega_-(1+a) - \omega_-(1-a_-) \leq \omega_-(1+a) \leq \varepsilon$. In conclusion,

$$\omega_+(1 - a_+) < \varepsilon \text{ and } \omega_-(a_+) \leq \varepsilon. \tag{3.2}$$

For each $t \in \check{A}$ let $m(t)$ and $M(t)$ be the smallest and the largest point in the spectrum $\sigma(h(t))$ of $h(t) \in A/t$. Since $\check{A}$ is Hausdorff by assumption, the two functions M and m (given by $M(t) = \|h(t)\|$ and $m(t) = \|h(t)^{-1}\|^{-1}$) are continuous [26, 4.4.5] and therefore define elements of the centre Z of A by the Dauns–Hoffman theorem. Set

$$b = Ma_+ + m(1 - a_+).$$

For each $t \in \check{A}$, the spectrum of $b(t)$ is $\sigma(m(t)1 + (M(t) - m(t))a_+(t))$ and is contained in $m(t) + (M(t) - m(t))[0, 1] \subseteq [m(t), M(t)]$ since $\sigma(a_+(t)) \subseteq \sigma(a_+) \subseteq [0, 1]$. Thus, the numerical range $W(b(t))$ of $b(t)$ (which for normal elements coincides with the convex hull of the spectrum) is contained in $W(h(t)) = [m(t), M(t)]$. Therefore, by [23, 4.1], b is in the norm closure of the set $\{\psi(h) : \psi \in \mathrm{E}(A)\}$, hence

$$\omega(b) \leq \alpha \tag{3.3}$$

by the first relation in (3.1). On the other hand, we can estimate $\omega(b)$ as

$$\begin{aligned}\omega(b) &= \omega_+(Ma_+) - \omega_-(Ma_+) + \omega_+(m(1-a_+)) - \omega_-(m(1-a_+)) \\ &= \omega_+(M) - \omega_-(m) - \omega_+((M-m)(1-a_+)) - \omega_-((M-m)a_+) \\ &\geq \omega_+(M) - \omega_-(m) - \|M-m\|(\omega_+(1-a_+) + \omega_-(a_+))\end{aligned}$$

(since $0 \leq (M-m)(1-a_+) \leq \|M-m\|(1-a_+)$ and $0 \leq (M-m)a_+ \leq \|M-m\|a_+$)

$$\geq \omega_+(M) - \omega_-(m) - 2\|M-m\|\varepsilon \quad \text{(by (3.2))}.$$

Thus, by (3.1), (3.3) and since $m \leq h \leq M$ implies that $\rho_+(h) \leq \rho_+(M)$ and $\rho_-(h) \geq \rho_-(m)$, we have now

$$\omega_+(M) - \omega_-(m) - 2\varepsilon\|M-m\| + \delta \leq \rho(h) = \rho_+(h) - \rho_-(h) \leq \rho_+(M) - \rho_-(m).$$

This can be rewritten as

$$(\omega_+ - \rho_+)(M) \leq (\omega_- - \rho_-)(m) + 2\varepsilon\|M-m\| - \delta \tag{3.4}$$

or (since $\omega|Z = \rho|Z$ implies that $(\omega_- - \rho_-)|Z = (\omega_+ - \rho_+)|Z$ and since $M, m \in Z$)

$$(\omega_+ - \rho_+)(M-m) \leq 2\varepsilon\|M-m\| - \delta.$$

Since this holds for all $\varepsilon > 0$, by choosing small enough ε, it follows that $(\omega_+ - \rho_+)(M-m) < 0$. But, $Z \ni M - m \geq 0$ and $\omega_+|Z \geq \rho_+|Z$ by Lemma 2.2, hence $(\omega_+ - \rho_+)(M-m) \geq 0$, which is a contradiction. □

The following corollary can be proved in the same way as Corollary 2.4, so we will omit the proof.

Corollary 3.2. *If ω and ρ are states on a C*-algebra A with Hausdorff primitive spectrum, then $\rho \in \overline{\omega \circ \mathrm{E}(A)}$ if and only if $\rho|Z = \omega|Z$.*

Before stating our main result in this section, we need a lemma. Recall that a projection p in the centre of the universal von Neumann envelope $\mathcal{R}$ of a C*-algebra A is called *open* if there is an ideal J in A such that $\overline{J} = p\mathcal{R}$, where $\overline{J}$ is the weak* closure of J in $\mathcal{R}$.

Lemma 3.3. *Let $\mathcal{R}$ be the universal von Neumann envelope of a C*-algebra A and $\mathcal{Z}$ the centre of $\mathcal{R}$. For each $h \in A_+$, the central carrier C_h of h in $\mathcal{R}$ can be approximated in norm by linear combinations of open central projections in $\mathcal{R}$, where the coefficients in each combination are positive.*

Proof. By definition, the central carrier z of h is the infimum of all c in $\mathcal{Z}$ such that $h \leq c$. If Δ is the maximal ideal space of $\mathcal{Z}$, then z corresponds (via the Gelfand isomorphism) to the function $\Delta \ni t \mapsto \|h(t)\|$, where $h(t)$ is the coset of h in $\mathcal{R}/t\mathcal{R}$.

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(This function is continuous by [14].) Thus, we will regard z as a function on Δ. Let $[m, M]$ be an interval containing the range of z, where $m \geq 0$ and $M = \|h\| = \|z\|$. Given $a \in A_+$, the set $U = \{t \in \delta : \ a(t) \neq 0\}$ is open since the function $\Delta \ni t \mapsto \|a(t)\| \in \mathbb{R}$ is continuous. The weak* closure of the ideal generated by a in $\mathcal{R}$ is of the form $p\mathcal{R}$ for a unique projection $p \in \mathcal{Z}$ and p is open by definition. Since the quotient algebras $\mathcal{R}/t\mathcal{R}$ have only scalars in their centres, $p(t) = 1$ for each $t \in U$, hence also for each $t \in \overline{U}$ by continuity, so $p \geq q$, where $q \in \mathcal{Z}$ is the projection that corresponds to the characteristic function of $\overline{U}$. But from the definition of U, we see that $qa = a$ and this implies that $qb = b$ for each b in the ideal generated by a. Hence, $qp = p$ and it follows that $q = p$. In particular, for each $r \in \mathbb{R}_+$ the projection that corresponds to the closure of the set $U_r = \{t \in \Delta : \ z(t) > r\}$ is open since U_r is just the set $\{t \in \Delta : \ a(t) \neq 0\}$, where $a = (h - r)_+$. (This has been observed already by Halpern in [18, proof of Lemma 6].) Given $\varepsilon > 0$, for each $k \in \mathbb{N}$ let p_k be the projection corresponding to the closure of the set $U_k = \{t \in \Delta : \ z(t) > M - k\varepsilon\}$. Then, $0 = p_0 \leq p_1 \leq p_2 \leq \ldots \leq p_n = 1$, where $n \in \mathbb{N}$ is such that $M - n\varepsilon < m$ and $M - (n-1)\varepsilon \geq m$. Now, from $1 = (p_1 - p_0) + (p_2 - p_1) + \ldots + (p_n - p_{n-1})$, we have that $F_k := \overline{U}_k \setminus \overline{U}_{k-1}$ are disjoint closed and open sets that cover Δ and for $t \in F_k$, we have that $M - k\varepsilon \leq z(t) \leq M - (k-1)\varepsilon$. Thus, if we choose in each interval $[M - k\varepsilon, M - (k-1)\varepsilon]$ a point $\lambda_k \geq 0$ and set $c := \sum_{k=1}^{n} \lambda_k (p_k - p_{k-1})$, it follows that $\|z - c\| \leq \varepsilon$. Finally, observe that

$$c = (\lambda_1 - \lambda_2)p_1 + (\lambda_2 - \lambda_3)p_2 + \ldots + (\lambda_{n-1} - \lambda_n)p_{n-1} + \lambda_n p_n$$

is a linear combination with positive coefficients of open projections. □

The following theorem is a special case of Theorem 3.7, but it is used in the proof of that theorem.

Theorem 3.4. *Let ω and ρ be states on a C*-algebra A. Then, ρ is in the weak* closure $\overline{\omega \circ \mathrm{E}(A)}$ of the set $\omega \circ \mathrm{E}(A) = \{\omega \circ \psi : \ \psi \in \mathrm{E}(A)\}$, where $\mathrm{E}(A)$ is the set of all unital completely positive elementary complete contractions on A, if and only if $\|\rho|J\| \leq \|\omega|J\|$ for each ideal J of A.*

Proof. Evidently, $\rho \in \overline{\omega \circ \mathrm{E}(A)}$ implies that $\|\rho|J\| \leq \|\omega|J\|$ for each ideal J in A since maps in $\mathrm{E}(A)$ are contractive and preserve ideals. For the converse, suppose that $\rho \notin \overline{\omega \circ \mathrm{E}(A)}$. Then, by the Hahn–Banach theorem there exist $h \in A_h$ and $\alpha \in \mathbb{R}$ such that (3.1) holds, that is $\omega(\psi(h)) \leq \alpha$ for all $\psi \in \mathrm{E}(A)$, while $\rho(h) > \alpha$. Replacing h by $h + \beta 1$ for a sufficiently large $\beta \in \mathbb{R}$ (and consequently α by $\alpha + \beta$), we may assume that h is positive.

Let $\mathcal{R}$ be the universal von Neumann envelope of A and denote the unique weak* continuous extensions of ω and ρ to $\mathcal{R}$ by the same two letters. We will use the same notation as in the proof of Lemma 3.3. Thus, z is the infimum of all c in $\mathcal{Z}$ such that $h \leq c$. Since $W(z(t)1) = \{z(t)\} \subseteq W(h(t))$ for each $t \in \Delta$, it follows by [23, 3.3] that $z \in \overline{\mathrm{co}_{\mathcal{R}}}(h)$ (= the weak* closure of the $\mathcal{R}$-convex hull of h), hence by the first relation in (3.1)

$$\omega(z) \leq \alpha, \tag{3.5}$$

since each map ψ of the form $x \mapsto \sum_i b_i^* x b_i$ ($b_i \in \mathcal{R}$, $\sum_i b_i^* b_i = 1$) can be approximated by maps of the form $x \mapsto \sum_i a_i^* x a_i$ ($a_i \in A$, $\sum_i a_i^* a_i = 1$). (This follows by using the

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Kaplansky density theorem in $\mathrm{M}_n(\mathcal{R})$ to approximate the column $b = (b_1, \ldots, b_n)^T$ by $(a_1, \ldots, a_n)^T$.) Since ω and ρ are states, the hypothesis $\|\rho|J\| \leq \|\omega|J\|$ for each ideal J in A means that $\rho(p) \leq \omega(p)$ for each open projection $p \in \mathcal{Z}$. Then, it follows by Lemma 3.3 that $\rho(z) \leq \omega(z)$. But from $h \leq z$ and using (3.5), we have now that $\rho(h) \leq \rho(z) \leq \omega(z) \leq \alpha$, which is in contradiction with the previously established relation $\rho(h) > \alpha$. □

The naive attempt to generalize Theorem 3.4 to hermitian functionals fails, as shown by the following example. The example also shows that the assumption in Theorem 3.1, that A has Hausdorff primitive spectrum, is not redundant and that in Theorem 2.3 the normality of ω and ρ is not redundant.

Example 3.5. For a separable Hilbert space $\mathcal{H}$, let ω_1 be a normal and ω_2 a singular state on $\mathrm{B}(\mathcal{H})$, ρ_1 and ρ_2 positive normal functionals on $\mathrm{B}(\mathcal{H})$ with orthogonal supports such that $\rho_1(1) = \frac{1}{2} = \rho_2(1)$. Set $\omega = \omega_1 - \omega_2$ and $\rho = \rho_1 - \rho_2$. Then, $\rho(1) = 0 = \omega(1)$, $\|\rho\| = \rho_1(1) + \rho_2(1) = 1 = \omega_1(1) \leq \|\omega\|$. Since ρ_1, ρ_2 and ω_1 are normal, while ω_2 is singular (which means that ω_2 annihilates the ideal $\mathrm{K}(\mathcal{H})$ of all compact operators on $\mathcal{H}$), we have $\|\rho|\mathrm{K}(\mathcal{H})\| = \|\rho\| = \rho_1(1) + \rho_2(1) = \omega_1(1) = \|\omega_1|\mathrm{K}(\mathcal{H})\| = \|\omega|\mathrm{K}(\mathcal{H})\| \leq \|\omega\|$. Thus, $\|\rho|J\| \leq \|\omega|J\|$ for each ideal J of $\mathrm{B}(\mathcal{H})$ and ω and ρ agree on the centre $\mathbb{C}1$ of $\mathrm{B}(\mathcal{H})$. But nevertheless, $\rho \notin \overline{\omega \circ \mathrm{E}(\mathrm{B}(\mathcal{H}))}$ since on $\mathrm{K}(\mathcal{H})$ all elements of $\overline{\omega \circ \mathrm{E}(\mathrm{B}(\mathcal{H}))}$ act as elements of $\overline{\omega_1 \circ \mathrm{E}(\mathrm{B}(\mathcal{H}))}|\mathrm{K}(\mathcal{H})$ and are therefore positive, while $\rho|\mathrm{K}(\mathcal{H}) = (\rho_1 - \rho_2)|\mathrm{K}(\mathcal{H})$ is not positive.

To generalize Theorem 3.4 to hermitian functionals, we need a lemma.

Lemma 3.6. *For each hermitian functional ω on a C^*-algebra A, we have*

$$\overline{\omega \circ \mathrm{E}(A)} = \overline{\omega_+ \circ \mathrm{E}(A)} - \overline{\omega_- \circ \mathrm{E}(A)}.$$

Proof. Suppose that $\rho \in \overline{\omega \circ \mathrm{E}(A)}$ and let (ψ_k) be a net in $\mathrm{E}(A)$ such that $\rho(a) = \lim_k \omega(\psi_k(a))$ for all $a \in A$. Extend ω, ρ and each ψ_k weak* continuously to the universal von Neumann envelope $\mathcal{R}$ of A and denote the extensions by the same symbols. Let ψ be a weak* limit point of the net (ψ_k) and note that ψ is a unital completely positive (hence contractive) module map over the centre $\mathcal{Z}$ of $\mathcal{R}$. Set $\rho_1 = \omega_+ \circ \psi|A$ and $\rho_2 = \omega_- \circ \psi|A$. Then $\rho_1 \in \overline{\omega_+ \circ \mathrm{E}(A)}$, $\rho_2 \in \overline{\omega_- \circ \mathrm{E}(A)}$ and $\rho = \omega \circ \psi|A = \rho_1 - \rho_2$. This proves the inclusion $\overline{\omega \circ \mathrm{E}(A)} \subseteq \overline{\omega_+ \circ \mathrm{E}(A)} - \overline{\omega_- \circ \mathrm{E}(A)}$.

To prove the reverse inclusion, suppose that $\rho_1 \in \overline{\omega_+ \circ \mathrm{E}(A)}$ and $\rho_2 \in \overline{\omega_- \circ \mathrm{E}(A)}$. Then, there exist nets of maps ϕ_k and ψ_k in $\mathrm{E}(A)$ such that $\rho_1 = \lim_k \omega_+ \circ \phi_k$ and $\rho_2 = \lim_k \omega_- \circ \psi_k$. Let p and q be the support projections in $\mathcal{R}$ of ω_+ and ω_- (where ω_+ and ω_- have been weak* continuously extended to $\mathcal{R}$). Let (a_n) be a net of positive contractions in A strongly converging to p in $\mathcal{R}$, set $b_n = \sqrt{1 - a_n^2}$ and define maps $\phi_{k,n}$ and $\psi_{k,n}$ on A by

$$\phi_{k,n}(x) = a_n \phi_k(x) a_n \text{ and } \psi_{k,n}(x) = b_n \psi_k(x) b_n.$$

The nets $(\omega_+(b_n^2)) = (\omega_+(1 - a_n^2))$ and $(\omega_-(a_n^2)) = (\omega_-(1 - b_n^2))$ all converge to 0. From this, we will verify in the following by using the Cauchy–Schwarz inequality for positive functionals that $\lim_{k,n} \omega_+ \circ \phi_{k,n} = \rho_1$, $\lim_{k,n} \omega_- \circ \psi_{k,n} = \rho_2$, $\lim_{k,n} \omega_+ \circ \psi_{k,n} =$

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0 and $\lim_{k,n} \omega_- \circ \phi_{k,n} = 0$ pointwise on A, hence

$$\rho := \rho_1 - \rho_2 = \lim_{k,n}[(\omega_+ - \omega_-) \circ (\phi_{k,n} + \psi_{k,n})] = \lim_{k,n} \omega \circ \theta_{k,n},$$

where $\theta_{k,n} := \phi_{k,n} + \psi_{k,n}$. Evidently each $\theta_{k,n}$ is elementary completely positive map and also unital since $\theta_{k,n}(1) = a_n\phi_k(1)a_n + b_n\psi_k(1)b_n = a_n^2 + b_n^2 = 1$. Thus, $\rho \in \overline{\omega \circ \mathrm{E}(A)}$, verifying the inclusion $\overline{\omega \circ \mathrm{E}(A)} \supseteq \overline{\omega_+ \circ \mathrm{E}(A)} - \overline{\omega_- \circ \mathrm{E}(A)}$. Now, we will verify that $\lim_{k,n} \omega_+\phi_{k,n} = \rho_1$, the verification of the other three limits that we have used is similar. For each $x \in A$, we estimate

$$\begin{aligned}
|\rho_1(x) - \omega_+(\phi_{k,n}(x))| &= |\rho_1(x) - \omega_+(a_n\phi_k(x)a_n)| \\
&\le |\rho_1(x) - \omega_+(\phi_k(x))| + |\omega_+((1-a_n)\phi_k(x))| \\
&\quad + |\omega_+(a_n\phi_k(x)(1-a_n))| \\
&\le |\rho_1(x) - \omega_+(\phi_k(x))| + \omega_+((1-a_n)^2)^{1/2}\omega_+(\phi_k(x)^*\phi_k(x))^{1/2} \\
&\quad + \omega_+(\phi_k(x)^* a_n^2 \phi_k(x))^{1/2}\omega_+((1-a_n)^2)^{1/2} \\
&\le |\rho_1(x) - \omega_+(\phi_k(x))| + 2\omega_+((1-a_n)^2)^{1/2}\|\omega_+\|^{1/2}\|x\|.
\end{aligned}$$

Both terms in the last line of the above expression converge to 0. □

Theorem 3.7. *Let ω and ρ be hermitian functionals on a C^*-algebra A. Then $\rho \in \overline{\omega \circ \mathrm{E}(A)}$ if and only if there exist positive functionals ρ_1 and ρ_2 on A satisfying the following condition*:

(B) $\rho = \rho_1 - \rho_2$, $\rho_1(1) = \omega_+(1)$, $\rho_2(1) = \omega_-(1)$, $\|\rho_1|J\| \le \|\omega_+|J\|$ and $\|\rho_2|J\| \le \|\omega_-|J\|$ for all ideals J in A.

(In particular $\|\rho|J\| \le \|\omega|J\|$.) If ω is positive, then the condition (B) simplifies to $\rho(1) = \omega(1)$ and $\|\rho|J\| \le \|\omega|J\|$ for all ideals J.

Proof. Suppose that $\rho \in \overline{\omega \circ \mathrm{E}(A)}$. Using the notation introduced in the first part of the proof of Lemma 3.6, we have observed that the map ψ on $\mathcal{R}$ introduced in that proof is a contractive unital $\mathcal{Z}$-bimodule map. Thus, for any ideal J in A, if $p \in \mathcal{Z}$ is the projection satisfying $\overline{J} = p\mathcal{R}$, then $\psi(\overline{J}) = \psi(p\mathcal{R}) = p\psi(\mathcal{R}) \subseteq \overline{J}$. Since ω (and hence also ω_+ and ω_-) are weak* continuous on $\mathcal{R}$, we have $\|\omega_+|\overline{J}\| = \|\omega_+|J\|$ and $\|\omega_-|\overline{J}\| = \|\omega_-|J\|$. With $\rho_1 = \omega_+ \circ \psi|A$ and $\rho_2 = \omega_- \circ \psi|A$ (as in the proof of Lemma 3.6), we have $\rho = \rho_1 - \rho_2$, $\rho_1(1) = \omega_+(1)$, $\rho_2(1) = \omega_-(1)$,

$$\|\rho_1|J\| \le \|\omega_+ \circ \psi|\overline{J}\| \le \|\omega_+|\overline{J}\| = \|\omega_+|J\|$$

and similarly $\|\rho_2|J\| \le \|\omega_-|J\|$. Therefore, also

$$\|\rho|J\| = \|\rho_1|J - \rho_2|J\| \le \|\rho_1|J\| + \|\rho_2|J\| \le \|\omega_+|J\| + \|\omega_-|J\| = \|\omega|J\|.$$

Conversely, assume the existence of positive functionals ρ_1 and ρ_2 on A satisfying the norm inequalities in condition (B). Then, by Theorem 3.4 $\rho_1 \in \overline{\omega_+ \circ \mathrm{E}(A)}$ and $\rho_2 \in \overline{\omega_- \circ \mathrm{E}(A)}$, hence by Lemma 3.6 $\rho \in \overline{\omega \circ \mathrm{E}(A)}$. □

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4. Maximally mixed states

For functionals ω and ρ on a C*-algebra A let us say that ρ is more mixed than ω if $\rho \in \overline{\omega \circ \mathrm{E}(A)}$ (where the bar denotes weak* closure). Applying Zorn's lemma to the family of all weak* closed $\mathrm{E}(A)$-invariant subsets of $\overline{\omega \circ \mathrm{E}(A)}$ we see that in $\overline{\omega \circ \mathrm{E}(A)}$, there exist minimal $\mathrm{E}(A)$-invariant compact non-empty subsets, which are evidently of the form $\overline{\rho \circ \mathrm{E}(A)}$ for some ρ and such ρ are called maximally mixed. Thus, a functional ω is *maximally mixed* if $\rho \in \overline{\omega \circ \mathrm{E}(A)}$ implies that $\omega \in \overline{\rho \circ \mathrm{E}(A)}$. If A has Hausdorff primitive spectrum, Corollary 3.2 implies that all states on A are maximally mixed. The same conclusion holds for liminal C*-algebras.

Corollary 4.1. *On a liminal C*-algebra A every state ω is maximally mixed.*

Proof. If $\rho \in \overline{\omega \circ \mathrm{E}(A)}$, then by Theorem 3.4 $\|\rho|J\| \leq \|\omega|J\|$ for each ideal J in A. Denoting by p the projection in $\mathcal{R} := A^{\sharp\sharp}$ such that $\overline{J} = p\mathcal{R}$, this means that $\rho(p) \leq \omega(p)$ for each open central projection p, where ω and ρ have been weak* continuously extended to $\mathcal{R}$. Since A is liminal, such projections are strongly dense in the set of all central projections by [10], hence it follows that $\rho(p^{\perp}) \leq \omega(p^{\perp})$. Since $\rho(p) + \rho(p^{\perp} = \rho(1) = 1 = \omega(1) = \omega(p) + \omega(p^{\perp})$, we conclude that $\rho(p) = \omega(p)$, that is $\|\rho|J\| = \|\omega|J\|$. By Theorem 3.4, this implies that $\omega \in \overline{\rho \circ \mathrm{E}(A)}$. □

Perhaps, the simplest C*-algebras on which not all states are maximally mixed are C*-algebras that have only one maximal ideal and this ideal is not 0.

Example 4.2. Suppose that a unital C*-algebra A has only one maximal ideal M (for example, A may be simple or a factor). Then, a state ω on A is maximally mixed if and only if $\omega|M = 0$.

Proof. Suppose that $\omega|M = 0$ and let $\rho \in \overline{\omega \circ \mathrm{E}(A)}$. Then, $\rho|M = 0$, hence also $\rho(J) = 0$ for each proper ideal J of A since $J \subseteq M$. Thus, $\|\omega|J\| = \|\rho|J\|$ for each ideal J of A, so $\omega \in \overline{\rho \circ \mathrm{E}(A)}$ by Theorem 3.4.

Suppose now that $\omega|M \neq 0$. Let ρ be any state on A such that $\rho|M = 0$. Then $\|\rho|J\| \leq \|\omega|J\|$ for all ideals J, hence $\rho \in \overline{\omega \circ \mathrm{E}(A)}$ by Theorem 3.4. But $\omega \notin \overline{\rho \circ \mathrm{E}(A)}$ since $\rho|M = 0$ and $\omega|M \neq 0$, thus ω is not maximally mixed. □

Remark 4.3. If K is an ideal of A, each state ω on A satisfying $\omega(K) = 0$ may be regarded as a state on A/K, say $\dot{\omega}$. Note that $\dot{\omega}$ is maximally mixed on A/K if and only if ω is maximally mixed on A. Indeed, denoting by $q : A \to A/K$ the natural map, $q(J)$ is an ideal in A/K for each ideal J in A and all ideals in A/K are of such a form. Moreover, $\|\omega|J\| = \|\dot{\omega}|q(J)\|$, hence the claim follows from Theorem 3.4.

Example 4.2 is generalized in Theorem 4.4. The proof of Theorem 4.4 is inspired by an idea from [4, 3.10], but we will avoid using a background result from [5], that is used in [4, 3.10], and present a short self-contained proof. Recall that the strong radical J_A of A is the intersection of all maximal ideals in A.

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Theorem 4.4. (i) *$\omega(J_A) = 0$ for each maximally mixed state ω on A.*

(ii) *If a state ω on A annihilates some intersection $M_1 \cap M_2 \cap \ldots \cap M_n$ of finitely many maximal ideals in A, then ω is maximally mixed.*

Thus, the set $S_m(A)$ of maximally mixed states on A is a weak dense subset of $S(A/J_A)$ (= the set of states on A that annihilate J_A).*

Proof. (i) Let $D = S(A/J_A)$ and ω a maximally mixed state on A. Suppose that $\omega \notin D$. Then, $\overline{\omega \circ \mathrm{E}(A)} \cap D = \emptyset$, otherwise this intersection would be a weak* closed proper $\mathrm{E}(A)$-invariant subset of $\overline{\omega \circ \mathrm{E}(A)}$, which would contradict the fact that ω is maximally mixed. Thus, by the Hahn–Banach theorem, there exist $\alpha, \beta \in \mathbb{R}$ and $h \in A_h$ such that

$$\rho(h) \leq \alpha \ \forall \rho \in D \text{ and } \omega(\psi(h)) \geq \beta > \alpha \ \forall \psi \in \mathrm{E}(A). \tag{4.1}$$

Replacing h by $h + \gamma 1$ for a sufficiently large $\gamma \in \mathbb{R}_+$ (and modifying α, β), we may assume that h is positive. Then, the first relation in (4.1) means that $\|\dot{h}\| \leq \alpha$, where $\dot{h}$ denotes the coset of h in A/J_A. The (algebraic) numerical range $W_{A/J_A}(h)$ of $\dot{h}$ is an interval, say $[c, d]$, contained in the numerical range $W_A(h)$ of h, which is an interval, say $[a, b]$; note that $a \leq c \leq d = \|\dot{h}\| \leq b = \|h\|$. Let $f : [a, b] \to [c, d]$ be the function, which act as the identity on $[c, d]$, and maps $[a, c]$ into $\{c\}$ and $[d, b]$ into $\{d\}$. For every proper ideal K in A the quotient $A/(K + J_A)$ is non-zero, for K is contained in a maximal ideal M and hence $K + J_A \subseteq M + J_A = M \neq A$. Since $W_{A/(K+J_A)}(h) \subseteq W_{A/K}(h) \cap W_{A/J_A}(h)$, this intersection is not empty, hence the interval $W_{A/K}(h)$ intersects $[c, d]$ and is therefore mapped by f into itself. The numerical range $W_{A/K}(f(h))$ of the coset of $f(h)$ in A/K is just the convex hull of the spectrum $\sigma_{A/K}(f(h)) = f(\sigma_{A/K}(h)$, hence $W_{A/K}(f(h)) \subseteq f(W_{A/K}(h)) \subseteq W_{A/K}(h)$. This inclusion implies that $f(h) \in \overline{\mathrm{E}(A)(h)}$ by [23], hence $\omega(f(h)) > \alpha$ by the second relation in (4.1). Since ω is a state, it follows that $W_A(f(h))$ intersects (α, ∞). But this is a contradiction since $W_A(f(h))$ is the convex hull of the spectrum $\sigma_A(f(h)) = f(\sigma_A(h)) \subseteq [c, d] = [c, \|\dot{h}\|] \subseteq [c, \alpha]$. Thus, $\omega \in D$.

(ii) By the Chinese remainder theorem [15, 6.3], there is a natural isomorphism $A/\cap_{j=1}^n M_j \cong \oplus_{j=1}^n A/M_j$, thus we may regard ω as a state on $\oplus_{j=1}^n A/M_j$. Since the algebras A/M_j are simple, all states on them are maximally mixed by Example 4.2. The same then holds for their direct sum, so all states on $A/\cap_{j=1}^n M_j$ are maximally mixed and (ii) follows by Remark 4.3.

The set of all states that annihilate some finite intersection of maximal ideals of A is convex and norming for A/J_A (since the natural map $A/J_A \to \oplus_M A/M$, where the sum is over all maximal ideals in A, is a monomorphism, thus isometric), hence weak* dense in $S(A/J_A)$ [20, 4.3.9]. □

Remark 4.5. A similar argument as in [4, 3.2] shows that the set $S_m(A)$ of all maximally mixed states on a C*-algebra A is always norm closed.

Recall that a C*-algebra A is *weakly central* if different maximal ideals of A have different intersections with the centre Z of A.

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Theorem 4.6. *If the set $S_m(A)$ of all maximally mixed states is weak* closed (which by Theorem 4.4 just means that $S_m(A) = S(A/J_A)$), then each primitive ideal of A containing J_A is maximal. If A is weakly central, then the converse also holds: if each primitive ideal containing J_A is maximal, then $S_m(A) = S(A/J_A)$.*

Proof. By Remark 4.3 a state ω on A/J_A is maximally mixed if and only if it is maximally mixed on A. By [3, 3.10], the quotients of weakly central C*-algebras are weakly central, so in particular A/J_A is weakly central. In this way, we reduce the proof to the algebra A/J_A (instead of A), which has strong radical 0. Thus, we may assume that $J_A = 0$.

Suppose now that $S_m(A) = S(A)$. Then, $S_m(A/P) = S(A/P)$ for each primitive ideal P of A by Remark 4.3. If M is a maximal ideal of A containing P, then A/M is a quotient of A/P, hence each state $\rho \in S(A/M)$ can be regarded as a state on A/P and therefore can be weak* approximated by convex combinations of vector states on A/P, where A/P has been faithfully represented on a Hilbert space. Since A/P is primitive, as a consequence of the Kadison transitivity theorem, each vector state is of the form $x \mapsto \theta(u^*xu)$ for a fixed state θ on A/P with $\theta(M/P) \neq 0$, where $u \in A/P$ is unitary [20, 5.4.5]. Thus, $\rho \in \overline{\theta \circ \mathrm{E}(A/P)}$. But $\rho(M/P) = 0$, while $\theta(M/P) \neq 0$ if $M \neq P$, hence $\theta \notin \overline{\rho \circ \mathrm{E}(A/P)}$ if $M \neq P$. Thus, ρ can not be maximally mixed (on A/P and hence also on A) if P is not maximal. This argument, which we have found in [4, proof of 3.15], shows that in general the equality $S_m(A) = S(A)$ can hold only if all primitive ideals containing J_A are maximal. If A is weakly central and by our reduction above $J_A = 0$, then the assumption that all primitive ideals are maximal implies that the primitive spectrum $\check{A}$ of A is homeomorphic to the maximal ideal space Δ of Z (via the map $\check{A} \ni M \mapsto M \cap Z \in \Delta$). Thus, $\check{A}$ is Hausdorff and in this case, Corollary 3.2 shows that all states on A are maximally mixed. $\square$

It is well known that each W*-algebra $\mathcal{R}$ is weakly central. If $\mathcal{R}$ is properly infinite, each primitive ideal P containing $J_{\mathcal{R}}$ is maximal. (Namely, by [16, 2.3] or [20, 8.7.21], the ideal $M := P + J_{\mathcal{R}} \supseteq \mathcal{R}(P \cap \mathcal{Z}) + J_{\mathcal{R}}$ is maximal, and $M = P$ if $P \supseteq J_{\mathcal{R}}$.) So, we can state the following corollary.

Corollary 4.7. *In a properly infinite von Neumann algebra $\mathcal{R}$ maximally mixed states are just the states that annihilate the strong radical $J_{\mathcal{R}}$.*

If $\mathcal{R}$ is finite, primitive ideals are not necessarily maximal. (By [17, 4.7], any ideal $\mathcal{R}t$, where t is a maximal ideal of the centre of $\mathcal{R}$, is primitive, while using the central trace, one can show that not all such ideals are maximal in $\mathcal{R} = \oplus_n \mathrm{M}_n(\mathbb{C})$, for example.) Thus, the set of maximally mixed states on $\mathcal{R}$ is not weak* closed.

Throughout the rest of the paper $\mathcal{R}$ is a W^*-algebra, $\mathcal{Z}$ its centre and Δ the maximal ideal space of $\mathcal{Z}$. For each $t \in \Delta$ let M_t be the unique maximal ideal of $\mathcal{R}$ that contains t [20, 8.7.15]). Note that $\phi(\mathcal{R}t) = \phi(\mathcal{R})t \subseteq t$ for each $\mathcal{Z}$-module map $\phi : \mathcal{R} \to \mathcal{Z}$.

To prove that tracial states are maximally mixed, we need a lemma.

Lemma 4.8. *A bounded $\mathcal{Z}$-module map $\phi : \mathcal{R} \to \mathcal{Z} \subseteq \mathcal{R}$ preserves all ideals of $\mathcal{R}$ if and only if $\phi(M_t) \subseteq t$ for each $t \in \Delta$. If $\mathcal{R}$ is properly infinite, this is equivalent to $\phi(J_{\mathcal{R}}) = 0$.*

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Proof. Let J be an ideal in $\mathcal{R}$ and $K = J \cap \mathcal{Z}$. As an ideal in $\mathcal{Z}$, K can be identified with the set of all continuous functions on Δ than vanish on some closed subset Δ_K of Δ, hence K is the intersection of a family $\{t : t \in \Delta_K\}$ of maximal ideals of $\mathcal{Z}$. By [20, 8.7.15], there exists the largest ideal $J(K)$ in $\mathcal{R}$ such that $J(K) \cap \mathcal{Z} = K$, and it follows from [20, 8.7.16] that $J(K) = \cap_{t\in\Delta_K} M_t$. Now $J \cap \mathcal{Z} = K$ implies that $J \subseteq J(K)$. Thus, if ϕ has the property that $\phi(M_t) \subseteq t$ for all $t \in \Delta$, then $\phi(J) \subseteq \phi(J(K)) \subseteq \cap_{t\in\Delta_K} \phi(M_t) \subseteq \cap_{t\in\Delta_K} t = K \subseteq J$.

If $\mathcal{R}$ is properly infinite, then $M_t = \mathcal{R}t + J_{\mathcal{R}}$ for each $t \in \Delta$ by [20, 8.7.21 (1)]. Thus, if $\phi(J_{\mathcal{R}}) = 0$, then we have $\phi(M_t) = \phi(\mathcal{R})t \subseteq t$ for all $t \in \Delta$. Conversely, if $\phi(M_t) \subseteq t$ for all t, then $\phi(J_{\mathcal{R}}) = \phi(\cap_{t\in\Delta} M_t) \subseteq \cap_{t\in\Delta} t = 0$. □

Corollary 4.9. *A unital positive $\mathcal{Z}$-module map $\phi : \mathcal{R} \to \mathcal{Z} \subseteq \mathcal{R}$ is in the point-norm closure of elementary such maps (that is, $\phi \in \overline{\mathrm{E}(\mathcal{R})}^{\mathrm{p.n.}}$) if and only if $\phi(M_t) \subseteq t$ for each $t \in \Delta$.*

Proof. By [24, 2.2] and [25, 2.1] each completely contractive map $\phi : \mathcal{R} \to \mathcal{Z} \subseteq \mathcal{R}$ which preserves all ideals of $\mathcal{R}$ is in the point-norm closure of maps of the form $x \mapsto a^*xb = \sum_{j=1}^{n} a_j^* x b_j$, where $n \in \mathbb{N}$, $a_j, b_j \in \mathcal{R}$, $a := (a_1, \ldots, a_n)^T$, $b := (b_1, \ldots, b_n)$, $\|a\| \leq 1$ and $\|b\| \leq 1$. If ϕ is unital, then we can modify such maps to unital maps in the same way as in the proof of Theorem 2.1, which shows that $\phi \in \overline{\mathrm{E}(\mathcal{R})}^{\mathrm{p.n.}}$. □

Corollary 4.10. *Let ω be a state of the form $\omega = \mu \circ \phi$, where $\mu = \omega|\mathcal{Z}$ and $\phi : \mathcal{R} \to \mathcal{Z}$ is a unital positive $\mathcal{Z}$-module map. If $\phi(M_t) \subseteq t$ for each $t \in \Delta$, then ω is maximally mixed. In particular, tracial states are maximally mixed.*

Proof. Suppose that $\rho \in \overline{\omega \circ \mathrm{E}(\mathcal{R})}$. Then, $\rho|\mathcal{Z} = \omega|\mathcal{Z} = \mu$, hence

$$\omega = \mu \circ \phi = (\rho|\mathcal{Z}) \circ \phi = \rho \circ \phi.$$

By Corollary 4.9 ϕ can be approximated in the point-norm topology by a net of maps $\phi_k \in \overline{\mathrm{E}(\mathcal{R})}^{\mathrm{p.n.}}$. Then, $\omega(x) = \lim_k(\rho(\phi_k(x)))$ for all $x \in \mathcal{R}$. This shows that $\omega \in \overline{\rho \circ \mathrm{E}(\mathcal{R})}$, so ω is maximally mixed.

Any tracial state ω annihilates the properly infinite part of $\mathcal{R}$, hence we assume that $\mathcal{R}$ is finite. Then, $\omega = (\omega|\mathcal{Z}) \circ \tau$, where τ is the central trace on $\mathcal{R}$ [20, 8.3.10]. Since M_t is of the form $M_t = \{a \in \mathcal{R} : \tau(a^*a) \in t\}$ by [20, 8.7.17], for $a \in M_t$, we have by the Schwarz inequality $\tau(a)^*\tau(a) \leq \tau(a^*a) \in t$. This implies that $\tau(a) \in t$. Thus, $\tau(M_t) \subseteq t$, hence ω is maximally mixed by the first part of the corollary. □

Are all maximally mixed states on W*-algebras of the form specified in Corollary 4.10? Not quite. To investigate this, we still need some preparation.

Lemma 4.11. *For each state ω on $\mathcal{R}$ there exists a positive $\mathcal{Z}$-module map $\phi : \mathcal{R} \to \mathcal{Z}$ such that $\omega = (\omega|\mathcal{Z}) \circ \phi$ and $p := \phi(1)$ is a projection with $\omega(p) = 1$.*

Proof. Let Φ be the universal representation of $\mathcal{R}$, so that $\mathcal{R}^{\sharp\sharp}$ is the weak* closure of $\Phi(\mathcal{R})$. Then, the $*$-homomorphism $\Phi^{-1} : \Phi(\mathcal{R}) \to \mathcal{R}$ can be weak* continuously extended to a $*$-homomorphism $\Psi : \mathcal{R}^{\sharp\sharp} \to \mathcal{R}$; set $\tilde{\omega} = \omega \circ \Psi$ [20, 10.1.1, 10.1.12]. Let $\tilde{\mathcal{Z}}$ be the centre of $\mathcal{R}^{\sharp\sharp}$. Since $\tilde{\omega}$ is weak* continuous, by [17] or [27, 1.4], there exists a unique

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$\tilde{\mathcal{Z}}$-module homomorphism $\psi : \mathcal{R}^{\sharp\sharp} \to \tilde{\mathcal{Z}}$ such that $\tilde{\omega} = (\tilde{\omega}|\tilde{\mathcal{Z}}) \circ \psi$ and $\psi(1)$ is the support projection q of $\tilde{\omega}|\tilde{\mathcal{Z}}$. It is not hard to verify that $\phi := (\Psi|\tilde{\mathcal{Z}}) \circ \psi \circ \Phi$ has the properties stated in the lemma. □

Let ω be a state on $\mathcal{R}$, $\mu = \omega|\mathcal{Z}$ and let ϕ, p be as in Lemma 4.11, so that $\omega = \mu \circ \phi$. Let J be an ideal of $\mathcal{R}$ and $K = J \cap \mathcal{Z}$. Let (e_k) and (f_l) be approximate units in J and K (respectively). Then

$$\|\omega|K\| = \|\mu|K\| = \lim_l \mu(f_l) \text{ and } \|\omega|J\| = \lim_k \mu(\phi(e_k)). \tag{4.2}$$

We may regard (f_l) and $(\phi(e_k))$ as two bounded increasing nets in the positive part of the unit ball of $C(\Delta)$ ($\cong \mathcal{Z}$), hence they converge pointwise to some lower semi-continuous functions f and g (respectively) on Δ. The ideal K of $C(\Delta)$ is of the form $K = \{a \in C(\Delta) : a|\Delta_K^c = 0\}$ for some open subset Δ_K of Δ and since (f_l) is an approximate unit for K, it follows that f is just the indicator function χ_{Δ_K} of Δ_K. Let Δ_p be the clopen subset of Δ that correspond to the projection $p = \phi(1)$ (that is, $p = \chi_{\Delta_p}$, the indicator function of Δ_p). Since $f_l \in J$ and (e_k) is an approximate unit for J, $\lim_k e_k f_l = f_l$, hence $gf_l = \lim_k \phi(e_k) f_l = \lim_k \phi(e_k f_l) = \phi(f_l) = f_l \phi(1) = f_l p$ and $gf = \lim_l gf_l = \lim_l f_l p = fp$, that is $(g - \chi_{\Delta_p})\chi_{\Delta_K} = 0$. This means that

$$g(t) = 1 \ \forall t \in \Delta_p \cap \Delta_K. \tag{4.3}$$

Since (e_k) is an approximate unit, for any k_1 and k_2, there exists $k_3 \geq k_1, k_2$ so that $e_{k_3} \geq e_{k_1}$ and $e_{k_3} \geq e_{k_2}$, and (f_l) have the analogous property. Thus, $f = \sup_l f_l$, $g = \sup_k \phi(e_k)$ and we may apply the version of the monotone convergence theorem for nets [12, 7.12]. Thus, denoting by $\hat{\mu}$, the Radon measure on Δ that corresponds to μ, we have $\lim_l \mu(f_l) = \sup_l \mu(f_l) = \sup_l \int_\Delta f_l \, d\hat{\mu} = \int_\Delta \sup_l f_l \, d\hat{\mu} = \int_\Delta f \, d\hat{\mu} = \hat{\mu}(f)$ and similarly $\lim_k \mu(\phi(e_k)) = \hat{\mu}(g)$. Therefore, by (4.2), the equality $\|\omega|J\| = \|\omega|K\|$ is equivalent to $\hat{\mu}(g) = \hat{\mu}(f) = \hat{\mu}(\Delta_K)$. By (4.3), this condition $\hat{\mu}(g) = \hat{\mu}(f))$ means that $0 = \hat{\mu}(g - f) = \int_{\Delta_K^c \cup \Delta_p^c} (g - f) \, d\hat{\mu} = \int_{\Delta_K^c} (g - \chi_{\Delta_K}) \, d\hat{\mu} = \int_{\Delta_K^c} g \, d\hat{\mu}$, since $\hat{\mu}(\Delta_p^c) = 0$ (because $\mu(p) = 1$). As $g \geq 0$, we conclude that $\|\omega|J\| = \|\omega|K\|$ if and only if $g(t) = 0$ for $\hat{\mu}$-almost all $t \in \Delta_K^c$. Since (e_k) is an approximate unit of J, $\phi(e_k)(t) > 0$ for some k if and only if $\phi(a)(t) \neq 0$ for some $a \in J$. Hence, since $g = \sup_k \phi(e_k)$,

$$\{t \in \Delta_K^c : g(t) > 0\} = \cup_k \{t \in \Delta_k^c : \phi(e_k)(t) > 0\} = \cup_{a \in J} \{t \in \Delta_K^c : \phi(a)(t) \neq 0\}.$$

This proves the following lemma. (Note that g is lower semi-continuous, hence the set $\Delta_{\phi(J)|\Delta_K^c \neq 0}$ in the lemma is $\hat{\mu}$-measurable.)

Lemma 4.12. *$\|\omega|J\| = \|\omega|(J \cap \mathcal{Z})\|$ if and only if $\hat{\mu}(\Delta_{\phi(J)|\Delta_K^c \neq 0}) = 0$, where*

$$\Delta_{\phi(J)|\Delta_K^c \neq 0} = \bigcup_{a \in J} \{t \in \Delta_K^c : \phi(a)(t) \neq 0\}. \tag{4.4}$$

Here, $K = J \cap \mathcal{Z}$ and Δ_K^c is the set of all common zeros of elements of K.

The following theorem says that maximally mixed states are those for which the corresponding ϕ almost (with respect to $\hat{\mu}$) preserve ideals.

Theorem 4.13. *Let ω be any state on $\mathcal{R}$. Let $\omega = \mu \circ \phi$, where $\mu = \omega|\mathcal{Z}$ and $\phi : \mathcal{R} \to \mathcal{Z}$ is a positive $\mathcal{Z}$-module map with $\phi(1)$ a projection. Denote by $\hat{\mu}$ the Radon measure on Δ that corresponds to μ. Then, ω is maximally mixed if and only if $\hat{\mu}(\Delta_{\phi(J)|\Delta_K^c \neq 0}) = 0$ for each ideal J in $\mathcal{R}$, where $K = J \cap \mathcal{Z}$, $\Delta_K^c = \{t \in \Delta : K \subseteq t\}$ and $\Delta_{\phi(J)|\Delta_K^c \neq 0}$ is the set defined in (4.4).*

Proof. Suppose that $\rho \in \overline{\omega \circ \mathrm{E}(\mathcal{R})}$. Then, $\rho|\mathcal{Z} = \omega|\mathcal{Z}$ and by Theorem 3.4 $\|\rho|J\| \leq \|\omega|J\|$ for each ideal J of $\mathcal{R}$. If $\hat{\mu}(\Delta_{\phi(J)|\Delta_K^c \neq 0}) = 0$ for each J, then by Lemma 4.12 $\|\omega|J\| = \|\omega|(J \cap \mathcal{Z})\|$ for each J, hence $\|\omega|J\| = \|\omega|(J \cap \mathcal{Z})\| = \|\rho|(J \cap \mathcal{Z})\| \leq \|\rho|J\|$. Therefore, by Theorem 3.4 $\omega \in \overline{\rho \circ \mathrm{E}(\mathcal{R})}$, which proves that ω is maximally mixed.

Conversely, if $\hat{\mu}(\Delta_{\phi(J_0)|\Delta_K^c \neq 0}) > 0$ for some ideal J_0, then by Lemma 4.12 $\|\omega|(J_0 \cap \mathcal{Z})\| < \|\omega|J_0\|$. Let $\psi : \mathcal{R} \to \mathcal{Z}$ be any positive unital $\mathcal{Z}$-module map that preserves ideals. (For example, the central trace, if $\mathcal{R}$ is finite, as we have seen in the proof of Corollary 4.10. If $\mathcal{R}$ is properly infinite, preservation of ideals is equivalent to $\psi(J_{\mathcal{R}}) = 0$ by Lemma 4.8, so we can take for ψ the composition $\mathcal{R} \xrightarrow{\eta} \mathcal{R}/J_{\mathcal{R}} \xrightarrow{\iota} \mathcal{Z}$, where η is the natural map and ι is an extension of the inclusion $\mathcal{Z} \to \mathcal{R}/J_{\mathcal{R}}$. Here $\mathcal{Z}$ is regarded as contained in $\mathcal{R}/J_{\mathcal{R}}$ since $\mathcal{Z} \cap J_{\mathcal{R}} = 0$, and ι exists by the C*-injectivity of $\mathcal{Z}$.) Let $\rho = \mu \circ \psi$. Since $\psi(J) \subseteq J \cap \mathcal{Z}$ for each J, the set $\Delta_{\psi(J)|\Delta_{J\cap\mathcal{Z}}^c \neq 0}$ is empty, hence by Lemma 4.12 $\|\rho|J\| = \|\rho|(J \cap \mathcal{Z})\|$. Since $\rho|\mathcal{Z} = \mu = \omega|\mathcal{Z}$, we have $\|\rho|J\| = \|\rho|(J \cap \mathcal{Z})\| = \|\omega|(J \cap \mathcal{Z})\| \leq \|\omega|J\|$ for all J, hence $\rho \in \overline{\omega \circ \mathrm{E}(\mathcal{R})}$ by Theorem 3.4. But $\|\omega|J_0\| > \|\omega|(J_0 \cap \mathcal{Z})\| = \|\rho|(J_0 \cap \mathcal{Z})\| = \|\rho|J_0\|$ implies that $\omega \notin \overline{\rho \circ \mathrm{E}(\mathcal{R})}$. Hence, ω is not maximally mixed. □

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