



# Maps with the Unique Extension Property and $C^*$ -Extreme Points

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**Abstract** We investigate boundary representations in the context where Hilbert spaces are replaced by  $C^*$ -modules over abelian von Neumann algebras and apply this to study  $C^*$ -extreme points. We present an (unexpected) example of a weak\* compact  $\mathcal{B}$ -convex subset of  $\mathbb{B}(\mathcal{H})$  without  $\mathcal{B}$ -extreme points, where  $\mathcal{B}$  is an abelian von Neumann algebra on a Hilbert space  $\mathcal{H}$ . On the other hand, if  $\mathcal{A}$  is a von Neumann algebra with a separable predual and whose finite part is injective, we show that each weak\* compact  $\mathcal{A}$ -convex subset of  $\ell^\infty(\mathcal{A})$  is generated by its  $\mathcal{A}$ -extreme points.

**Keywords** Completely positive maps · Operator bimodules ·  $C^*$ -extreme points

**Mathematics Subject Classification** Primary 46L07 · 47L07; Secondary 46L05

## 1 Introduction

If  $V$  is a unital self-adjoint space generating a commutative  $C^*$ -algebra  $A$ , then pure states of  $V$  norm  $V$  and each such state has a unique extension to a pure state on  $A$ , which is necessarily multiplicative [3, 4.1.2]. In an analogous non-commutative setting, a unital completely positive (u.c.p.) map  $\varphi$  from an operator system  $X$ , generating a  $C^*$ -algebra  $B$ , into  $\mathbb{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ , is said to have the *unique*

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*extension property (u.e.p.)* if  $\phi$  is the restriction of a representation  $\pi: B \rightarrow \mathbb{B}(\mathcal{H})$  and  $\pi$  is the only c.p. extension of  $\phi$  to  $B$ . If in addition  $\pi$  is irreducible, then  $\pi$  is called a *boundary representation for  $X$* . Arveson [1] proved for a separable  $X$  that boundary representations completely norm  $X$  in the sense that for all  $n$  and all  $x \in \mathbb{M}_n(X)$  the equality  $\|x\| = \sup_{\phi} \|\phi_n(x)\|$  holds, where the sup is over boundary representations  $\phi$  and  $\phi_n$  means  $\phi$  applied to the entries of matrices. Davidson and Kennedy [5] showed that this is true for general  $X$ . Maps with the u.e.p. are usually constructed by dilations (see [1, 5, 6]) and so tend to have ranges in  $\mathbb{B}(\mathcal{L})$  for large Hilbert spaces  $\mathcal{L}$ . On the other hand, pure states all have their ranges in the same space  $\mathbb{C}$ . If  $C \subseteq \mathbb{B}(\mathcal{K})$  is a unital  $C^*$ -algebra and  $X \subseteq \mathbb{B}(\mathcal{K})$  is a closed self-adjoint  $C$ -subbimodule of  $\mathbb{B}(\mathcal{K})$  containing  $C$ , it is natural to define the *operator-valued quasi-state space* of  $X$  to be the set  $\text{CCP}_C(X, \mathbb{B}(\mathcal{H}))$  of all contractive c.p.  $C$ -bimodule maps from  $X$  into  $\mathbb{B}(\mathcal{H})$ , where  $\mathcal{H}$  is some standard Hilbert space associated to  $C$  [such that  $C \subseteq \mathbb{B}(\mathcal{H})$  and all states on  $C$  and all normal states on  $C'$  are vector states], while the *operator valued state space* of  $X$  is the subset  $\text{UCP}_C(X, \mathbb{B}(\mathcal{H}))$  of all unital such maps. Now a natural question is:

Suppose that  $X$  can be represented isometrically in  $\ell^\infty(\mathbb{B}(\mathcal{H}))$  as a self-adjoint unital  $C$ -bimodule. Do then maps in  $\text{CCP}_C(X, \mathbb{B}(\mathcal{H}))$  with the u.e.p. norm  $X$ ?

The positive answer in the classical case  $C = \mathbb{C} = \mathcal{H}$  might lead us to believe that the answer is positive in general, however, it turns out that this is not the case. The question turns out to be connected with the problem of existence of  $C^*$ -extreme points in  $C^*$ -convex sets of operators. In the past such points were studied first for subsets of  $\mathcal{A}$  (see [10, 15] and references there). Later a version of a non-commutative Krein–Milman theorem was proved in [33] and operator valued state spaces were studied in several papers, including [8–11] and recently (in the context of quantum information theory) in [16]. However, the coefficients in convex combinations were restricted mainly to be in finite dimensional algebras  $\mathbb{M}_n(\mathbb{C})$  and those techniques do not usually apply to infinite dimensional algebras. The theory of c.p. maps with the u.e.p. [1, 5, 6] (together with some results on operator bimodules) enables a progress also in studying  $C^*$ -convexity over infinite-dimensional operator algebras. Finally, it is worthwhile to mention that the study of this generalized form of convexity is not limited to functional analysis; at least matrix convex sets play important role in semi-algebraic geometry [14].

To conclude this introduction, let us briefly summarize the content of the paper. After some preliminaries in Sect. 2, we will provide in Sect. 3 an unexpected example of a weak\* compact set  $K$  of Hilbert space operators that is  $\mathcal{B}$ -convex over an abelian von Neumann algebra  $\mathcal{B}$ , but does not have any  $\mathcal{B}$ -extreme points. We also study  $C^*$ -extreme points for singly generated weak\* compact  $C^*$ -convex sets of compact operators.

A significant difference between the theory of Banach spaces and the theory of operator bimodules over a von Neumann algebra  $\mathcal{A}$  is that each Banach space can be embedded into  $\ell^\infty(\mathbb{C})$ , while not every operator  $\mathcal{A}$ -bimodule  $X$  can be embedded into  $\ell^\infty(\mathcal{A})$ . Therefore it is reasonable to hope that, although the non-commutative version of the Krein–Milman theorem does not hold in general, it may hold at least for weak\* compact  $\mathcal{A}$ -convex subsets of  $\ell^\infty(\mathcal{A})$ . In Sect. 5 we will prove such a theorem for  $\ell^\infty_{\mathbb{N}}(\mathcal{A})$  in the case when  $\mathcal{A}$  has separable predual, under the assumption that the finite

part in the central decomposition of  $\mathcal{A}$  is injective. (The author does not know if this assumption can be removed.) For this, we will have to pay much attention to the special case when  $\mathcal{A}$  is of the form  $\mathbb{M}_n(\mathcal{Z})$  for an abelian  $\mathcal{Z}$  and finite  $n$ . To study this special case we will first need to extend the theory of maps with the unique extension property ([1, 5, 6]) to the context when Hilbert spaces are replaced by self-dual  $C^*$ -modules over  $\mathcal{Z}$ ; this is the content of Sect. 4.

## 2 Preliminaries

A representation of a unital  $C^*$ -algebra  $A$  on a Hilbert space  $\mathcal{K}$  makes  $\mathcal{K}$  a *Hilbert  $A$ -module*.  $\mathcal{K}$  is called *cyclic* if there exists a vector  $\xi \in \mathcal{K}$  such that  $\mathcal{K} = [A\xi]$ , where  $[\cdot]$  denotes the closure of the linear span. The set of all bounded  $A$ -module maps on  $\mathcal{K}$  is denoted by  $\mathbb{B}_A(\mathcal{K})$ . A Hilbert module  $\mathcal{K}$  over a von Neumann algebra  $\mathcal{A}$  is called *normal* if the corresponding representation  $\mathcal{A} \rightarrow \mathbb{B}(\mathcal{K})$  is weak\* continuous. (We will denote von Neumann algebras by letters  $\mathcal{A}, \mathcal{B}, \dots, \mathcal{Z}$ , while general  $C^*$ -algebras by  $A, B, \dots$ )

Given a  $C^*$ -algebra  $C$  faithfully represented on a Hilbert space  $\mathcal{K}$ , a *faithful operator  $C$ -system* is a norm closed self-adjoint  $C$ -subbimodule  $X$  of  $\mathbb{B}(\mathcal{K})$  containing  $C$  (see [29] for an abstract characterization).

A faithful operator  $C$ -system  $X$  over a von Neumann algebra  $\mathcal{C}$  is called *strong* if  $X$  is an operator  $C$ -system in  $\mathbb{B}(\mathcal{K})$  for a faithful normal Hilbert  $\mathcal{C}$ -module  $\mathcal{K}$  such that  $X$  contains every  $b \in \mathbb{B}(\mathcal{K})$  for which there exist two families of projections  $(e_i)$  and  $(f_j)$  in  $\mathcal{C}$  with  $\sum_i e_i = 1 = \sum_j f_j$  and  $e_i b f_j \in X$  for all  $i, j$ . (In other words,  $X$  is strong as a  $\mathcal{C}$ -bimodule in the sense of [20, 2.2], [21, p. 385].)

Let  $\mathcal{H}$  be a Hilbert  $C$ -module,  $X$  a faithful operator  $C$ -system contained in a  $C^*$ -algebra  $B$  so that  $C$  and  $B$  have the same unit 1. By the well-known multiplicative domain argument [29, 3.18] any c.p. extension to  $B$  of a map  $\varphi \in \text{UCP}_C(X, \mathbb{B}(\mathcal{H}))$  must be a  $C$ -bimodule map (since it extends the representation  $\varphi|_C$ ).

**Definition 2.1** Let  $B$  be a  $C^*$ -algebra generated by an operator  $C$ -system  $X$ . A map  $\varphi \in \text{CCP}_C(X, \mathbb{B}(\mathcal{H}))$  (= the set of all contractive c.p.  $C$ -bimodule maps) is said to have the *u.e.p.* if  $p := \varphi(1)$  is a projection and  $\varphi$ , as a unital map into  $\mathbb{B}(p\mathcal{H})$ , has the u.e.p. as defined in the sect. 1.

*Remark 2.2* If  $\varphi \in \text{CCP}_C(X, \mathbb{B}(\mathcal{H}))$  has the u.e.p.,  $p = \varphi(1)$  and  $\pi: B = C^*(X) \rightarrow \mathbb{B}(p\mathcal{H}) \subseteq \mathbb{B}(\mathcal{H})$  is the representation extending  $\varphi$ , then  $\pi$  is the only c.c.p. extension of  $\varphi$  as a map into  $\mathbb{B}(\mathcal{H})$ . Indeed, if  $\psi: B \rightarrow \mathbb{B}(\mathcal{H})$  is any c.p. extension, then  $0 \leq \psi(b) \leq p$  for all  $b \in B$  with  $0 \leq b \leq 1$ , hence  $\psi(B) = p\psi(B)p$ . So  $\psi$  maps into  $\mathbb{B}(p\mathcal{H})$  and therefore must coincide with  $\pi$  by the u.e.p.

We denote by  $\overline{G}$  the weak\* closure and by  $\overline{\overline{G}}$  the norm closure of a set  $G$ .

**Definition 2.3** If  $\mathcal{K}$  is a Hilbert module over a  $C^*$ -algebra  $A$ , a subset  $K \subseteq \mathbb{B}(\mathcal{K})$  is called  *$A$ -convex* if

$$\sum_{j=1}^n a_j^* y_j a_j \in K \tag{2.1}$$

whenever

$$y_j \in K, \quad a_j \in A \quad \text{and} \quad \sum_{j=1}^n a_j^* a_j = 1. \quad (2.2)$$

We denote by  $\text{co}_A G$  the smallest  $A$ -convex set containing a set  $G$ .

A map  $f: K \rightarrow \tilde{K}$  between two  $A$ -convex sets is  $A$ -affine if

$$f \left( \sum_{j=1}^n a_j^* y_j a_j \right) = \sum_{j=1}^n a_j^* f(y_j) a_j,$$

where  $y_j$  and  $a_j$  are as in (2.2).

Using the Kaplansky density theorem [for  $\mathbb{M}_n(A)$ ] it is easy to show that a weak\* closed subset  $K \subseteq \mathbb{B}(K)$  is  $A$ -convex if and only if it is  $\bar{A}$ -convex [25, proof of 3.3].

**Definition 2.4** A set  $\mathcal{F}$  of maps  $\phi: X \rightarrow \mathbb{B}(\mathcal{H}_\phi)$  norms  $X$  if  $\|x\| = \sup_{\phi \in \mathcal{F}} \|\phi(x)\|$  for all  $x \in X$ . If the equality  $\|x\| = \sup_{\phi \in \mathcal{F}} \|\phi_n(x)\|$  holds for all  $x \in \mathbb{M}_n(X)$  and all  $n = 1, 2, \dots$  (where  $\phi_n = \phi \otimes 1_{\mathbb{M}_n(\mathbb{C})}$ ), then  $\mathcal{F}$  completely norms  $X$ .

A subset  $K_0$  of a weak\* closed  $A$ -convex set  $K$  generates  $K$  if  $K = \overline{\text{co}}_A K_0$  (the weak\* closure of  $\text{co}_A K_0$ ).

The following generalization from [25, 7.2] of a familiar fact [17, 4.3.9 (ii), (iii)] reformulates the question about generating a set  $K$  to a perhaps more tractable problem concerned with norming  $K$ . The conditions on  $\mathcal{H}$  in the Theorem 2.5 are satisfied if  $\mathcal{H}$  is a Hilbert space on which  $\mathcal{C}$  is in the standard form [32, Section IX.1]; we will call such a  $\mathcal{H}$  a *standard  $\mathcal{C}$ -module*.

**Theorem 2.5** Let  $\mathcal{H}$  be a Hilbert module over a von Neumann algebra  $\mathcal{C}$  such that all normal states on  $\mathcal{C}$  and on  $\mathcal{A} := \mathbb{B}_{\mathcal{C}}(\mathcal{H})$  are vector states (arising from vectors in  $\mathcal{H}$ ),  $X$  a faithful strong operator  $\mathcal{C}$ -system,  $X_h$  the self-adjoint part of  $X$ , and let  $S_0$  be a subset of  $Q := \text{CCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$  such that either  $S_0 \subseteq S := \text{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$  or  $0 \in S_0$ . Then  $S_0$  is norming for  $X_h$  if and only if  $\overline{\text{co}}_A S_0 \supseteq S$ .

### 3 A $C^*$ -Convex Set without $C^*$ -Extreme Points and Singly Generated $C^*$ -Convex Sets

**Definition 3.1** A point  $x$  in an  $A$ -convex set  $K$  is called an  $A$ -extreme point of  $K$  if the condition

$$x = \sum_{j=1}^n a_j^* x_j a_j, \quad \text{where } x_j \in K, \quad a_j \in A, \quad \sum_{j=1}^n a_j^* a_j = 1 \quad (n \text{ finite}) \quad (3.1)$$

and  $a_j$  are invertible, implies that there exist unitary elements  $u_j \in A$  such that  $x_j = u_j^* x u_j$ . If (3.1), where the  $a_j$  are assumed to be positive and invertible, implies that  $x_j = x$  and  $a_j x = x a_j$  for all  $j$ , then  $x$  is called a *strong  $A$ -extreme point* of  $K$ .

If (3.1) (where  $a_j$  are not necessarily invertible) implies that  $a_j^*a_j$  commutes with  $x$ ,  $a_ja_j^*$  commutes with  $x_j$  and  $a_j^*x_ja_j = |a_j|x|a_j|$  for all  $j$ , then  $x$  is called a *Choquet  $A$ -point* of  $K$ .

It suffices to check the above condition (3.1) in the case  $n = 2$ , [25, 5.5]. It is not hard to verify (using the polar decomposition of the  $a_j$ ) that Choquet  $A$ -points are  $A$ -extreme in  $K$ . Recently  $A$ -extreme points have been studied in the context of quantum information theory in [16], but in the case  $K \subseteq A$  such points were studied much earlier (under the name  $C^*$ -extreme points) e.g. in [11, 15, 28]. In particular, in the case  $A = M_n(\mathbb{C})$  various non-commutative versions of the Krein–Milman theorem have been proved by Morenz [28] and Webster and Winkler [33]. For a more general  $A$ , it can be proved that weak\* compact  $A$ -convex sets  $K \subseteq \mathbb{B}(\mathcal{K})$  are generated by their  $A$ -extreme points if  $\mathcal{K}$  and  $A$  are separable and  $A$  has large socle [26]. By all these positive results it is very tempting to conjecture that each weak\* compact  $\mathcal{A}$ -convex set of Hilbert space operators is generated by its  $\mathcal{A}$ -extreme points, but the following simple example shows that this conjecture is not true.

*Example 3.2* Let  $\mathcal{K} = L^2(\mu)$  and  $\mathcal{R} = L^\infty(\mu)$  regarded as the von Neumann algebra of multiplication operators on  $\mathcal{K}$ , where  $\mu$  is a finite Borel measure, say on  $[0, 1]$ . Let  $X$  be the set of all bounded operators on  $\mathcal{K} \oplus \mathbb{C}$  that can be represented in the form

$$\begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix}, \tag{3.2}$$

where  $\xi \in \mathcal{K}$  is identified with the operator from  $\mathbb{C}$  to  $\mathcal{K}$  mapping 1 to  $\xi$ . Clearly  $X \subseteq \mathbb{B}(\mathcal{K} \oplus \mathbb{C})$  is a weak\* closed operator bimodule over the von Neumann algebra  $\mathcal{B} := \mathcal{R} \oplus \mathbb{C}$ , where

$$\begin{bmatrix} a & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a\xi \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & \xi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & \lambda\xi \\ 0 & 0 \end{bmatrix}.$$

Let  $K$  be the unit ball of  $X$  and identify each  $\xi$  with the corresponding matrix (3.2). Then  $K$  is a weak\* compact  $\mathcal{B}$ -convex set.

*We claim that  $K$  has no  $\mathcal{B}$ -extreme points if  $\mu$  has no atoms.*

To show this, take any  $\xi \in K$ . If  $\|\xi\| < 1$ , then obviously  $\xi$  is a proper convex combination of two elements of  $K$  with the norms different from  $\|\xi\|$ , hence  $\xi$  is not a  $\mathcal{B}$ -extreme point of  $K$ . So assume now that  $\|\xi\| = 1$ . Denote by  $\chi$  the function on  $[0, 1]$  defined by  $\chi(t) = \frac{1}{3}(1 + t)$  and let  $a_1 = \chi$ ,  $a_2 = \sqrt{1 - \chi^2}$ ,  $\lambda_i = \|a_i\xi\|$ ,  $\xi_i = \lambda_i^{-1}a_i\xi$  and  $b_i = a_i \oplus \lambda_i$  ( $i = 1, 2$ ). Then the  $b_i$  are positive and invertible with  $\sum_{i=1}^2 b_i^2 = 1$  and

$$\sum_{i=1}^2 b_i \xi_i b_i = \sum_{i=1}^2 \lambda_i a_i \xi_i = \sum_{i=1}^2 a_i^2 \xi = \xi.$$

But there is no unitary  $v \in \mathcal{B}$  satisfying  $\xi_1 = v^*\xi v$ , so that  $\xi$  is not  $\mathcal{B}$ -extreme in  $K$ . To see this, observe that the existence of such  $v$  would mean the existence of a unitary  $u \in \mathcal{R}$  satisfying  $\xi_1 = u\xi$ , implying that  $|\xi_1(t)| = |\xi(t)|$  for almost all  $t \in [0, 1]$ . But

$|\xi_1(t)| = \lambda_1^{-1} \chi(t) |\xi(t)|$ , so it would follow that  $\chi(t) = \lambda_1$  for almost all  $t$  such that  $\xi(t) \neq 0$ . However, this is impossible since  $\chi$  is not constant on any set of positive measure (because  $\mu$  has no atoms by the hypothesis).

By [25, 3.5]  $K$  can be represented as  $K \cong S =: \text{UCP}_{\mathcal{C}}(X, \mathbb{B}(\mathcal{H}))$ , where  $\mathcal{H}$  is a standard Hilbert  $\mathcal{B}$ -module ( $\mathcal{H} = \mathcal{K} \oplus \mathbb{C}$ ),  $\mathcal{C} = \mathbb{B}_{\mathcal{B}}(\mathcal{H})$  and  $X$  is a strong operator  $\mathcal{C}$ -system. Since maps with the u.e.p. are easily seen to be  $\mathcal{B}$ -extreme points in  $S$  [25], it follows that in this case  $S$  does not contain any map with the u.e.p.

The  $\mathcal{B}$ -bimodule  $X$  and its unit ball  $K$  are not selfadjoint, so perhaps it would be preferable to replace  $K$  in the above arguments with the isomorphic set

$$\tilde{K} = \left\{ \begin{bmatrix} 0 & \xi \\ \xi^* & 0 \end{bmatrix} \mid \xi \in \mathcal{K}, \|\xi\| \leq 1 \right\}, \quad (3.3)$$

which also is a  $\mathcal{B}$ -convex and weak\* compact subset of  $\mathbb{B}(\mathcal{K} \oplus \mathbb{C})$ . [Here  $\xi^*$  is the adjoint of  $\xi \in \mathcal{K}$ , where  $\xi$  is regarded as an operator from  $\mathbb{C}$  to  $\mathcal{K}$ , thus  $\xi^* \in \mathbb{B}(\mathcal{K}, \mathbb{C})$ .] However,  $\tilde{K}$  is not the unit ball of an operator  $\mathcal{B}$ -bimodule, and the author does not know of any example of a selfadjoint dual operator bimodule over a von Neumann algebra  $\mathcal{A}$  the unit ball of which has no  $\mathcal{A}$ -extreme points.

It is interesting that each  $x \in K$  is a  $\mathcal{B}$ -extreme point of  $\overline{\text{co}}_{\mathcal{B}}(x)$ . Rather than to verify this in this particular case, we will prove below a more general result.

**Proposition 3.3** *Let  $\mathcal{A} \subseteq \mathbb{B}(\mathcal{K})$  be a von Neumann algebra and  $\mathcal{A}'$  its commutant. For each  $x \in \mathbb{B}(\mathcal{K})$  we have the equality*

$$\overline{\text{co}}_{\mathcal{A}}(x) = \{\phi(x) : \phi \in \text{UCP}_{\mathcal{A}'}(\mathbb{B}(\mathcal{K}))\}.$$

*Proof* By [7] each completely contractive  $\mathcal{A}'$ -bimodule endomorphism  $\phi$  of  $\mathbb{B}(\mathcal{K})$  can be approximated in the point weak\* topology by maps of the form  $y \mapsto (a \odot b)(y) := a^*yb = \sum_{j=1}^n a_j^*y b_j$ , where  $a = (a_j)_{j=1}^n$  and  $b = (b_j)_{j=1}^n$  are contractive columns (that is,  $a^*a = \sum_{j=1}^n a_j^*a_j \leq 1$  and  $b^*b \leq 1$ ) with components  $a_j, b_j \in \mathcal{A}$ . Since the set of such maps is convex and the same linear functionals on  $\mathbb{B}(\mathcal{K})$  are continuous in the weak\* topology as in the strong\* topology, we may assume that the approximation is point-wise in the strong\* topology. Moreover, if  $\phi$  is unital, then  $a^*b$  approximates  $\phi(1) = 1$  in the strong\* topology, hence it follows from  $(a-b)^*(a-b) = a^*a + b^*b - a^*b - b^*a \leq 2 - a^*b - b^*a \approx 0$  that  $\phi$  can be approximated by maps of the form  $a^* \odot a$  point-wise in the strong operator topology. In particular  $a^*a$  approximates 1, hence  $\phi(x)$  is approximated by  $\mathcal{A}$ -convex combinations of the form  $a^*xa + \sqrt{1-a^*a}x\sqrt{1-a^*a}$ . This implies that  $\phi(x) \in \overline{\text{co}}_{\mathcal{A}}(x)$  and thus proves the inclusion

$$Sx := \{\phi(x) : \phi \in \text{UCP}_{\mathcal{A}'}(\mathbb{B}(\mathcal{K}))\} \subseteq \overline{\text{co}}_{\mathcal{A}}(x).$$

The reverse inclusion follows at once from the following two facts: (i) each  $y \in \text{co}_{\mathcal{A}}(x)$  is of the form  $\phi(x)$  for a map  $\phi \in \text{UCP}_{\mathcal{A}'}(\mathbb{B}(\mathcal{K}))$  of the form  $\phi(y) = \sum_{j=1}^n a_j^*y a_j$ ; (ii) the set  $Sx$  is weak\* closed since the set  $\text{UCP}_{\mathcal{A}'}(\mathbb{B}(\mathcal{K}))$  is weak\* compact [in  $\text{CB}_{\mathcal{A}'}(\mathbb{B}(\mathcal{K}))$ ] and the map  $\phi \mapsto \phi(x)$  is weak\* continuous from  $\text{CB}_{\mathcal{A}'}(\mathbb{B}(\mathcal{K}))$  to  $\mathbb{B}(\mathcal{K})$ .  $\square$

**Proposition 3.4** *Let  $\mathcal{A} \subseteq \mathbb{B}(\mathcal{K})$  be an abelian von Neumann algebra, where  $\mathcal{K}$  is a separable Hilbert space. Then each  $x \in \mathbb{K}(\mathcal{K})$  (compact operators on  $\mathcal{K}$ ) is a Choquet  $\mathcal{A}$ -point (hence also an  $\mathcal{A}$ -extreme point) of  $\overline{\text{co}}_{\mathcal{A}}(x)$ .*

*Proof* Let  $K = \overline{\text{co}}_{\mathcal{A}}(x)$  and suppose that (3.1) holds. Then by Proposition 3.3 each  $x_j$  is of the form  $x_j = \sigma_j(x)$ , where  $\sigma_j \in \text{UCP}_{\mathcal{A}}\mathbb{B}(\mathcal{K})$ . Since  $x \in \mathbb{K}(\mathcal{K})$  and the restriction of every bounded linear map to  $\mathbb{K}(\mathcal{K})$  is weak\* continuous, it follows from the Kraus theorem for completely positive such maps that  $x_j = \sigma_j(x) = \sum_{i \in \mathbb{N}} b_{i,j}^* x b_{i,j}$  for some elements  $b_{i,j} \in \mathcal{A}$  satisfying  $\sum_{i \in \mathbb{N}} b_{i,j}^* b_{i,j} = 1$ . Thus we have

$$x = \sum_{j=1}^n \sum_{i \in \mathbb{N}} a_j^* b_{i,j}^* x b_{i,j} a_j, \quad (3.4)$$

where  $\sum_{j=1}^n \sum_{i \in \mathbb{N}} a_j^* b_{i,j}^* b_{i,j} a_j = 1$ . Since all the  $b_{i,j} a_j \in \mathcal{A}$  and  $\mathcal{A}$  is abelian, it now follows from (3.4) by a known result about fixed points of normal completely positive unital maps [31], [24, 4.6] that all the  $b_{i,j} a_j$  commute with  $x$  (hence so does also  $a_j^* b_{i,j}^*$  by the Putnam–Fuglede theorem). Then each  $a_j^* a_j = \sum_{i \in \mathbb{N}} a_j^* b_{i,j}^* b_{i,j} a_j$  also commutes with  $x$ . Further, since  $x$  commutes with  $b_{i,j} a_j$ , we have

$$\begin{aligned} x_j(a_j a_j^*) &= \sum_{i \in \mathbb{N}} b_{i,j}^* x b_{i,j} a_j a_j^* = \sum_{i \in \mathbb{N}} b_{i,j}^* b_{i,j} a_j x a_j^* = a_j x a_j^* \\ &= a_j \sum_{i \in \mathbb{N}} x a_j^* b_{i,j}^* b_{i,j} = a_j \sum_{i \in \mathbb{N}} a_j^* b_{i,j}^* x b_{i,j} = (a_j a_j^*) x_j. \end{aligned}$$

Finally, since  $|a_j| = \sqrt{a_j^* a_j}$  commutes with  $x$ , we also have

$$|a_j| x |a_j| = x a_j^* a_j = \sum_{i \in \mathbb{N}} x a_j^* b_{i,j}^* b_{i,j} a_j = \sum_{i \in \mathbb{N}} a_j^* b_{i,j}^* x b_{i,j} a_j = a_j^* x a_j.$$

Thus  $x$  is a Choquet  $\mathcal{A}$ -point of  $K$ . □

## 4 Boundary Representations on $C^*$ -Modules Over Abelian von Neumann Algebras

In this section we will show that the main results of [1] and [5] can be extended to the context where Hilbert spaces are replaced by  $C^*$ -modules over an abelian von Neumann algebra  $\mathcal{Z}$  in the sense of Kaplansky [18] and Paschke [30] (also called Hilbert  $C^*$ -modules). For a theory of such modules we refer to [3, 19] or [27]. Informally, such modules are like Hilbert spaces, except that the inner product takes its values in  $\mathcal{Z}$ . Given such a module  $\mathcal{E}$ , we denote by  $\langle \cdot, \cdot \rangle$  the  $\mathcal{Z}$ -valued inner product on  $\mathcal{E}$  and by  $|e| := \sqrt{\langle e, e \rangle}$  the corresponding  $\mathcal{Z}$ -valued norm. A  $C^*$ -module  $\mathcal{E}$  over  $\mathcal{Z}$  is called *self-dual* if each bounded  $\mathcal{Z}$ -module map from  $\mathcal{E}$  to  $\mathcal{Z}$  has the form  $e \mapsto \langle e, f \rangle$  for an  $f \in \mathcal{E}$ . By  $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$  we denote the von Neumann algebra of all bounded  $\mathcal{Z}$ -module endomorphisms of  $\mathcal{E}$  (which are automatically adjointable since  $\mathcal{E}$  is assumed self-dual).

*Remark 4.1* If  $\mathcal{E} \subseteq \mathcal{F}$  are self-dual  $C^*$ -modules over  $\mathcal{Z}$ , then  $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}^\perp$  [27, 2.5.4], and  $|e + g|^2 = |e|^2 + |g|^2$  for all  $e \in \mathcal{E}$  and  $g \in \mathcal{E}^\perp$ .

We will consider u.c.p.  $\mathcal{Z}$ -module maps  $\varphi$  from an operator  $\mathcal{Z}$ -system  $X$  into  $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ , where we assume that  $X$  is *central* in the sense that  $X \subseteq \mathcal{Z}'$ , where  $\mathcal{Z}'$  is the commutant of  $\mathcal{Z}$  in  $\mathbb{B}(\mathcal{K})$  for some normal faithful Hilbert  $\mathcal{Z}$ -module  $\mathcal{K}$ .

**Definition 4.2** A  $\mathcal{Z}$ -dilation of a map  $\varphi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  is a map  $\psi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$  for a self-dual  $C^*$ -module  $\mathcal{F} \supseteq \mathcal{E}$  over  $\mathcal{Z}$  such that  $p\psi(x)|_{\mathcal{E}} = \varphi$ , where  $p: \mathcal{F} \rightarrow \mathcal{E}$  is the orthogonal projection. We write  $\psi \succeq_{\mathcal{Z}} \varphi$  if  $\psi$  is a  $\mathcal{Z}$ -dilation of  $\varphi$  and we will say that  $\varphi$  is  $\mathcal{Z}$ -maximal at  $(x, e) \in X \times \mathcal{E}$  if  $\psi(x)e = \varphi(x)e$  whenever  $\psi \succeq_{\mathcal{Z}} \varphi$ .

**Lemma 4.3** A map  $\varphi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  is  $\mathcal{Z}$ -maximal at  $(x, e)$  if and only if  $|\psi(x)e| = |\varphi(x)e|$  for all  $\psi \succeq_{\mathcal{Z}} \varphi$ .

*Proof* This follows from Remark 4.1. □

**Lemma 4.4** Suppose that  $X$  is a central operator  $\mathcal{Z}$ -system. For each  $(x, e) \in X \times \mathcal{E}$  and  $\varphi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  there exists a  $\mathcal{Z}$ -dilation  $\psi \in T := \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E} \oplus \mathcal{Z}))$  of  $\varphi$  which is  $\mathcal{Z}$ -maximal at  $(x, e)$ .

*Proof* Given a self-dual Hilbert  $\mathcal{Z}$ -module  $\mathcal{F}$  and  $f \in \mathcal{F}$ , let  $[\mathcal{Z}f]$  be the smallest self-dual Hilbert  $\mathcal{Z}$ -submodule containing  $f$ . The map

$$[\mathcal{Z}f] \rightarrow \mathcal{Z}, \quad zf \mapsto z|f|$$

is an isometric  $\mathcal{Z}$ -module map and can be uniquely extended to a  $\mathcal{Z}$ -module map of  $[\mathcal{Z}f]$  onto  $[\mathcal{Z}|f|] \subseteq \mathcal{Z}$  by [30, 3.6], which (by applying the same argument to the inverse) must be an isometric isomorphism of Hilbert  $\mathcal{Z}$ -modules. Thus  $[\mathcal{Z}f]$  can be identified with a self-dual  $\mathcal{Z}$ -submodule in  $\mathcal{Z}$ .

Let  $\rho$  be any  $\mathcal{Z}$ -dilation of  $\varphi$  and  $(x, e) \in X \times \mathcal{E}$ . Then  $\rho(x)e$  has the form  $\rho(x)e = \varphi(x)e \oplus f$  for some  $f$  orthogonal to  $\varphi(x)e$  and it follows from the argument of the previous paragraph (by first considering the compression of  $\rho$  to the self-dual module  $\mathcal{E} \oplus \mathcal{Z}f$ ) that there exists a  $\mathcal{Z}$ -dilation  $\theta \in T$  with  $|\theta(x)e| = |\rho(x)e|$ . Since  $\mathcal{Z}$  is an abelian von Neumann algebra, there exists  $c := \sup\{|\theta(x)e| : \theta \succeq_{\mathcal{Z}} \varphi, \theta \in T\}$ . We have to prove that there exists  $\psi \in T$  with  $|\psi(x)e| = c$ , for such a  $\psi$  is necessarily  $\mathcal{Z}$ -maximal at  $(x, e)$  by Lemma 4.3.

Each  $\theta \in T$  with  $\theta \succeq_{\mathcal{Z}} \varphi$  can be represented by a  $2 \times 2$  matrix of maps  $\theta = [\theta_{i,j}]$ , where  $\theta_{1,1} = \varphi$ ,  $\theta_{2,1}$  is a map from  $X$  into  $\mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{Z})$ ,  $\theta_{1,2} = \theta_{2,1}^*$  and  $\theta_{2,2}$  is a map from  $X$  into  $\mathbb{B}_{\mathcal{Z}}(\mathcal{Z})$ . Let  $\theta_{2,1}(x)e = \sigma|\theta_{2,1}(x)e|$  be the polar decomposition of  $\theta_{2,1}(x)e$  in  $\mathcal{Z}$ , with  $\sigma$  unitary. Replacing  $\theta$  by the map  $(1 \oplus \sigma^*)\theta(1 \oplus \sigma)$ , changes  $\theta_{2,1}(x)e$  to  $|\theta_{2,1}(x)e|$ , hence, in searching for  $\mathcal{Z}$ -dilations of  $\varphi$  that are  $\mathcal{Z}$ -maximal at  $(x, e)$ , we may restrict to maps  $\theta$  with  $\theta_{2,1}(x)e \geq 0$ . Observe further that, if  $\theta$  and  $\tilde{\theta}$  are two such maps, then there exists such a dilation  $\hat{\theta}$  with

$$\hat{\theta}_{2,1}(x)e = \max\{\theta_{2,1}(x)e, \tilde{\theta}_{2,1}(x)e\}.$$



[Indeed, since  $\mathcal{Z}$  is abelian, there is a projection  $p \in \mathcal{Z}$  such that  $p\theta_{2,1}(x)e \geq p\tilde{\theta}_{2,1}(x)e$  and  $p^\perp\theta_{2,1}(x)e \leq p^\perp\tilde{\theta}_{2,1}(x)e$ , and then we may take  $\hat{\theta} = p\theta + p^\perp\tilde{\theta}$ .]

Set  $d = \sqrt{c^2 - |\varphi(x)e|^2}$ . Given  $\varepsilon > 0$ , by the definition of  $c$  for each point  $t$  in the maximal ideal space  $\Delta$  of  $\mathcal{Z}$  there exists a  $\mathcal{Z}$ -dilation  $\theta^t \in T$  of  $\varphi$  such that  $(\theta^t_{2,1}(x)e)(t) > d(t) - \varepsilon$ . Hence by continuity and compactness and extreme disconnectedness of  $\Delta$  there exists finitely many projections  $p_k \in \mathcal{Z}$  ( $k = 1, \dots, n$ ) with the sum 1 and  $\mathcal{Z}$ -dilations  $\theta^k \in T$  of  $\varphi$  such that  $p_k\theta^k_{2,1}(x)e \geq p_kd - \varepsilon$ . Then by the observation from the previous paragraph there exists a dilation  $\theta \in T$  with  $\theta_{2,1}(x)e \geq d - \varepsilon$ . Thus we can choose for each  $n = 1, 2, \dots$  a dilation  $\theta_n \in T$  of  $\varphi$  such that  $(\theta_n)_{2,1}(x)e \geq d - \frac{1}{n}$ . Denoting by  $\psi$  a weak\* limit point of the sequence  $(\theta_n)$  in  $T$ , it follows that  $\psi$  is a  $\mathcal{Z}$ -dilation of  $\varphi$  with  $\psi_{2,1}(x)e = d$ , hence  $|\psi(x)e| = c$ .  $\square$

**Lemma 4.5** *Let  $\mathcal{K}$  be a faithful normal Hilbert  $\mathcal{Z}$ -module,  $K \subseteq \mathbb{B}(\mathcal{K})$  a  $\mathcal{Z}$ -convex subset contained in the commutant  $\mathcal{Z}' =: \mathcal{R}$  of  $\mathcal{Z}$  in  $\mathbb{B}(\mathcal{K})$  and let  $x$  be an (ordinary) extreme point of  $K$ . Then the condition*

$$x = ay + (1 - a)w, \quad y, w \in K, \quad a \in \mathcal{Z}, \quad 0 \leq a \leq 1 \quad (4.1)$$

*implies that  $r_ax = r_ay$ , where  $r_a$  is the range projections of  $a$ . In particular  $x$  is a strong  $\mathcal{Z}$ -extreme point of  $K$ .*

*Proof* Let us first recall the Glimm decomposition of  $\mathcal{R}$  [12]. Let  $\Delta$  be the maximal ideal space of  $\mathcal{Z}$  and for each  $t \in \Delta$  let  $\mathcal{R}t$  be the closed ideal in  $\mathcal{R}$  generated by  $t$  and let  $\mathcal{R}(t) := \mathcal{R}/(\mathcal{R}t)$  be the quotient  $C^*$ -algebra. For each  $x \in \mathcal{R}$  denote by  $x(t)$  the coset of  $x$  in  $\mathcal{R}(t)$ . We will need the fact that the function  $\Delta \ni t \mapsto \|x(t)\|$  is continuous [12]. Suppose that (4.1) holds. Let  $U = \{t \in \Delta : a(t) < \frac{1}{2}\}$  and let  $p \in \mathcal{Z}$  be the projection that corresponds to the clopen set  $\overline{U}$ . The elements

$$u := (2pa + p^\perp)y + p(1 - 2a)w \quad \text{and} \quad v := p^\perp(2a - 1)y + (p + 2p^\perp(1 - a))w$$

are in  $K$  since both are  $\mathcal{Z}$ -convex combinations of  $y$  and  $w$ , and  $\frac{1}{2}(u + v) = ay + (1 - a)w = x$ . Hence  $u = x = v$  since  $x$  is an extreme point of  $K$ . The equality  $u = v$  can be written as

$$[a(p - p^\perp) + p^\perp]y = [a(p - p^\perp) + p^\perp]w,$$

from which we deduce (multiplying by  $p$  and by  $p^\perp$ ) the two identities  $apy = apw$  and  $(1 - a)p^\perp y = (1 - a)p^\perp w$ . The first identity means that  $y(t) = w(t)$  for all  $t \in \overline{U}$  such that  $a(t) \neq 0$ , while the second identity means that  $y(t) = w(t)$  for all  $t \in \overline{U}^c$  such that  $a(t) \neq 1$ . It follows that the identity  $y(t) = w(t)$  holds for all  $t \in \Delta$ , except maybe for those  $t$  for which  $a(t) = 0$  or  $a(t) = 1$ . By functional calculus this is equivalent to  $qy = qw$ , where  $q$  is the spectral projection of  $a$  corresponding to the interval  $(0, 1)$ . Using (4.1) we now see that  $qx = qy$ . Let  $n_{1-a}$  be the null projection of  $1 - a$ ; thus  $n_{1-a}(1 - a) = 0$ ,  $n_{1-a}a = n_{1-a}$  and [using (4.1)]  $n_{1-a}x = n_{1-a}y$ . Since  $r_a = q + n_{1-a}$ , adding the identities  $qx = qy$  and  $n_{1-a}x = n_{1-a}y$  we obtain  $r_ax = r_ay$ . In particular, if  $a$  is invertible, then  $r_a = 1$ , hence in this case (4.1) implies that  $x = y = w$ , therefore  $x$  is a strong  $\mathcal{Z}$ -extreme point of  $K$ .  $\square$

*Remark 4.6* If  $e, f$  are elements of a  $C^*$ -module  $\mathcal{E}$  over  $\mathcal{Z}$  and if  $|f|$  is invertible in  $\mathcal{Z}$ , then in the Cauchy Schwarz inequality

$$|\langle e, f \rangle| \leq |e||f|$$

the equality holds if and only if  $e = cf$  for some  $c \in \mathcal{Z}$ . The proof is the same as in the case of the usual Cauchy-Schwarz inequality. (Namely, using that  $\mathcal{Z}$  is abelian, from the inequality  $0 \leq \langle e - cf, e - cf \rangle = |e|^2 - 2\operatorname{Re}(c\langle f, e \rangle) + |c|^2|f|^2$  ( $c \in \mathcal{Z}$ ) we obtain by putting  $c = |f|^{-2}\langle e, f \rangle$  and using the definiteness of  $\langle \cdot, \cdot \rangle$  that  $e = cf$ .)

**Definition 4.7** A map  $\varphi \in \operatorname{CCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  is called  $\mathcal{Z}$ -pure if all maps  $\psi \in \operatorname{CCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  satisfying  $\psi \leq \varphi$  are of the form  $\psi = c\varphi$ , where  $c \in \mathcal{Z}$ .

Given a  $\mathcal{Z}$ -pure map  $\varphi \in \operatorname{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  and a pair  $(x, e) \in X \times \mathcal{E}$ , let  $c = \sup\{|\rho(x)e| : \rho \succeq_{\mathcal{Z}} \varphi\}$ ,  $d = \sqrt{c^2 - |\varphi(x)e|^2}$  and

$$S_{\varphi} = \{\psi \in \operatorname{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E} \oplus \mathcal{Z})) : \psi \succeq_{\mathcal{Z}} \varphi \text{ and } \psi(x)e = \psi(x)e \oplus d\}. \quad (4.2)$$

Let  $\psi_0$  be an extreme point of  $S_{\varphi}$ . Note that by Lemma 4.5  $\psi_0$  is a  $\mathcal{Z}$ -extreme point of  $S_{\varphi}$ . [Indeed, as any normal operator bimodule, the  $\mathcal{Z}$ -bimodule  $\operatorname{CB}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E} \oplus \mathcal{Z}))$  can be represented as a subbimodule  $\mathcal{M}$  in  $\mathbb{B}(\mathcal{K})$  for some faithful normal Hilbert  $\mathcal{Z}$ -module  $\mathcal{K}$ . Since  $X$  is central,  $\mathcal{M}$  is contained in the commutant of  $\mathcal{Z}$  in  $\mathbb{B}(\mathcal{K})$ , so Lemma 4.5 can be applied.] Using Remark 4.6 and Lemma 4.4, the same arguments as in the proof of [5, 2.3] prove the following generalization.

**Lemma 4.8** *Let  $\mathcal{Z}$  be an abelian von Neumann algebra,  $X$  a central operator  $\mathcal{Z}$ -system,  $\mathcal{E}$  a self-dual  $C^*$ -module over  $\mathcal{Z}$ ,  $(x, e) \in X \times \mathcal{E}$  and  $\varphi \in \operatorname{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  a  $\mathcal{Z}$ -pure map which is not  $\mathcal{Z}$ -maximal at  $(x, e)$ . Then there exists a  $\mathcal{Z}$ -pure dilation  $\psi \in \operatorname{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E} \oplus \mathcal{Z}))$  which is  $\mathcal{Z}$ -maximal at  $(x, e)$ ; in fact we may take for  $\psi$  any extreme point of the set  $S_{\varphi}$  defined by (4.2).*

We will omit the proof of Lemma 4.8 since it is the same as the proof of [5, 2.3]. We only note that instead of the Cholesky algorithm mentioned in [5] we use in our context the following fact, which is well-known (at least in a context which does not involve  $C^*$ -modules), but for convenience we will include a short sketch of the proof.

**Lemma 4.9** *Let  $\mathcal{E}$  be a self-dual Hilbert  $\mathcal{Z}$ -module (where  $\mathcal{Z}$  is not necessarily abelian). A  $2 \times 2$  operator matrix*

$$g = \begin{bmatrix} a & e \\ e^* & c \end{bmatrix},$$

with the entries  $a \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E})_+$ ,  $c \in \mathcal{Z}_+$  and  $e \in \mathcal{E}$ , is positive (if and) only if there exists  $f \in \mathcal{E}$  with  $\|f\| \leq 1$  such that  $e = \sqrt{a}f\sqrt{c}$ ; in which case  $g$  admits a factorization

$$g = \begin{bmatrix} \sqrt{a} & 0 \\ x^* & b \end{bmatrix} \begin{bmatrix} \sqrt{a} & x \\ 0 & b \end{bmatrix},$$

where  $x = f\sqrt{c}$  and  $b = \sqrt{c - x^*x}$ .

*Proof* Let  $a_n = a + \frac{1}{n}$ ,  $c_n = c + \frac{1}{n}$  and let  $d_n$  be the diagonal matrix  $d_n = a_n^{-1/2} \oplus c_n^{-1/2}$  (where  $n = 1, 2, \dots$ ). If  $g$  is positive, then so is also

$$h_n := d_n g d_n = \begin{bmatrix} a_n^{-1} a & a_n^{-1/2} e c_n^{-1/2} \\ (a_n^{-1/2} e c_n^{-1/2})^* & c_n^{-1} c \end{bmatrix}.$$

As  $n \rightarrow \infty$ , the sequences  $(a_n^{-1} a)$  and  $(c_n^{-1} c)$  converge to the range projections  $p$  and  $q$  of  $a$  and  $c$  (respectively). Since  $a_n^{-1} a \leq 1$ ,  $c_n^{-1} c \leq 1$  and  $h_n \geq 0$ , the matrix

$$\begin{bmatrix} 1 & a_n^{-1/2} e c_n^{-1/2} \\ (a_n^{-1/2} e c_n^{-1/2})^* & 1 \end{bmatrix}$$

is positive, hence  $\|a_n^{-1/2} e c_n^{-1/2}\| \leq 1$  by [29, 3.1]. Thus the sequence  $(a_n^{-1/2} e c_n^{-1/2})$  has a weak\* limit point  $f$  in  $\mathcal{E}$  ( $\mathcal{E}$  is a dual Banach space with an appropriate weak\* topology). It follows that the matrix

$$h = \begin{bmatrix} p & f \\ f^* & q \end{bmatrix}$$

is a weak\* limit point (in the linking von Neumann algebra of  $\mathcal{E}$ ) of the sequence of positive matrices  $h_n$ , hence  $h \geq 0$ . The positivity of  $h$  implies that  $p^\perp f = 0 = f q^\perp$ , hence  $f = p f q$ , and that  $\|f\| \leq 1$ . Since  $f$  is a weak\* limit point of  $(a_n^{-1/2} e c_n^{-1/2})$  (and the  $a_n$  and  $b_n$  converge to  $a$  and  $b$ , respectively, in norm), it follows that  $e = a^{1/2} f c^{1/2}$ .  $\square$

**Definition 4.10** A map  $\varphi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  is called  $\mathcal{Z}$ -maximal if it is  $\mathcal{Z}$ -maximal at each  $(x, e) \in X \times \mathcal{E}$ . This means that every  $\psi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ , where  $\mathcal{F}$  is a self-dual  $C^*$ -module over  $\mathcal{Z}$ , satisfying  $\psi \succeq_{\mathcal{Z}} \varphi$ , decomposes as  $\psi = \varphi \oplus \theta$  for some map  $\theta \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}^\perp))$ .

Every self-dual  $C^*$ -module is a direct summand in the self-dual completion of a direct sum of sufficiently many copies of  $\mathcal{Z}$ . Using Zorn's lemma (or transfinite induction as in [5, 2.4]) one can deduce from Lemma 4.8 the following theorem.

**Theorem 4.11** Each  $\mathcal{Z}$ -pure map  $\varphi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  (where  $X$  is a central operator  $\mathcal{Z}$ -system and  $\mathcal{E}$  is a self-dual  $C^*$ -module over  $\mathcal{Z}$ , as before) has a  $\mathcal{Z}$ -pure  $\mathcal{Z}$ -maximal dilation  $\psi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ , where  $\mathcal{F}$  is a self-dual Hilbert  $C^*$ -module over  $\mathcal{Z}$ .

*Remark 4.12* Since an abelian von Neumann algebra  $\mathcal{Z}$  is injective and each self-dual  $C^*$ -module  $\mathcal{E}$  over  $\mathcal{Z}$  can be embedded into  $\mathcal{Z}^{\mathbb{I}}$  as an orthogonally complemented submodule for a cardinal  $\mathbb{I}$ , the von Neumann algebra  $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$  is also injective [as a corner in  $\mathbb{M}_{\mathbb{I}}(\mathcal{Z}) = \mathbb{M}_{\mathbb{I}}(\mathbb{C}) \overline{\otimes} \mathcal{Z}$ ]. Therefore, if an operator  $\mathcal{Z}$ -system  $X$  is contained in a  $C^*$ -algebra  $B$  generated by  $X$  and containing  $\mathcal{Z}$  in its center, any map  $\varphi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  can be extended to a map  $\tilde{\varphi} \in \text{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ . By an appropriate version of the of the Stinespring's dilation theorem  $\tilde{\varphi}$  can be represented as  $\tilde{\varphi}(b) =$

$V^*\pi(b)V$  ( $b \in B$ ), where  $\pi: B \rightarrow \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$  is a representation on a self-dual  $C^*$ -module  $\mathcal{F}$  over  $\mathcal{Z}$  and  $V \in \mathbb{B}_{\mathcal{Z}}(\mathcal{E}, \mathcal{F})$  is an isometry such that  $[\pi(B)V\mathcal{E}] = \mathcal{F}$  (see [19, 5.6] and [30, 3.7]). From this it follows (as in the classical case of Hilbert spaces [1], [3, 4.1.12]) that a  $\mathcal{Z}$  maximal map  $\varphi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  necessarily has the  $\mathcal{Z}$ -unique extension property in the sense of the following definition.

**Definition 4.13** A map  $\varphi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  has the  $\mathcal{Z}$ -unique extension property ( $\mathcal{Z}$ -u.e.p.) if the only c.p. extension of  $\varphi$  to a map in  $\text{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ , where  $B = C^*(X)$ , is a (necessarily unique) representation  $\pi: B \rightarrow \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$ .

*Remark 4.14* If  $\varphi$  in Remark 4.12 is  $\mathcal{Z}$ -pure, then the commutant  $\pi(B)'$  of  $\pi(B)$  in  $\mathbb{B}_{\mathcal{Z}}(\mathcal{E})$  must coincide with  $\pi(\mathcal{Z})$  since for every  $s \in \pi(B)'$  with  $0 \leq s \leq 1$  the c.p. map  $\psi(b) := \pi(b)s$  satisfies  $\psi|_X \leq \varphi$ .

**Definition 4.15** A representation (that is, a homomorphism of  $C^*$ -algebras)  $\pi \in \text{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  is  $\mathcal{Z}$ -irreducible if  $\pi(B)' = \pi(\mathcal{Z})$ . A  $\mathcal{Z}$ -boundary representation of  $B$  for  $X$  is a  $\mathcal{Z}$ -irreducible representation  $\pi: B \rightarrow \mathbb{B}_{\mathcal{Z}}(\mathcal{E})$  (for some self-dual  $\mathcal{E}$ ) such that  $\pi|_X$  has the  $\mathcal{Z}$ -u.e.p.

*Remark 4.16* If  $B$  is a  $C^*$ -algebra containing  $\mathcal{Z}$  in its center, each map  $\rho \in \text{UCP}_{\mathcal{Z}}(B, \mathcal{Z})$  induces a representation  $\pi_{\rho}$  of  $B$  onto a self-dual  $C^*$ -module  $\mathcal{F}$  over  $\mathcal{Z}$  [that is, a  $*$ -homomorphism  $\pi_{\rho} \in \text{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ ] by a GNS-like construction, so that there is an element  $\xi \in \mathcal{F}$  such that  $\rho(b) = \langle \pi_{\rho}(b)\xi, \xi \rangle$  for all  $b \in B$  and  $\mathcal{F} = [\pi_{\rho}(B)\xi]$  [that is,  $\mathcal{F}$  is generated by  $\pi_{\rho}(B)\xi$  as a self-dual  $C^*$ -module over  $\mathcal{Z}$ ]. Moreover, essentially by the same arguments as in the well-known case  $\mathcal{Z} = \mathbb{C}$  it can be shown that  $\rho$  is  $\mathcal{Z}$ -pure if and only if  $\pi_{\rho}$  is  $\mathcal{Z}$ -irreducible. (If  $B$  is a von Neumann algebra this is proved in [13].)

*Remark 4.17* If  $B$  as in Remark 4.16 is contained in a  $C^*$ -algebra  $A$  in such a way that  $\mathcal{Z}$  is contained in the center of  $A$ , it follows from Remark 4.16 that for every  $\mathcal{Z}$ -irreducible representation  $\pi$  of  $B$  on a self dual  $C^*$ -module  $\mathcal{F}$  over  $\mathcal{Z}$  there exist a self-dual  $C^*$ -module  $\mathcal{E}$  over  $\mathcal{Z}$  and a  $\mathcal{Z}$ -irreducible representation  $\sigma \in \text{UCP}_{\mathcal{Z}}(A, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  such that  $\mathcal{F} \subseteq \mathcal{E}$  and  $\pi(b) = \sigma(b)|_{\mathcal{F}}$  for all  $b \in B$ , hence  $\pi$  is a subrepresentation of  $\sigma|_B$ . (Here the injectivity of  $\mathcal{Z}$  is used to extend  $\mathcal{Z}$ -states on  $B$  to  $\mathcal{Z}$ -states on  $A$ .)

Using the above terminology we can reformulate Theorem 4.11 as follows.

**Theorem 4.18** Each  $\mathcal{Z}$ -pure map  $\varphi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  (where  $X$  is a central operator  $\mathcal{Z}$ -system generating a  $C^*$ -algebra  $B$ ) has a  $\mathcal{Z}$ -pure  $\mathcal{Z}$ -maximal dilation  $\psi \in \text{UCP}_{\mathcal{Z}}(X, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$  which is the restriction of a  $\mathcal{Z}$ -boundary representation  $\pi \in \text{CB}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ .

**Lemma 4.19** If  $X$  is a central operator  $\mathcal{Z}$ -system, then  $\mathcal{Z}$ -pure maps into  $\mathcal{Z}$  norm the self-adjoint part  $X_h$  of  $X$ . That is, for each  $x \in X_h$  there exists a  $\mathcal{Z}$ -pure map  $\rho \in T = \text{UCP}_{\mathcal{Z}}(X, \mathcal{Z})$  such that  $\|\rho(x)\| = \|x\|$ .

*Proof* Since  $X$  is central,  $X \subseteq \mathcal{Z}' =: \mathcal{R}$ , the commutant of  $\mathcal{Z}$  on some normal faithful Hilbert  $\mathcal{Z}$ -module. Considering the Glimm decomposition of  $\mathcal{R}$  along the

maximal ideal space  $\Delta$  of  $\mathcal{Z}$  (as in the proof of Lemma 4.5), let  $c \in \mathcal{Z}$  be defined by  $c(t) = \|x(t)\|$  ( $t \in \Delta$ ). Then  $\|c\| = \|x\|$  by [12] and there exists  $\theta \in \text{UCP}_{\mathcal{Z}}(\mathcal{R}, \mathcal{Z})$  such that  $\theta(x) = c$  by [22, 3.4]. Thus the set  $T_c := \{\psi \in \text{UCP}_{\mathcal{Z}}(X, \mathcal{Z}) : \psi(x) = c\}$  is nonempty since it contains  $\theta|_X$ . Since  $T_c$  is weak\* compact and  $\mathcal{Z}$ -convex, it contains an extreme point  $\rho$ . Now it suffices to show that such a  $\rho$  is a  $\mathcal{Z}$ -pure map. Suppose that  $\psi \in \text{CCP}_{\mathcal{Z}}(X, \mathcal{Z})$  and  $\psi \leq \rho$ . Denote  $\theta = \rho - \psi$  and  $a = \psi(1)$ . Then we may write  $\psi = a\psi_1$  and  $\theta = (1-a)\theta_1$ , where  $\psi_1$  and  $\theta_1$  are unital (that is,  $\psi_1, \theta_1 \in T$  [25, 6.5]), hence  $\rho = a\psi_1 + (1-a)\theta_1$ . It follows now by Lemma 4.5 that  $r_a\rho = r_a\psi_1$ , hence  $\psi = a\psi_1 = ar_a\psi_1 = ar_a\rho = a\rho$ .  $\square$

Now from Theorem 4.18 and Lemma 4.19 we can deduce the following result.

**Theorem 4.20** *If  $X$  is a central operator  $\mathcal{Z}$ -system generating a  $C^*$ -algebra  $B$ , then  $\mathcal{Z}$ -boundary representations of  $B$  for  $X$  on self-dual  $C^*$ -modules over  $\mathcal{Z}$  completely norm  $X$ .*

*Proof* It suffices to show that for each  $n$  and each self-adjoint  $x \in \mathbb{M}_n(X)$  there exists a  $\mathcal{Z}$ -boundary representation  $\pi$  of  $B = C^*(X)$  such that  $\|\pi_n(x)\| = \|x\|$ . By Lemma 4.19 [applied to  $\mathbb{M}_n(X)$  instead of  $X$ ] there exists a  $\mathcal{Z}$ -pure map  $\rho \in \text{UCP}_{\mathcal{Z}}(\mathbb{M}_n(X), \mathcal{Z})$  such that  $\|\rho(x)\| = \|x\|$ . By Theorem 4.18  $\rho$  can be dilated to a  $\mathcal{Z}$ -pure map  $\tilde{\rho} \in \text{UCP}_{\mathcal{Z}}(\mathbb{M}_n(X), \mathbb{B}_{\mathcal{Z}}(\mathcal{F}))$ , for some self-dual  $C^*$ -module  $\mathcal{F} \supseteq \mathcal{Z}$  over  $\mathcal{Z}$ , such that  $\tilde{\rho}$  is the restriction to  $\mathbb{M}_n(X)$  of a  $\mathcal{Z}$ -boundary representation  $\sigma: C^*(\mathbb{M}_n(X)) = \mathbb{M}_n(B) \rightarrow \mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ . Standard argument (using matrix units) shows that  $\mathcal{F}$  is necessarily of the form  $\mathcal{F} = \mathcal{E}^n$  for a self-dual  $C^*$ -module  $\mathcal{E}$  over  $\mathcal{Z}$  [so that  $\mathbb{B}_{\mathcal{Z}}(\mathcal{F}) = \mathbb{M}_n(\mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ ], and that  $\sigma$  is of the form  $\pi_n$  for a  $\mathcal{Z}$ -irreducible representation  $\pi \in \text{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ . Since  $\pi_n$  is the only extension in  $\text{UCP}_{\mathcal{Z}}(\mathbb{M}_n(B), \mathbb{M}_n(\mathbb{B}_{\mathcal{Z}}(\mathcal{E})))$  of  $\tilde{\rho} = \pi_n|_{\mathbb{M}_n(X)}$ ,  $\pi$  must be the unique extension of  $\pi|_X$  to a map in  $\text{UCP}_{\mathcal{Z}}(B, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$ . Thus  $\pi$  is a  $\mathcal{Z}$ -boundary representation of  $B$  (relative to  $X$ ). Finally, from  $\|\pi_n(x)\| \geq \|\rho(x)\| = \|x\|$  and since  $\pi_n$  is contractive we conclude that  $\|\pi_n(x)\| = \|x\|$ .  $\square$

## 5 $\mathcal{A}$ -Extreme Points of Weak\* Compact $\mathcal{A}$ -Convex Subsets of $\ell_{\mathbb{N}}^{\infty}(\mathcal{A})$

In Sect. 3 we have seen that general  $\mathcal{A}$ -convex weak\* compact sets need not have enough  $\mathcal{A}$ -extreme points; in this section we will show that for subsets of  $\ell_{\mathbb{N}}^{\infty}(\mathcal{A})$  the situation is better.

**Lemma 5.1** *Suppose that  $\mathcal{C} \subseteq \mathbb{B}(\mathcal{K})$  is a von Neumann algebra and  $X \subseteq \mathbb{B}(\mathcal{K})$  is an operator system such that  $cx = xc$  for all  $x \in X$  and  $c \in \mathcal{C}$ . Let  $\mathcal{X}_0 = \overline{CX}$  (the operator  $\mathcal{C}$ -system generated by  $X$ ),  $B$  the  $C^*$ -algebra generated by  $\mathcal{X}_0$  and  $\mathcal{H}$  a normal Hilbert  $\mathcal{C}$ -module through a normal representation  $\sigma: \mathcal{C} \rightarrow \mathbb{B}(\mathcal{H})$ . Given  $\varphi \in \text{UCP}_{\mathcal{C}}(\mathcal{X}_0, \mathbb{B}(\mathcal{H}))$ , consider the Stinespring representation of  $\varphi$ , that is, a representation  $\pi: B \rightarrow \mathbb{B}(\mathcal{L})$  on a Hilbert space  $\mathcal{L} \supseteq \mathcal{H}$  such that  $\varphi(x) = q\pi(x)|_{\mathcal{H}}$  ( $x \in \mathcal{X}_0$ ), where  $q$  is the projection of  $\mathcal{L}$  onto  $\mathcal{H}$ , and such that  $\mathcal{L} = [\pi(B)\mathcal{H}]$ . Then  $\pi|_{\mathcal{C}}$  is weak\* continuous (hence  $\mathcal{L}$  is a normal Hilbert  $\mathcal{C}$ -module via  $\pi|_{\mathcal{C}}$ ).*

*Proof* Let  $\bar{\pi}: B^{**} \rightarrow \mathbb{B}(\mathcal{L})$  be the weak\* continuous extension of  $\pi$  and  $p$  the projection in the center  $\mathcal{Z}_{C^{**}}$  of  $C^{**}$  such that the kernel of the canonical epimorphism  $C^{**} \rightarrow C$  is equal to  $p^\perp C^{**}$  (see [17, Section 10.1]). Then the map  $C \ni c \mapsto \bar{\pi}(cp) = \pi(c)\bar{\pi}(p)$  is a *normal* representation of  $C$  on the Hilbert space  $\mathcal{L}_0 := \bar{\pi}(p)\mathcal{L}$  (this follows from [17, 10.13]), (hence  $\mathcal{L}_0$  is a normal Hilbert  $C$ -module). Moreover, since  $C$  and  $X$  commute (element-wise), so do also their weak\* closures  $C^{**}$  and  $X^{**}$  inside  $B^{**}$ . (Here we have identified the weak\* closure  $\bar{X}$  of  $X$  in  $B^{**}$  with  $X^{**}$  in the familiar way, namely through the map  $\iota^{**}$ , where  $\iota: X \rightarrow B$  is the inclusion. Further  $B$  has been identified with its image in  $B^{**}$  via the canonical map. Similarly for  $C$ .) This implies that  $p$  ( $\in \mathcal{Z}_{C^{**}}$ ) commutes in particular with elements of the copy of  $B$  inside  $B^{**}$ . Consequently  $\bar{\pi}(p)$  commutes with  $\pi(B)$  and therefore  $\mathcal{L}_0$  is invariant under  $\pi(B)$ . It suffices now to show that  $\mathcal{H} \subseteq \mathcal{L}_0$ , for then  $\mathcal{L} = [\pi(B)\mathcal{H}] \subseteq \mathcal{L}_0$ , hence  $\mathcal{L}_0 = \mathcal{L}$ .

Let  $\bar{\varphi}: \mathcal{X}_0^{**} \rightarrow \mathbb{B}(\mathcal{H})$  be the weak\* continuous extension of  $\varphi$ . By weak\* continuity the equality  $\bar{\varphi}(x) = q\bar{\pi}(x)|\mathcal{H}$  must hold for all  $x \in \mathcal{X}_0^{**}$  since it holds for  $x \in \mathcal{X}_0$ . From weak\* continuity of  $\varphi|C = \sigma$  (and the definition of  $p$ ) we have that  $\bar{\varphi}(p^\perp) = 0$  (see[17, 10.13]). By the Schwarz inequality for c.c.p. maps we see that

$$\bar{\varphi}(xp^\perp)^* \bar{\varphi}(xp^\perp) \leq \bar{\varphi}(p^\perp x^* x p^\perp) \leq \|x\|^2 \bar{\varphi}(p^\perp) = 0,$$

hence, since  $\bar{\varphi}$  is a  $C^{**}$ -module map,  $\varphi(x)\bar{\pi}(p)^\perp = \bar{\varphi}(xp^\perp) = 0$  for all  $x \in \mathcal{X}_0$  and so  $\varphi(x) = \bar{\varphi}(xp)$ . Thus we have  $q\bar{\pi}(x)|\mathcal{H} = \varphi(x) = \bar{\varphi}(xp) = q\bar{\pi}(xp)|\mathcal{H}$  ( $x \in \mathcal{X}_0$ ); putting  $x = 1$  we get  $1_{\mathcal{H}} = q|\mathcal{H} = q\bar{\pi}(p)|\mathcal{H}$ , which implies [since  $q$  and  $\bar{\pi}(p)$  are projections] that  $\bar{\pi}(p)|\mathcal{H} = 1_{\mathcal{H}}$  and consequently  $\mathcal{H} \subseteq \bar{\pi}(p)\mathcal{L} = \mathcal{L}_0$ .  $\square$

The following result is proved in [26, 4.10], but we we will need the proof in Lemma 5.7 below, so we present it here.

**Lemma 5.2** *For a  $C^*$ -algebra  $C$ , an operator  $C$ -system  $\mathcal{X}$  and Hilbert  $C$ -modules  $\mathcal{H}_i$  ( $i = 1, 2$ ), let  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ , let  $p_i: \mathcal{H} \rightarrow \mathcal{H}_i$  be the projections,  $\mathcal{A}_i = \mathbb{B}_C(\mathcal{H}_i)$  and  $\mathcal{A} = \mathbb{B}_C(\mathcal{H})$ . Observe that  $p_i \in \mathcal{A}$  and  $\mathcal{A}_i \cong p_i \mathcal{A} p_i$  ( $i = 1, 2$ ). Let  $\varphi_1 \in S_1 := \text{UCP}_C(\mathcal{X}, \mathbb{B}(\mathcal{H}_1))$  be such that  $\varphi_1 \oplus 0$  is a Choquet  $\mathcal{A}$ -point of  $Q := \text{CCP}_C(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ , let  $\varphi_2$  be an  $\mathcal{A}_2$ -extreme point of  $S_2 := \text{UCP}_C(\mathcal{X}, \mathbb{B}(\mathcal{H}_2))$  and denote  $\varphi = \varphi_1 \oplus \varphi_2$ . Then  $\varphi$  is an  $\mathcal{A}$ -extreme point of  $S := \text{UCP}_C(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ .*

*Proof* Suppose that

$$\varphi_1 \oplus \varphi_2 = \sum_{j=1}^n a_j^* \psi_j a_j, \quad (5.1)$$

where  $\psi_j \in S$  and  $a_j \in \mathcal{A}$  are invertible with  $\sum_{j=1}^n a_j^* a_j = 1$ . We have to prove that there exist unitary elements  $z_j \in \mathcal{A}$  such that  $\psi_j = z_j(\varphi_1 \oplus \varphi_2)z_j^*$ . Using the polar decompositions of  $a_j$ , we may assume that all  $a_j$  are positive. The projections  $p_i: \mathcal{H} \rightarrow \mathcal{H}_i$  are in  $\mathcal{A}$  since  $\mathcal{H}_i$  is invariant under  $C$ . Let  $a_j p_1 = u_j |a_j p_1|$  be the polar decomposition. Since  $a_j$  is positive and invertible the range projection of  $|a_j p_1|$  is  $p_1$ . From (5.1) we have  $\varphi_1 \oplus 0 = \sum_{j=1}^n p_1 a_j \psi_j a_j p_1 + p_2 0 p_2$ , which implies (since  $\varphi_1 \oplus 0$  is a Choquet  $\mathcal{A}$ -point of  $Q$ ) that  $|a_j p_1|$  commutes with  $\varphi_1 \oplus 0$ ,  $|p_1 a_j|^2 = a_j p_1 a_j$

commutes with  $\psi_j$ , hence so does also the range projection  $q_j$  of  $a_j p_1$ , and (noting that the range projection of  $|a_j p_1|$  is  $p_1$ ) that

$$\varphi_1 = u_j^* \psi_j u_j. \quad (5.2)$$

Since  $q_j \psi_j = \psi_j q_j$ , relative to the decomposition  $\mathcal{H} = q_j \mathcal{H} \oplus (1 - q_j) \mathcal{H}$  we can represent  $\psi_j$  as  $\psi_j = \sigma_j \oplus \theta_j$  for suitable u.c.p.  $C$ -bimodule maps  $\sigma_j$  and  $\theta_j$ . Since  $q_j$  is the range projection of  $a_j p_1$ ,  $q_j \mathcal{H} = a_j \mathcal{H}_1$  and  $(1 - q_j) \mathcal{H} = (a_j \mathcal{H}_1)^\perp$ .

With respect to the decompositions  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \rightarrow a_j \mathcal{H}_1 \oplus (a_j \mathcal{H}_1)^\perp = \mathcal{H}$  we can represent  $a_j$  by an operator matrix

$$a_j = \begin{bmatrix} b_j & c_j \\ 0 & d_j \end{bmatrix}. \quad (5.3)$$

Since  $a_j$  is invertible, so must be  $d_j$ . Indeed, if  $d_j$  were not surjective, then  $a_j \mathcal{H}$  could not contain  $(a_j \mathcal{H}_1)^\perp$ . To show the injectivity, suppose that  $d_j \eta = 0$  ( $\eta \in \mathcal{H}_2$ ). Then [by the form of the matrix  $a_j$  and the decomposition  $\mathcal{H} = a_j \mathcal{H}_1 \oplus (a_j \mathcal{H}_1)^\perp$ ]  $a_j \eta = c_j \eta \in a_j \mathcal{H}_1$ , so  $a_j \eta = a_j \xi$  for a  $\xi \in \mathcal{H}_1$ . Now injectivity of  $a_j$  implies that  $\eta = \xi$  and, since  $\mathcal{H}_1 \cap \mathcal{H}_2 = 0$ , it follows that  $\eta = 0$ .

Further, from (5.1)  $0 \oplus \varphi_2 = \sum_{j=1}^n p_2 a_j^* \psi_j a_j p_2$ , that is

$$\varphi_2 = \sum_{j=1}^n (c_j^* \sigma_j c_j + d_j^* \theta_j d_j) = \sum_{j=1}^n (|c_j| (w_j^* \sigma_j w_j) |c_j| + |d_j| (s_j^* \theta_j s_j) |d_j|), \quad (5.4)$$

where  $c_j = w_j |c_j|$  and  $d_j = s_j |d_j|$  are the polar decompositions. Note that  $s_j$  is unitary since  $d_j$  is invertible. Observe that  $|c_j|, |d_j| \in p_2 \mathcal{A} p_2 \cong \mathcal{A}_2$  and  $\sum_{j=1}^n (|c_j|^2 + |d_j|^2) = \sum_{j=1}^n (a_j p_2)^* (a_j p_2) = p_2$ , so the right side of (5.4) is an  $\mathcal{A}_2$ -convex combination of elements  $w_j^* \sigma_j w_j, s_j^* \theta_j s_j \in \text{CCP}_C(\mathcal{X}, \mathbb{B}(\mathcal{H}_2))$ . Using the column  $h := (\frac{1}{\sqrt{2}} |d_1|, |d_2|, \dots, |d_n|, |c_1|, \dots, |c_n|)^T$  and the diagonal matrix  $\Lambda$  with the entries  $s_j^* \theta_j s_j$  and  $w_j^* \sigma_j w_j$  along the diagonal, (5.4) can be rewritten as

$$\varphi_2 = \frac{1}{2} |d_1| (s_1^* \theta_1 s_1) |d_1| + h^* \Lambda h$$

or equivalently as

$$\varphi_2 = \frac{1}{2} |d_1| (s_1^* \theta_1 s_1) |d_1| + |h| \tau |h|, \quad (5.5)$$

where  $h = t|h|$  is the polar decomposition of  $h$  and  $\tau := t^* \Lambda t$ . Since  $d_1$  is invertible, so is  $|h|$  ( $= \sqrt{h^* h} \geq \frac{1}{\sqrt{2}} |d_1|$ ). Since  $\varphi_2$  is unital, while the maps  $s_1^* \theta_1 s_1$  and  $\tau$  are positive contractions and  $\frac{1}{2} |d_1|^2 + |h|^2 = 1$ , (5.5) easily implies that  $s_1^* \theta_1 s_1$  and  $\tau$  are unital. Since by assumption  $\varphi_2$  is an  $\mathcal{A}_2$ -extreme point of  $S_2$ , it follows now from (5.5) that  $\varphi_2 = v_1^* s_1^* \theta_1 s_1 v_1$  for a unitary  $v_1 \in \mathcal{A}_2$ . Similarly there are unitary  $v_j \in \mathcal{A}_2$  such that

$$\varphi_2 = v_j^* s_j^* \theta_j s_j v_j \quad (j = 1, 2, \dots, n). \quad (5.6)$$

Utilizing now (5.2) (which can be written also as  $\varphi_1 = u_j^* \sigma_j u_j$ ) we see that  $\varphi_1 \oplus \varphi_2 = z_j^*(\sigma_j \oplus \theta_j)z_j = z_j^* \psi_1 z_j$ , where  $z_j := u_j \oplus s_j v_j \in \mathcal{A}$  is a unitary.  $\square$

**Theorem 5.3** *Let  $\mathcal{A}$  be a von Neumann algebra with a separable predual and without central summands of type  $II_1$ . Then each weak\* compact  $\mathcal{A}$ -convex subset  $K$  of  $\ell_{\mathbb{N}}^{\infty}(\mathcal{A})$  is generated by its  $\mathcal{A}$ -extreme points.*

*Beginning of the proof of Theorem 5.3* Let  $\chi_i$  be the restriction to  $K$  of the  $i$ -th coordinate projection  $\ell_{\mathbb{N}}^{\infty}(\mathcal{A}) \rightarrow \mathcal{A}$  ( $i \in \mathbb{N}$ ),  $\mathcal{H}$  a standard Hilbert  $\mathcal{A}$ -module,  $\mathcal{C} = \mathbb{B}_{\mathcal{A}}(\mathcal{H})$  (so  $\mathcal{C}$  is anti-isomorphic to  $\mathcal{A}$ ) and denote by  $\mathcal{X}_0$  the operator  $\mathcal{C}$ -system generated inside  $\mathbb{A}_{\mathcal{A}}^{w*}(K, \mathbb{B}(\mathcal{H}))$  [the space of all  $\mathcal{A}$ -affine weak\* continuous functions from  $K$  to  $\mathbb{B}(\mathcal{H})$ ] by all the functions  $\chi_i$ . Thus  $\mathcal{X}_0$  is the norm closure of  $\mathcal{C} + \sum_{i \in \mathbb{N}} (\mathcal{C}\chi_i + \mathcal{C}\chi_i^*)$  and is contained in  $\ell_K^{\infty}(\mathbb{B}(\mathcal{H}))$  since all the  $\chi_i$  are bounded  $\mathbb{B}(\mathcal{H})$ -valued maps on  $K$ . Note that the usual operator system  $X$  generated by the functions  $\chi_i$  commutes with  $\mathcal{C}$  since  $\chi_i(K) \subseteq \mathcal{A}$ . Let  $B$  be the  $C^*$ -algebra generated by  $\mathcal{X}_0$  [inside  $\ell_K^{\infty}(\mathbb{B}(\mathcal{H}))$ ] and let  $\mathcal{X}$  be the strong operator  $\mathcal{C}$ -system generated by  $\mathcal{X}_0$ , thus  $\mathcal{X} := \overline{\mathcal{X}_0}^{\mathcal{C}} \subseteq \mathbb{A}_{\mathcal{A}}^{w*}(K, \mathbb{B}(\mathcal{H})) = \mathbb{A}_{\mathcal{A}}^{w*}(K, \mathbb{B}(\mathcal{H}))$ . Since the  $\chi_i$  (hence also  $\mathcal{X}$ ) separate points of  $K$ , the map  $K \ni y \mapsto \varepsilon_y \in \tilde{\mathcal{S}} := \text{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ , where  $\varepsilon_y$  is the evaluation,  $\varepsilon_y(f) = f(y)$  ( $f \in \mathcal{X}$ ), is injective. It is also  $\mathcal{A}$ -affine and weak\* continuous. Therefore the same argument as in the proof of [25, 3.5] shows that  $\varepsilon: K \rightarrow \tilde{\mathcal{S}}$  is also surjective, hence a weak\* homeomorphism and we may therefore identify the two sets  $\tilde{\mathcal{S}}$  and  $K$ . Since  $B \subseteq \ell_K^{\infty}(\mathbb{B}(\mathcal{H}))$ , the coordinate projections  $\eta_k: \ell_K^{\infty}(\mathbb{B}(\mathcal{H})) \rightarrow \mathbb{B}(\mathcal{H})$  ( $k \in K$ ) norm  $B$ . Now the proof will be divided into two cases.  $\square$

*Proof of Theorem 5.3 (for properly infinite  $\mathcal{A}$ )* Composing each  $\eta_k$  with each map of the form  $x \mapsto ex|e\mathcal{H}$ , where  $e \in \mathcal{A}$  is a projection onto a cyclic subspace  $[\eta_k(B)\xi]$  of  $\mathcal{H}$ , we obtain a family of maps that norm  $B$ . Since  $B = \overline{\mathcal{C}B_0}$ , where  $B_0$  is the unital  $C^*$ -algebra generated by the countable set  $\chi_i$  ( $i \in \mathbb{N}$ ), and since  $\mathcal{C}$  has a countable weak\* dense subset, all Hilbert spaces  $[\eta_k(B)\xi]$  ( $\xi \in \mathcal{H}$ ) are separable. Thus the family of all maps in  $\text{UCP}_{\mathcal{C}}(\mathcal{X}_0, \mathbb{B}(\mathcal{K}))$ , where  $\mathcal{K}$  is a separable normal Hilbert  $\mathcal{C}$ -module, completely norms  $\mathcal{X}_0$ . By [1] and [5] each such map can be dilated to a map with the u.e.p.; moreover, by Lemma 5.1 we may take such dilations to be on normal Hilbert  $\mathcal{C}$ -modules  $\mathcal{L}$ . For each such  $\mathcal{L}$  the corresponding representation  $\pi_{\mathcal{L}}: B \rightarrow \mathbb{B}(\mathcal{L})$  (such that  $\pi_{\mathcal{L}}$  is the u.c.p. extension of  $\pi_{\mathcal{L}}|_{\mathcal{X}_0}$ ) can be decomposed into a direct sum of cyclic representations, which are normal on  $\mathcal{C}$ . Hence (again since  $\mathcal{C}$  is countably generated and  $B_0$  separable) the Hilbert spaces of such cyclic representations must be separable, hence countable direct sums of cyclic Hilbert  $\mathcal{C}$ -modules and therefore contained in  $\mathcal{H}^{\aleph_0}$  as Hilbert  $\mathcal{C}$ -modules (since  $\mathcal{H}$  contains all normal cyclic Hilbert  $\mathcal{C}$ -modules). Since  $\mathcal{C}$  has properly infinite commutant  $\mathcal{A}$ ,  $\mathcal{H}^{\aleph_0}$  is isomorphic to  $\mathcal{H}$  as a Hilbert  $\mathcal{C}$ -module. This shows that maps in  $\tilde{\mathcal{Q}} := \text{CCP}_{\mathcal{C}}(\mathcal{X}_0, \mathbb{B}(\mathcal{H}))$  with the u.e.p. completely norm  $\mathcal{X}_0$ . Then it follows from [25, 2.1] that such maps completely norm also  $\mathcal{X}$ , and these maps are Choquet  $\mathcal{A}$ -points of  $\tilde{\mathcal{Q}}$  by [25, 6.4]. Now it suffices to show that for each such map  $\varphi_1 \in \tilde{\mathcal{Q}}$  with the u.e.p. there exists a  $(p_2\mathcal{A}p_2\text{-convex})$   $p_2\mathcal{A}p_2$ -extreme point  $\varphi_2$  of the set  $p_2\tilde{\mathcal{S}}p_2 = \text{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(p_2\mathcal{H}))$ , where  $p_2 := \varphi_1(1)^{\perp}$  (a projection, since  $\varphi_1|_{\mathcal{C}}$  is multiplicative), for then by Lemma 5.2  $\varphi_1 \oplus \varphi_2$  is a  $\mathcal{A}$ -extreme point of  $\tilde{\mathcal{S}} = \text{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ , while by Theorem 2.5 the set of all such maps  $\varphi_1 \oplus \varphi_2$



generates  $\tilde{S}$  since it norms  $\mathcal{X}$ . Since  $\varphi_1$  is a  $\mathcal{C}$ -bimodule map,  $\varphi_1(1) \in \mathcal{C}' = \mathcal{A}$ , hence  $p_2 \in \mathcal{A}$  and so  $p_2 \tilde{S} p_2 \cong p_2 K p_2$  is a weak\* compact  $\mathcal{R}$ -convex subset of  $\ell_{\mathbb{N}}^{\infty}(\mathcal{R})$ , where  $\mathcal{R} = p_2 \mathcal{A} p_2$ . Any weak\* compact  $\mathcal{R}$ -convex subset of every von Neumann algebra  $\mathcal{R}$  has an  $\mathcal{R}$ -convex  $\mathcal{R}$ -extreme point by [22, 3.6], from which it is easy to deduce the same also for such subsets of  $\ell_{\mathbb{N}}^{\infty}(\mathcal{R})$  [Indeed, let  $q_j : \ell_{\mathbb{N}}^{\infty}(\mathcal{R}) \rightarrow \mathcal{R}$  be the projection onto the  $j$ -th summand and let  $\mathcal{Z}$  be the center of  $\mathcal{R}$ . Then  $q_0(K)$  is a weak\* compact  $\mathcal{R}$  convex subset of  $\mathcal{R}$ , hence  $q_0(K) \cap \mathcal{Z}$  contains an  $\mathcal{R}$ -extreme point  $z_0$  of  $q_0(K)$  by [22, 3.6]. Let  $K_1 = \{y \in K : q_0(y) = z_0\}$ . By the same argument  $q_1(K_0) \cap \mathcal{Z}$  contains an  $\mathcal{R}$ -extreme point  $z_1$  of  $q_1(K_0)$ . Continuing, we find a decreasing sequence of nonempty weak\* compact  $\mathcal{R}$ -convex subsets of  $K$  such that for each  $j$  the set  $K_j$  contains only elements of the form  $(z_0, \dots, z_{j-1}, y_j, y_{j+1}, \dots)$  ( $y_k \in \mathcal{R}$ ). The only point in the intersection  $H := \bigcap_{j \in \mathbb{N}} K_j \subseteq K$  (which is not empty by compactness) is  $z := (z_0, z_1, \dots)$ . Since all  $z_j$  are in  $\mathcal{Z}$ ,  $z$  is  $\mathcal{R}$ -convex. Since each  $z_j$  is an  $\mathcal{R}$ -extreme point of  $q_j(K)$  and is in  $\mathcal{Z}$ ,  $z_j$  is a strong  $\mathcal{R}$ -extreme point of  $q_j(K)$ . Now it can be verified that  $z$  is a strong  $\mathcal{R}$ -extreme point of  $K$ .]  $\square$

In the proof of Theorem 5.3 in the case  $\mathcal{A} = \mathbb{M}_n(\mathcal{Z})$  we will need the following simple lemma.

**Lemma 5.4** *Let  $\mathcal{D} \supseteq \mathcal{Z}$  be an abelian  $C^*$ -algebra and let  $\pi \in \text{UCP}_{\mathcal{Z}}(\mathcal{D}, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  be a  $\mathcal{Z}$ -irreducible representation of  $\mathcal{D}$  on a (self-dual)  $C^*$ -module  $\mathcal{E}$  over  $\mathcal{Z}$ . Then  $\mathcal{E}$  is isometrically isomorphic to a  $\mathcal{Z}$ -submodule of  $\mathcal{Z}$ .*

*Proof* Note that  $\pi(\mathcal{D})' = \pi(\mathcal{Z})$  by  $\mathcal{Z}$ -irreducibility, hence  $\pi(\mathcal{Z})' = \pi(\mathcal{D})'' = \overline{\pi(\mathcal{D})}$ , since the bicommutant theorem holds for von Neumann algebras on self-dual  $C^*$ -modules (essentially by the same proof as on Hilbert spaces). Since  $\mathcal{D}$  is abelian, we also have  $\overline{\pi(\mathcal{D})} \subseteq \pi(\mathcal{D})'$  and it follows that  $\pi(\mathcal{Z})' \subseteq \pi(\mathcal{D})' = \pi(\mathcal{Z})$  and so  $\pi(\mathcal{Z})' = \pi(\mathcal{Z})$  since  $\mathcal{Z}$  is abelian. It is known [30] that each self-dual  $\mathcal{E}$  is of the form  $\mathcal{E} = \overline{\bigoplus_{i \in \mathbb{I}} p_i \mathcal{Z}}$ , where  $(p_i)_{i \in \mathbb{I}}$  is a family of projections in  $\mathcal{Z}$ . Then  $\pi$  as a  $\mathcal{Z}$ -module map decomposes accordingly and it is easy to see that  $\pi(\mathcal{Z})'$  can not be abelian if any two projections  $p_i, p_j$  ( $i \neq j$ ) have a non-zero common subprojection. Thus the family  $(p_i)_{i \in \mathbb{I}}$  must be orthogonal. But then  $\mathcal{E}$  is evidently isomorphic to  $\mathcal{Z} \sum_{i \in \mathbb{I}} p_i$ .  $\square$

*Proof of Theorem 5.3 (in the case  $\mathcal{A} = \mathbb{M}_n(\mathcal{Z})$ ,  $\mathcal{Z}$  abelian)* Denote, as above, by  $\chi_i$  the restrictions to  $K$  of the  $i$ -th coordinate projection  $\ell_{\mathbb{N}}^{\infty}(\mathcal{A}) \rightarrow \mathcal{A}$ . In addition to  $\mathcal{X}_0$  above, we consider also the norm closure  $X$  of  $\mathcal{Z} + \sum_{i \in \mathbb{N}} (\mathcal{Z} \chi_i + \mathcal{Z} \chi_i^*)$ ; thus  $X$  is a central operator  $\mathcal{Z}$ -system. Let  $B_0$  be the  $C^*$ -algebra generated by  $X$  inside  $\ell_K^{\infty}(\mathcal{A}) \cong \mathbb{M}_n(\ell_K^{\infty}(\mathcal{Z}))$ . So  $B_0$  contains  $\mathcal{Z}$  in the center. Each  $\mathcal{Z}$ -boundary representation of  $B_0$  (with respect to  $X$ ) is a  $\mathcal{Z}$ -irreducible representation  $\sigma$  of  $B_0$  on a self-dual  $C^*$ -module  $\mathcal{F}$  over  $\mathcal{Z}$ . Set  $\mathcal{D} := \ell_K^{\infty}(\mathcal{Z})$ . Since  $B_0 \subseteq \mathbb{M}_n(\mathcal{D})$ , by Remark 4.17 [and since every representation of  $\mathbb{M}_n(\mathcal{D})$  is of the form  $\pi_n$  for a representation  $\pi$  of  $\mathcal{D}$ ]  $\sigma$  is a subrepresentation of  $\pi_n|_{B_0}$ , where  $\pi \in \text{UCP}_{\mathcal{Z}}(\mathcal{D}, \mathbb{B}_{\mathcal{Z}}(\mathcal{E}))$  is a  $\mathcal{Z}$ -irreducible representation of  $\mathcal{D}$  on a self-dual  $C^*$ -module  $\mathcal{E}$  over  $\mathcal{Z}$ .

By Lemma 5.4  $\mathcal{E}$  can be regarded as a submodule in  $\mathcal{Z}$ , so  $\pi$  can be considered as a map into  $\mathcal{Z}$  and therefore  $\pi_n$  and  $\sigma$  are regarded as maps into  $\mathbb{M}_n(\mathcal{Z})$ . Note that  $\mathcal{F}$  (being self-dual) is a direct summand in  $\mathcal{Z}^n$ , hence  $\mathbb{B}_{\mathcal{Z}}(\mathcal{F})$  is a corner in

$\mathbb{M}_n(\mathcal{Z})$  and consequently  $\sigma$  is still the unique c.p. extension of  $\sigma|X$  regarded as a  $\mathcal{Z}$ -module map into  $\mathcal{A} = \mathbb{M}_n(\mathcal{Z})$ . [Indeed, if  $\theta: B_0 \rightarrow \mathbb{M}_n(\mathcal{Z})$  is any c.p. extension of  $\sigma|X$ , denoting by  $p: \mathcal{Z}^n \rightarrow \mathcal{F}$  the projection, we have that  $p\theta p = \sigma$  by the u.e.p. of  $\sigma|X$ . Since  $\sigma$  is multiplicative, a well-known argument based on the Schwarz inequality for  $\theta$  shows that  $\theta(b)$  commutes with  $p$  for all  $b \in B_0$ . Thus, relative to the decomposition  $\mathcal{Z}^n = \mathcal{F} \oplus \mathcal{F}^\perp$  the map  $\theta$  decomposes into a direct sum  $\theta = \theta_1 \oplus \theta_2$ . Since  $\sigma$  is the unique c.p. extension of  $\sigma|X$  as a map into  $\mathbb{B}_{\mathcal{Z}}(\mathcal{F})$ ,  $\theta_1 = \sigma$ . From  $\theta_1(1) \oplus \theta_2(1) = \theta(1) = (\sigma|X)(1) = \sigma(1)$ , we see that  $\theta_2(1) = 0$ , hence  $\theta_2 = 0$  by positivity and  $\theta = \sigma$ .] The family of all such maps  $\sigma$  completely norms  $X$  since by Theorem 4.20  $\mathcal{Z}$ -boundary representations of  $B_0$  completely norm  $X$ .

Let  $\mathcal{H}_{\mathcal{Z}}$  be the standard Hilbert  $\mathcal{Z}$ -module. Then  $\mathcal{H} = \mathbb{C}^n \otimes \mathcal{H}_{\mathcal{Z}} \otimes \mathbb{C}^n$  is the standard Hilbert module over  $\mathcal{A} = \mathbb{M}_n(\mathcal{Z}) \cong \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{Z} \otimes 1$  and over  $\mathcal{C} = \mathcal{A}' = 1 \otimes \mathcal{Z} \otimes \mathbb{M}_n(\mathbb{C})$ . Regard  $\mathcal{Z}$  as contained in  $\ell_K^\infty(\mathcal{Z})$  (as the subspace of constant maps from  $K$  to  $\mathcal{Z}$ ) and note that  $B_0 \subseteq \mathbb{M}_n(\mathbb{C}) \otimes \ell_K^\infty(\mathcal{Z}) \cong \mathbb{M}_n(\mathbb{C}) \otimes \ell_K^\infty(\mathcal{Z}) \otimes 1$ ,  $B_0\mathcal{C} = B_0 \otimes \mathbb{M}_n(\mathbb{C})$ . Each map  $\sigma: B_0 \rightarrow \mathbb{M}_n(\mathcal{Z}) = \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{Z}$  from the previous paragraph can obviously be extended to a  $\mathbb{M}_n(\mathbb{C})$ -bimodule map

$$\sigma \otimes 1_{\mathbb{M}_n(\mathbb{C})}: B_0\mathcal{C} = B_0 \otimes \mathbb{M}_n(\mathbb{C}) \rightarrow \mathbb{M}_n(\mathbb{C}) \otimes \mathcal{Z} \otimes \mathbb{M}_n(\mathbb{C}) \cong \mathbb{M}_n(\mathcal{Z})\mathcal{C}$$

in a unique way. This extension is a  $\mathcal{C}$ -bimodule map since  $\sigma$  is a  $\mathcal{Z}$ -module map. Further, since  $\mathcal{C}$  and  $B_0$  commute, any  $\mathcal{C}$ -bimodule map  $\tilde{\sigma}: B_0\mathcal{C} \rightarrow \mathbb{B}(\mathcal{H})$  must map  $B_0$  into the commutant  $\mathcal{A}$  of  $\mathcal{C}$  in  $\mathbb{B}(\mathcal{H})$ . Consequently, if  $\tilde{\sigma} \in \text{CCP}_{\mathcal{C}}(B_0\mathcal{C}, \mathbb{B}(\mathcal{H}))$  is an extension of  $(\sigma \otimes 1_{\mathbb{M}_n(\mathbb{C})})|X\mathcal{C}$ , then  $\tilde{\sigma}|B_0$  extends  $\sigma|X$  as a map into  $\mathcal{A}$ , hence by the previous paragraph  $\tilde{\sigma}|B_0 = \sigma$  and therefore  $\tilde{\sigma} = \sigma \otimes 1_{\mathbb{M}_n(\mathbb{C})}$  by the  $\mathcal{C}$ -bimodule property of both maps. Thus  $\tilde{\sigma} := \sigma \otimes 1_{\mathbb{M}_n(\mathbb{C})}$  is the unique extension in  $\text{CCP}_{\mathcal{C}}(B_0\mathcal{C}, \mathbb{B}(\mathcal{H}))$  of  $\tilde{\sigma}|X\mathcal{C}$ . Since the maps  $\sigma$  together completely norm  $X$ , the maps  $\tilde{\sigma}$  completely norm  $X \otimes \mathbb{M}_n(\mathbb{C}) \cong X\mathcal{C} = \mathcal{X}_0$ . Thus we have shown the existence of a family of maps in  $\text{CCP}_{\mathcal{C}}(\mathcal{X}_0, \mathbb{B}(\mathcal{H}))$  with the u.e.p. that completely norm  $\mathcal{X}_0$ , hence also  $\mathcal{X}$  by [25, 2.1]. From this it follows by the same argument as in the properly infinite case above, that  $K$  ( $\cong \tilde{S} = \text{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ ) is generated by its  $\mathcal{A}$ -extreme points, each of which is (as an element of  $\tilde{S}$ ) of the form  $\varphi_1 \oplus \varphi_2$ , where  $\varphi_1 \in \text{CCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$  has the  $\mathcal{Z}$ -u.e.p.,  $p_2 := 1 - \varphi_1(1)$  is a projection and  $\varphi_2$  is a  $p_2\mathcal{A}p_2$ -convex  $p_2\mathcal{A}p_2$ -extreme point of  $p_2\tilde{S}p_2$ .  $\square$

*Conclusion of the proof of Theorem 5.3* If  $\mathcal{A}$  is as in the theorem, then  $\mathcal{A}$  decomposes into a direct sum of the properly infinite part and algebras of the form  $\mathbb{M}_n(\mathcal{Z})$  for an abelian  $\mathcal{Z}$  [17, 6.5.2]. Since  $K \subseteq \ell_{\mathbb{N}}^\infty(\mathcal{A})$ ,  $K$  contains an  $\mathcal{A}$ -convex  $\mathcal{A}$ -extreme point  $z$  (by the argument at the end of the proof in the properly infinite case). Hence, replacing  $K$  by  $K - z$ , we may assume that  $0 \in K$ . Then (by  $\mathcal{A}$ -convexity and weak\* compactness)  $K$  also decomposes accordingly and the proof follows easily from the cases considered above.  $\square$

Let us remark why we had to use representations on  $C^*$ -modules over  $\mathcal{Z}$  (rather than on usual Hilbert spaces) in the above proof for finite type  $I$  algebras. Every irreducible representation  $\pi$  of  $\ell_K^\infty(\mathcal{Z})$  is on  $\mathbb{C}$ , but  $\mathbb{C}$  (as a Hilbert  $\mathcal{Z}$  module) is contained in the standard Hilbert  $\mathcal{Z}$ -module only if  $\pi$  is a normal representation. However, there is usually not enough normal characters on  $\mathcal{Z}$  if  $\mathcal{Z}$  is not atomic.

The author does not know if Theorem 5.3 holds also for algebras  $\mathcal{A}$  of type  $II_1$ , but at least for injective ones we have the following proposition.

**Proposition 5.5** *If  $\mathcal{A}$  is an injective von Neumann algebra of type  $II_1$  with a separable predual, then each weak\* compact  $\mathcal{A}$ -convex subset of  $\ell_{\mathbb{N}}^{\infty}(\mathcal{A})$  is generated by its strong  $\mathcal{A}$ -extreme points.*

The proof of Proposition 5.5 is based on approximate finite dimensionality and the following property of strong  $\mathcal{A}$ -extreme points, which is perhaps not true for the usual  $\mathcal{A}$ -extreme points.

**Lemma 5.6** *Let  $\mathcal{M} \subseteq \mathcal{A}$  be a  $C^*$ -subalgebra containing the unit of  $\mathcal{A}$  such that for each non-zero  $x \in \mathcal{A}$  there exists a conditional expectation  $E: \mathcal{A} \rightarrow \mathcal{M}$  with  $E(x) \neq 0$ . If  $K$  is an  $\mathcal{A}$ -convex subset of  $\ell_{\mathbb{N}}^{\infty}(\mathcal{A})$  such that  $\hat{\varphi}(K) \subseteq K$  for each u.c.p. map  $\varphi: \mathcal{A} \rightarrow \mathcal{M}$ , where  $\hat{\varphi}: \ell_{\mathbb{N}}^{\infty}(\mathcal{A}) \rightarrow \ell_{\mathbb{N}}^{\infty}(\mathcal{M})$  is defined by  $\hat{\varphi}((a_j)) = (\varphi(a_j))$ , then each strong  $\mathcal{M}$ -extreme point of  $K \cap \ell_{\mathbb{N}}^{\infty}(\mathcal{M})$  is a strong  $\mathcal{A}$ -extreme point of  $K$ .*

The proof of Lemma 5.6 is essentially the same as [23, 1.3] (where strong  $\mathcal{A}$ -extreme points are called just  $\mathcal{A}$ -extreme points), so it will be omitted. Let  $\mathcal{Z}$  be the center of  $\mathcal{A}$ . To prove Proposition 5.5 we will need to find, for a subalgebra  $\mathcal{M}$  of the form  $\mathcal{M} = \mathbb{M}_m(\mathcal{Z})$  in  $\mathcal{A}$ ,  $\mathcal{M}$ -extreme points of  $K \cap \ell_{\mathbb{N}}^{\infty}(\mathcal{M})$ , which are either strong  $\mathcal{M}$ -extreme or can be generated by strong  $\tilde{\mathcal{M}}$ -extreme points of  $K \cap \ell_{\mathbb{N}}^{\infty}(\tilde{\mathcal{M}})$  for some larger  $\tilde{\mathcal{M}}$  of the form  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ . For this we will need a refinement of Lemma 5.2. Given  $a \in \mathbb{B}(K)$  [or a map  $f$  into  $\mathbb{B}(K)$ ], we denote by  $a^{(n)}$  [or by  $f^{(n)}$ ] the direct sum of  $n$  copies of  $a$  [or the map defined by  $f^{(n)}(x) = f(x)^{(n)}$ ].

**Lemma 5.7** *In the context of Lemma 5.2 assume in addition that: (i)  $C$  is a maximal abelian von Neumann algebra  $\mathcal{Z}$  on a Hilbert space  $\mathcal{H}_{\mathcal{Z}}$ , (ii)  $\mathcal{H} = \mathcal{H}_{\mathcal{Z}}^m$  for some  $m \in \mathbb{N}$ , so that  $\mathcal{A} = \mathbb{B}_{\mathcal{Z}}(\mathcal{H}) = \mathbb{M}_m(\mathcal{Z})$ , (iii)  $\mathcal{X} = X$  is a central  $\mathcal{Z}$ -bimodule, (iv) the point  $\varphi_2$  is  $\mathcal{A}_2$ -convex and (v) the map  $\varphi_1$  has the  $\mathcal{Z}$ -u.e.p. Then  $(\varphi)^{(m)}$  is in the  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -convex hull of strong  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -extreme points of  $UCP_{\mathcal{Z}}(X, \mathbb{M}_{\tilde{m}}(\mathcal{Z}))$ , where  $\tilde{m} = mm!$ .*

*Proof* Let us first see, what we can extract from the proof of Lemma 5.2 under the additional assumption that  $\varphi_2$  is  $\mathcal{A}_2$ -convex and  $\mathcal{X}$  is generated (as an operator  $C$ -bimodule) by a central  $\mathcal{Z}$ -bimodule  $X$ , where  $\mathcal{Z}$  is the center of  $\mathcal{A}$ . (Other assumptions of Lemma 5.7 will be needed only later in the proof.) Since  $\varphi_2$  is  $\mathcal{A}_2$ -convex,  $u^* \varphi_2 u = \varphi_2$  for all unitary  $u \in \mathcal{A}_2$ , hence  $\varphi_2$  commutes with  $\mathcal{A}_2$ . In particular (5.6) simplifies to  $s_j^* \theta_j s_j = \varphi_2$ . Since  $u_j$  is a unitary from  $\mathcal{H}_1$  onto  $a_j \mathcal{H}_1$  (and  $\psi_j = \sigma_j \oplus \theta_j$ ), (5.2) is equivalent to  $\sigma_j = u_j \varphi_1 u_j^*$ . Using all this and the identity  $\sum_{j=1}^n (|c_j|^2 + |d_j|^2) = 1$ , we can now write (5.4) as

$$c^2 \varphi_2 = \sum_{j=1}^n |c_j| w_j^* u_j \varphi_1 u_j^* w_j |c_j|, \quad \text{where } c = \left( \sum_{j=1}^n |c_j|^2 \right)^{\frac{1}{2}}. \quad (5.7)$$

Let  $e \in \mathcal{A}_2$  be the range projection of  $c$ . Since  $\| |c_j| \xi \| \leq \| c \xi \|$  for all  $\xi \in \mathcal{H}_2$ , the map  $c \xi \mapsto |c_j| \xi$  is contractive; extending it by 0 on  $e^{\perp} \mathcal{H}_2$ , we obtain a contraction  $g_j$  in  $\mathcal{A}_2$  such that  $|c_j| = g_j c$ . Denoting  $v_j = u_j^* w_j g_j$ , we have from (5.7)

$$\varphi_2 e = \sum_{j=1}^n v_j^* \varphi_1 v_j. \quad (5.8)$$

Since  $c \sum_{j=1}^n v_j^* v_j c = \sum_{j=1}^n c g_j^* w_j^* u_j u_j^* w_j g_j c = \sum_{j=1}^n |c_j| w_j^* w_j |c_j| = \sum_{j=1}^n c_j^* c_j = c^2$  and  $v_j e^\perp = 0$ , we have that

$$\sum_{j=1}^n v_j^* v_j = e. \quad (5.9)$$

Let  $P$  be the central carrier of  $e$  in  $\mathcal{A}$ . It is well known that  $P$  is an orthogonal sum of a family of subprojections each of which is equivalent in  $\mathcal{A}$  to a subprojection of  $e$ . Thus  $P = \sum_{l \in L} h_l^* h_l$  for some elements  $h_l \in \mathcal{A}$  such that  $e_l := h_l h_l^*$  is a subprojection of  $e$ . Note that if  $\mathcal{A} = \mathbb{M}_m(\mathcal{Z})$ ,  $L$  can be taken to be finite (because  $P$  is a finite sum of abelian projections, which are equivalent to subprojections of  $e$  [17, 6.4.6]). Since  $\varphi_2 P = \varphi_2 p_2 P p_2$ ,  $e_l \leq e \leq p_2$  and  $\varphi_2$  commutes with elements of  $p_2 \mathcal{A} p_2 \cong \mathcal{A}_2$  we have, using (5.8),

$$\begin{aligned} \varphi_2 P &= \varphi_2 p_2 \sum_{l \in L} h_l^* h_l p_2 = \varphi_2 \sum_{l \in L} p_2 h_l^* e_l h_l p_2 \\ &= \sum_{l \in L} p_2 h_l^* e_l \varphi_2 e_l h_l p_2 = \sum_{j=1}^n \sum_{l \in L} p_2 h_l^* v_j^* \varphi_1 v_j h_l p_2. \end{aligned}$$

With  $t_k := v_j h_l p_2$  for each  $k = (l, j) \in I := L \times \{1, \dots, n\}$ , we thus obtain

$$P \varphi_2 = \sum_{k \in I} t_k^* P \varphi_1 t_k. \quad (5.10)$$

Since  $\varphi(x)P = P\varphi(x)$  for all  $x \in \mathcal{X}$ ,  $\varphi$  decomposes into the orthogonal sum  $P\varphi \oplus P^\perp\varphi$ . The element  $c_j$  in the matrix (5.3) of  $a_j$  satisfies  $c_j P^\perp = 0$  since  $c P^\perp = c e P^\perp = 0$  and  $|c_j| \leq c$ . Thus the matrix of  $a_j P^\perp$  is block-diagonal and so  $(a_j P^\perp)^2 = (a_j P^\perp)^*(a_j P^\perp)$  maps  $P^\perp \mathcal{H}_1$  into  $P^\perp \mathcal{H}_1$  and  $P^\perp \mathcal{H}_2$  into  $P^\perp \mathcal{H}_2$ . Since  $a_j$ , hence also  $a_j P^\perp$ , is positive, we have  $a_j P^\perp = \sqrt{(a_j P^\perp)^2}$ , so the two spaces  $P^\perp \mathcal{H}_1$  and  $P^\perp \mathcal{H}_2$  are invariant also for  $a_j P^\perp$ , and consequently  $b_j P^\perp = 0$  and  $d_j P^\perp$  are positive and invertible. From (5.1) (with  $a_j \geq 0$ ) we have now

$$P^\perp \varphi = P^\perp \varphi_1 \oplus P^\perp \varphi_2 = \sum_{j=1}^n (P^\perp b_j \oplus P^\perp d_j)(P^\perp \sigma_j \oplus P^\perp \theta_j)(P^\perp b_j \oplus P^\perp d_j). \quad (5.11)$$

Using the facts that  $\varphi_1$  is strong  $\mathcal{A}_1$ -extreme point of  $S_1$  (as a Choquet  $\mathcal{A}_1$ -point) and  $\varphi_2$  is a strong  $\mathcal{A}_2$ -extreme point of  $S_2$  (as an  $\mathcal{A}_2$ -convex  $\mathcal{A}_2$ -extreme point) and that  $P$  commutes with everything interesting, it follows readily from (5.11) that  $P^\perp \sigma_j =$

$P^\perp\varphi_1, P^\perp\theta_j = P^\perp\varphi_2$  and that  $b_j P^\perp$  commutes with  $P^\perp\varphi_1$  and  $d_j P^\perp$  commutes with  $P^\perp\varphi_2$ . Thus

$$P^\perp\varphi \text{ is a strong } P^\perp\mathcal{A}\text{-extreme point of } P^\perp S. \quad (5.12)$$

Now note that by the hypothesis of the lemma  $\mathcal{A} = \mathbb{M}_m(\mathbb{B}_{\mathcal{Z}}(\mathcal{H}_{\mathcal{Z}})) = \mathbb{M}_m(\mathcal{Z})$ . Further, since  $X$  is central over  $\mathcal{Z}$ , every  $\mathcal{Z}$ -bimodule map of  $X$  into  $\mathbb{B}(\mathcal{H}_{\mathcal{Z}}^m)$  must have its range in the commutant of  $\mathcal{Z}$  in  $\mathbb{B}(\mathcal{H}_{\mathcal{Z}}^m)$ , hence  $\text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_m(\mathbb{B}(\mathcal{H}_{\mathcal{Z}}))) = \text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_m(\mathcal{Z}))$ . Since the commutant of  $\mathcal{A}$  in  $\mathbb{B}(\mathcal{H}_{\mathcal{Z}}^m)$  is  $\mathcal{Z}$ , the commutants of  $\mathcal{A}_1 \cong p_1\mathcal{A}p_1$  and  $\mathcal{A}_2 \cong p_2\mathcal{A}p_2$  are  $\mathcal{Z}p_1$  and  $\mathcal{Z}p_2$  (respectively), isomorphic to  $\mathcal{Z}P_1$  and  $\mathcal{Z}P_2$ , where  $P_1, P_2 \in \mathcal{Z}$  are the central carriers of  $p_1$  and  $p_2$  [17, 5.5.5]. Therefore in the central decompositions of  $\mathcal{A}_1$  and  $\mathcal{A}_2$  into homogeneous algebras of type  $I_k$  ( $k \leq m$ ) the summands appear with multiplicity one only (otherwise their commutants would not be abelian by [17, 9.3.2]). It follows that the identity 1 is a finite sum of projections  $P_i \in \mathcal{Z}$  ( $i = 1, \dots, n$ ) such that both  $\mathcal{A}_1 P_i$  and  $\mathcal{A}_2 P_i$  are of the form  $\mathbb{M}_{r_i}(P_i\mathcal{Z})$  and  $\mathbb{M}_{s_i}(P_i\mathcal{Z})$  for some integers  $r_i$  and  $s_i$ , and  $r_i + s_i = m$  since the sum of the units  $p_j P_i \in \mathcal{A}_j P$  ( $j = 1, 2$ ) is the unit  $P_i$  of  $\mathcal{A}P_i \cong \mathbb{M}_m(\mathcal{Z}P_i)$ . Then, up to a conjugation by a unitary in  $\mathbb{M}_m(\mathcal{Z})$ ,  $p_1 P_i$  and  $p_2 P_i$  are just the projections of  $P_i\mathcal{H}_{\mathcal{Z}}^m$  onto the direct sums  $P_i\mathcal{H}_{\mathcal{Z}}^{r_i}$  and  $P_i\mathcal{H}_{\mathcal{Z}}^{s_i}$  of the first  $r_i$  and the last  $s_i$  copies of  $P_i\mathcal{H}_{\mathcal{Z}}$ , respectively. [This can be deduced also from the fact that  $p_1, p_2$  can be diagonalized in  $\mathbb{M}_m(\mathcal{Z})$  [17, 6.9.35].] Then observe that  $P_i\varphi_1^{(m)}$  and  $P_i(\varphi_1 \oplus \varphi_2)^{(r_i)}$  are both elements of  $\text{UCP}_{\mathcal{Z}}(X, P_i\mathbb{M}_{mr_i}(\mathbb{B}(\mathcal{H}_{\mathcal{Z}})))$ . Since  $P_i\varphi^{(r_i)} = (P_i\varphi_1 \oplus P_i\varphi_2)^{(r_i)}$  is unitarily equivalent to  $P_i\varphi_1^{(r_i)} \oplus P_i\varphi_2^{(r_i)}$  via a suitable permutation matrix, it suffices now to prove the following

**Claim** *Each  $P_i$  can be decomposed as  $P_i = P'_i + P''_i$ , with  $P'_i, P''_i \in \mathcal{Z}$ , so that  $P'_i\varphi^{(r_i)}$  is in the  $P'_i\mathbb{M}_{r_i m_i}(\mathcal{Z})$ -convex hull of  $P'_i\varphi_1^{(m)}$  and  $P'_i\varphi$  is a strong  $P'_i\mathbb{M}_m(\mathcal{Z})$ -extreme point of  $\text{UCP}_{P'_i\mathcal{Z}}(X, P'_i\mathbb{M}_m(\mathcal{Z}))$ .*

Namely, assuming the claim, we will have, with  $l_i := \frac{m!}{r_i}$ , that  $P'_i\varphi^{(m!)} = P'_i(\varphi^{(r_i)})^{(l_i)}$  is in the  $P'_i\mathbb{M}_{r_i m_i}(\mathcal{Z})$ -convex hull of  $P'_i(\varphi_1^{(m)})^{(l_i)} = P'_i\varphi_1^{(m!)}$ , where  $r_i m_i = m! = \tilde{m}$ . Since  $\varphi_1$  has the  $\mathcal{Z}$ -u.e.p., its central summand  $P'_i\varphi_1$  must have the  $P'_i\mathcal{Z}$ -u.e.p., and then the same must hold for  $P'_i\varphi_1^{(m!)} = (P'_i\varphi_1)^{(m!)}$ . (Namely, a direct sum of maps with the  $\mathcal{Z}$ -u.e.p. has a  $\mathcal{Z}$ -u.e.p.; the proof is essentially the same as for the usual u.e.p. in [2, 2.2].) Hence  $P'_i\varphi_1^{(m!)}$  is a Choquet  $P'_i\mathcal{A}$ -point and so a strong  $P'_i\mathcal{A}$ -extreme point of  $\text{UCP}_{P'_i\mathcal{Z}}(X, P'_i\mathbb{M}_{\tilde{m}}(\mathcal{Z}))$ . But then the direct sum  $\sigma := \sum_i P'_i\varphi_1^{(m!)}$  is also a strong  $P'\mathcal{A}$ -extreme point of  $\text{UCP}_{P'\mathcal{Z}}(X, P'\mathbb{M}_{\tilde{m}}(\mathcal{Z}))$ , where  $P' := \sum_i P'_i$ , since  $\text{UCP}_{P'\mathcal{Z}}(X, P'\mathbb{M}_{\tilde{m}}(\mathcal{Z})) \cong \bigoplus_{i=1}^n \text{UCP}_{P'_i\mathcal{Z}}(X, P'_i\mathbb{M}_{\tilde{m}}(\mathcal{Z}))$  because  $X$  is a central bimodule over  $\mathcal{Z}$ . It follows from the claim that  $P'_i\varphi^{(m!)} = P'_i(\varphi^{(r_i)})^{(l_i)}$  is in the  $P'_i\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -convex hull of the point  $P'_i(\varphi_1^{(m)})^{(l_i)} = P'_i\varphi_1^{(m!)}$ ; since  $P'_i\varphi_1^{(m!)} = P'_i\sigma P'_i$  and  $\sum_i P'_i = P'$ , this implies that  $P'\varphi^{(m!)} (= \sum_i P'_i\varphi^{(m!)} P_i)$  is in the  $P'\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -convex hull of the point  $\sigma$ . Further, since each  $P'_i\varphi$  is a strong  $P'_i\mathbb{M}_m(\mathcal{Z})$ -extreme point of  $\text{UCP}_{P'_i\mathcal{Z}}(X, P'_i\mathbb{M}_m(\mathcal{Z}))$ , the direct sum

$$P''\varphi = \sum_i P_i''\varphi$$

is a strong  $P''\mathbb{M}_m(\mathcal{Z})$ -extreme point of

$$\text{UCP}_{P''\mathcal{Z}}(X, P''\mathbb{M}_m(\mathcal{Z})) \cong \bigoplus_{i=1}^n \text{UCP}_{P_i''\mathcal{Z}}(X, P_i''\mathbb{M}_m(\mathcal{Z})),$$

where  $P'' = \sum_i P_i''$ . But then  $\sigma \oplus P''\varphi^{(m)}$  is a strong  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -extreme point of

$$\text{UCP}_{P'\mathcal{Z}}(X, P'\mathbb{M}_{\tilde{m}}(\mathcal{Z})) \oplus \text{UCP}_{P''\mathcal{Z}}(X, P''\mathbb{M}_{\tilde{m}}(\mathcal{Z})) \cong \text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_{\tilde{m}}(\mathcal{Z})),$$

and it contains  $\varphi^{(m)} = P_i'\varphi^{(m)} + P_i''\varphi^{(m)}$  in its  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -convex hull.

To prove the claim, let  $i$  be fixed and set  $r = r_i, s = s_i$ . Then, to further simplify the notation, we may assume that  $P_i = 1$ . We must prove that there exist projections  $P'$  and  $P''$  in  $\mathcal{Z}$  with  $P' + P'' = 1$  such that: (1)  $P'\varphi^{(r)} \cong P'\varphi_1^{(r)} \oplus P'\varphi_2^{(r)}$  is in the  $P'\mathbb{M}_{r_m}(\mathcal{Z})$ -convex hull of  $P'\varphi_1^{(r)} \oplus P'\varphi_1^{(s)} = P'\varphi_1^{(m)}$  and (2)  $P''\varphi$  is a strong  $P''\mathbb{M}_m(\mathcal{Z})$ -extreme point of  $\text{UCP}_{P''\mathcal{Z}}(X, P''\mathbb{M}_m(\mathcal{Z}))$ . For this, let  $P'$  be the projection  $P$  obtained as in the first part of the proof so that we have (5.10) and (5.12). Then  $P'' = P^\perp$  and (2) follows immediately from (5.12). To prove (1), we may now assume (to simplify the notation) that  $P = 1$ . Then it suffices to show that  $\varphi_2^{(r)}$  is in the  $\mathbb{M}_{r_s}(\mathcal{Z})$ -convex hull of  $\varphi_1^{(s)}$ , since this implies that  $\varphi_1^{(r)} \oplus \varphi_2^{(r)}$  is in the  $\mathbb{M}_{r_m}(\mathcal{Z})$ -convex hull of  $\varphi_1^{(r)} \oplus \varphi_1^{(s)}$ . Denote by  $f$  the projection of  $(\mathcal{H}_{\mathcal{Z}}^s)^s$  onto its first summand  $\mathcal{H}_{\mathcal{Z}}^s$  and by  $e_{i,1}$  the inclusion map of  $\mathcal{H}_{\mathcal{Z}}^s$  onto the  $i$ -th summand  $\mathcal{H}_{\mathcal{Z}}^s$  of  $(\mathcal{H}_{\mathcal{Z}}^s)^r$  ( $i = 1, \dots, r$ ). Then  $f^*$  is the inclusion of  $\mathcal{H}_{\mathcal{Z}}^s$  onto the first summand  $\mathcal{H}_{\mathcal{Z}}^s$  of  $(\mathcal{H}_{\mathcal{Z}}^s)^s$  and  $e_{i,1}^*$  is the projection of  $(\mathcal{H}_{\mathcal{Z}}^s)^r$  onto the  $i$ -th summand  $\mathcal{H}_{\mathcal{Z}}^s$ . Since  $\varphi_1$  and  $\varphi_2$  are unital maps, (5.10) (with  $P = 1$ ) implies that  $p_2 = \sum_{k \in K} t_k^* p_1 t_k$ , where we may assume that  $t_k = p_1 t_k p_2$ , so that each  $t_k$  can be regarded as a map from  $p_2 \mathcal{H}_{\mathcal{Z}}^m = \mathcal{H}_{\mathcal{Z}}^s$  into  $p_1 \mathcal{H}_{\mathcal{Z}}^m = \mathcal{H}_{\mathcal{Z}}^r$ . Note that each product  $h_{i,k} := t_k^{(s)} f^* e_{i,1}^*$  is a  $\mathcal{Z}$ -bimodule map from  $\mathcal{H}_{\mathcal{Z}}^s = (\mathcal{H}_{\mathcal{Z}}^s)^r$  into  $(\mathcal{H}_{\mathcal{Z}}^r)^s = \mathcal{H}_{\mathcal{Z}}^{sr}$ , hence an element in  $\mathbb{M}_{r_s}(\mathcal{Z})$ . Finally, we have (using (5.10) with  $P = 1$ ) that

$$\begin{aligned} \sum_{i,k} h_{i,k}^* \varphi_1^{(s)} h_{i,k} &= \sum_i \sum_k e_{i,1} f(t_k^{(s)})^* \varphi_1^{(s)} t_k^{(s)} f^* e_{i,1}^* = \sum_i e_{i,1} f \varphi_2^{(s)} f^* e_{i,1}^* \\ &= \sum_i e_{i,1} \varphi_2 e_{i,1}^* = \varphi_2^{(r)}. \end{aligned}$$

Applying both sides to the identity 1, this also implies that  $\sum_{i,k} h_{i,k}^* h_{i,k} = 1_{\mathbb{M}_{r_s}(\mathcal{Z})}$ . Thus  $\varphi_2^{(r)}$  is in the  $\mathbb{M}_{r_s}(\mathcal{Z})$ -convex hull of  $\varphi_1^{(s)}$ .

*Proof of Proposition 5.5* Recall that an injective von Neumann algebra  $\mathcal{A}$  with a separable predual is of the form  $\mathcal{A} = \mathcal{R}_0 \overline{\otimes} \mathcal{Z}$ , where  $\mathcal{Z}$  is abelian and  $\mathcal{R}_0$  is the hyperfinite factor of type II<sub>1</sub> [32, part III, p. 237]. Further, it is known [17, Section 12.2] that that  $\mathcal{R}_0$  contains an increasing sequence of subalgebras of the form  $\mathcal{B}_n \cong \mathbb{M}_{m_n}(\mathbb{C})$ , such that  $\cup_n \mathcal{B}_n$  is dense in  $\mathcal{R}_0$  in the ultrastrong operator topology, so that the corresponding trace preserving conditional expectations  $F_n: \mathcal{R}_0 \rightarrow \mathcal{B}_n$  converge to the

identity map on  $\mathcal{R}_0$ . Let  $\mathcal{A}_n = \mathcal{B}_n \otimes \mathcal{Z} \cong \mathbb{M}_{m_n}(\mathcal{Z})$  and  $E_n = F_n \otimes 1_{\mathcal{Z}} : \mathcal{A} \rightarrow \mathcal{A}_n$ . Each conditional expectation  $E : \mathcal{A} \rightarrow \mathcal{B}$  for a subalgebra  $\mathcal{B}$  of the form  $\mathbb{M}_m(\mathcal{Z})$  is a c.c.p.  $\mathcal{Z}$ -bimodule map on  $\mathcal{A}$ , hence can be weak\* approximated by elementary operators by [4, 4.5], hence also by u.c.p. elementary operators (that is, by  $\mathcal{A}$ -convex combinations, see the proof of Proposition 3.3). This holds then also for the induced map  $\hat{E} : \ell_{\mathbb{N}}^{\infty}(\mathcal{A}) \rightarrow \ell_{\mathbb{N}}^{\infty}(\mathcal{B})$  ( $\hat{E}((a_j)) := (E(a_j))$ ), thus  $\hat{E}(K) \subseteq K \cap \ell_{\mathbb{N}}^{\infty}(\mathcal{B}) =: K_{\mathcal{B}}$ . (The reverse inclusion is obvious, thus  $\hat{E}(K) = K_{\mathcal{B}}$ .) Since  $\mathcal{A} = \mathcal{B} \otimes \mathcal{D}$  for a von Neumann subalgebra  $\mathcal{D}$  of  $\mathcal{A}$ , conditional expectations of  $\mathcal{A}$  onto  $\mathcal{B}$  separate points of  $\mathcal{A}$ , hence by Lemma 5.6 strong  $\mathcal{B}$ -extreme points of  $K_{\mathcal{B}}$  are strong  $\mathcal{A}$ -extreme in  $K$ . Since for each  $x \in K$  we have  $x = \lim \hat{E}_n(x)$ , where  $\hat{E}_n(x) \in K_{\mathcal{A}_n}$ , and since  $K_{\mathcal{A}_n}$  is generated by its  $\mathcal{A}_n$ -extreme points of the kind as in the proof of Theorem 5.3, it suffices now to prove the following

**Claim** *For each such point  $y$  of  $K_{\mathcal{A}_n}$  there exists a subalgebra  $\tilde{\mathcal{A}}_n$  in  $\mathcal{A}$ , containing  $\mathcal{A}_n$  and isomorphic to  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$  for some  $\tilde{m}$ , such that  $y$  can be expressed as an  $\tilde{\mathcal{A}}_n$ -convex combination of strong  $\tilde{\mathcal{A}}_n$ -extreme points of  $K_{\tilde{\mathcal{A}}_n}$ .*

Namely, such points are strong  $\mathcal{A}$ -extreme in  $K$  by Lemma 5.6. To prove the Claim, let  $n$  be fixed and denote  $m := m_n$ ,  $\mathcal{M} = \mathcal{A}_n$ . It is well-known that each subfactor of the form  $\mathbb{M}_m(\mathbb{C})$  in  $\mathcal{R}_0$  is contained in a subfactor of the form  $\mathbb{M}_l(\mathbb{C})$ , where  $l$  is any positive integer; we choose  $l = m!$ . Thus  $\mathcal{M}$  is contained in the von Neumann subalgebra  $\tilde{\mathcal{M}}$  in  $\mathcal{A}$  of the form  $\tilde{\mathcal{M}} \cong \mathbb{M}_{mm!}(\mathcal{Z})$  in  $\mathcal{A} = \mathcal{R}_0 \bar{\otimes} \mathcal{Z}$ . Set  $\tilde{m} = mm!$  and  $\tilde{H} = K \cap \ell_{\mathbb{N}}^{\infty}(\tilde{\mathcal{M}}) = \hat{E}_{\tilde{\mathcal{M}}, \mathcal{A}}(\tilde{H})$ , where  $E_{\tilde{\mathcal{M}}, \mathcal{A}}$  is the (central trace preserving) conditional expectation from  $\mathcal{A}$  onto  $\tilde{\mathcal{M}}$ . Then  $\tilde{H} = \hat{E}_{\tilde{\mathcal{M}}, \mathcal{A}}(K)$  by the approximation argument already mentioned above; moreover, by the same reasoning  $H := K_n = K \cap \ell_{\mathbb{N}}^{\infty}(\tilde{\mathcal{M}}) = \hat{E}_{\mathcal{M}, \tilde{\mathcal{M}}}(\tilde{H})$ , where  $E_{\mathcal{M}, \tilde{\mathcal{M}}}$  is the (central trace preserving) conditional expectation from  $\tilde{\mathcal{M}}$  onto  $\mathcal{M}$ . As in the proof of Theorem 5.3 we realize  $\tilde{H}$  as a set of the form  $\text{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H}))$ , where  $\mathcal{H}$  is the standard Hilbert module over  $\tilde{\mathcal{M}} \cong \mathbb{M}_{\tilde{m}}(\mathcal{Z})$ ,  $\mathcal{C} := \mathbb{B}_{\tilde{\mathcal{M}}}(\mathcal{H})$  and  $\mathcal{X}$  is an operator  $\mathcal{C}$ -system generated by a subspace that commutes with  $\mathcal{C}$ . Now  $\mathcal{H} = \mathbb{C}^{\tilde{m}} \otimes \mathcal{H}_{\mathcal{Z}} \otimes \mathbb{C}^{\tilde{m}}$ , where  $\mathcal{H}_{\mathcal{Z}}$  is the standard Hilbert  $\mathcal{Z}$ -module, hence  $\mathbb{B}(\mathcal{H}) = \mathbb{M}_{\tilde{m}}(\mathbb{C}) \otimes \mathbb{B}(\mathcal{H}_{\mathcal{Z}}) \otimes \mathbb{M}_{\tilde{m}}(\mathbb{C})$ . Identify  $\tilde{\mathcal{M}}$  with  $\mathbb{M}_{\tilde{m}}(\mathbb{C}) \otimes \mathcal{Z} \otimes 1_{\mathbb{M}_{\tilde{m}}(\mathbb{C})} \cong \mathbb{M}_{\tilde{m}}(\mathcal{Z}) \otimes 1_{\mathbb{M}_{\tilde{m}}(\mathbb{C})}$ , hence  $\mathcal{C}$  (the commutant of  $\mathcal{A}$ ) with  $1_{\mathbb{M}_{\tilde{m}}(\mathbb{C})} \otimes \mathcal{Z} \otimes \mathbb{M}_{\tilde{m}}(\mathbb{C}) \cong 1_{\mathbb{M}_{\tilde{m}}(\mathcal{Z})} \otimes \mathbb{M}_{\tilde{m}}(\mathbb{C})$ . Using the standard matrix units  $e_{i,j} \in \mathbb{M}_{\tilde{m}}(\mathbb{C}) \subseteq \mathbb{M}_{\tilde{m}}(\mathcal{Z})$ , it is not hard to show that each operator  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -system is of the form  $\mathbb{M}_{\tilde{m}}(X)$  for an operator  $\mathcal{Z}$ -system  $X$  (namely,  $X \cong e_{1,1} \mathcal{X} e_{1,1}$ , so that  $X$  commutes with  $\mathcal{Z}$ ), and that each  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -bimodule map  $\mathbb{M}_{\tilde{m}}(X) \rightarrow \mathbb{M}_{\tilde{m}}(Y)$  between two such systems is of the form  $\psi \otimes 1_{\mathbb{M}_{\tilde{m}}(\mathbb{C})}$  for a  $\mathcal{Z}$ -bimodule map  $\psi : X \rightarrow Y$ . Thus we have

$$\begin{aligned} \tilde{H} &\cong S_{\tilde{m}} =: \text{UCP}_{\mathcal{C}}(\mathcal{X}, \mathbb{B}(\mathcal{H})) = \text{UCP}_{\mathbb{M}_{\tilde{m}}(\mathcal{Z})}(\mathbb{M}_{\tilde{m}}(X), \mathbb{M}_{\tilde{m}}(\mathbb{M}_{\tilde{m}}(\mathbb{B}(\mathcal{H}_{\mathcal{Z}})))) \\ &\cong \text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_{\tilde{m}}(\mathbb{B}(\mathcal{H}_{\mathcal{Z}}))). \end{aligned}$$

Since  $X$  commutes with  $\mathcal{Z}$ , we have  $\text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_{\tilde{m}}(\mathbb{B}(\mathcal{H}_{\mathcal{Z}}))) = \text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_{\tilde{m}}(\mathcal{Z}))$  (as in the proof of Lemma 5.7), therefore

$$\tilde{H} \cong S_{\tilde{m}} = \text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_{\tilde{m}}(\mathcal{Z})). \quad (5.13)$$

If we identify  $\tilde{\mathcal{M}}$  with  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$  and denote by  $\alpha: \tilde{H} \rightarrow S_{\tilde{m}}$  the  $\tilde{\mathcal{M}}$ -affine isomorphism in (5.13), we note that  $\alpha$  commutes with  $\tilde{\mathcal{M}}$ -convex combinations, hence it commutes also with  $E_{\mathcal{M}, \tilde{\mathcal{M}}}$  since  $E_{\mathcal{M}, \tilde{\mathcal{M}}}$  can be weak\* approximated by such combinations and  $\alpha$  is weak\* continuous. (In fact, in our context it can be proved that  $E_{\mathcal{M}, \tilde{\mathcal{M}}}$  is just an  $\tilde{\mathcal{M}}$ -convex combination.) Thus, for each  $x \in \tilde{H}$  we have  $\alpha(\hat{E}_{\mathcal{M}, \tilde{\mathcal{M}}}(x)) = E_{\mathcal{M}, \tilde{\mathcal{M}}} \circ \alpha(x)$ , hence from (5.13) [and since under the identification of  $\tilde{\mathcal{M}}$  with  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$  the subalgebra  $\mathcal{M}$  is identified with an appropriate copy  $\mathbb{M}_m(\mathcal{Z})^{(m^1)}$  of  $\mathbb{M}_m(\mathcal{Z})$  inside  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ ] we have

$$H = \hat{E}_{\mathcal{M}, \tilde{\mathcal{M}}}(\tilde{H}) \cong \text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_m(\mathcal{Z})). \quad (5.14)$$

It follows from the proof of Theorem 5.3 that  $S_m := \text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_m(\mathcal{Z}))$  is generated by its  $\mathbb{M}_m(\mathcal{Z})$ -extreme points of the form  $\varphi = \varphi_1 \oplus \varphi_2$ , where  $\varphi_1 \in \text{CCP}_{\mathcal{Z}}(X, \mathbb{M}_m(\mathcal{Z}))$  is a map with the  $\mathcal{Z}$ -u.e.p., so that in particular  $p_1 := \varphi_1(1)$  and  $p_2 := 1 - p_1$  are projections in  $\mathbb{M}_m(\mathcal{Z})$ , and  $\varphi_2$  is a  $p_2 \mathbb{M}_m(\mathcal{Z}) p_2$ -convex  $p_2 \mathbb{M}_m(\mathcal{Z}) p_2$ -extreme point of  $p_2 S_m p_2$ . By Lemma 5.7  $\varphi^{(m^1)}$  is in the  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -convex hull of strong  $\mathbb{M}_{\tilde{m}}(\mathcal{Z})$ -extreme points of  $\text{UCP}_{\mathcal{Z}}(X, \mathbb{M}_{\tilde{m}}(\mathcal{Z})) \cong \tilde{H}$ . By the identifications (5.13) and (5.14) this proves the Claim above and concludes the proof of the proposition.  $\square$

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