MAPPINGS PRESERVING SUBMODULES OF HILBERT C*-MODULES

BOJAN MAGAJNA

1. *Introduction and the main result*

A Hilbert module over a C*-algebra B is a right B-module X, equipped with an inner product $\langle \cdot, \cdot \rangle$ which is linear over B in the second factor, such that X is a Banach space with the norm $\|x\| := \|\langle x, x\rangle\|^{1/2}$. (We refer to **[8]** for the basic theory of Hilbert modules; the basic example for us will be $X = B$ with the inner product $\langle x, y\rangle = x^*y$.) We denote by $B(X)$ the algebra of all bounded linear operators on X, and we denote by $L(X)$ the C*-algebra of all adjointable operators. (In the basic example $X = B, L(X)$ is just the multiplier algebra of B.) Let A be a C*-subalgebra of $L(X)$, so that X is an A-B-bimodule. We always assume that A is nondegenerate in the sense that $[AX] = X$, where $[AX]$ denotes the closed linear span of AX.

Denote by A_X the algebra of all mappings on X of the form

$$(x) = \sum_{i=1}^{m} a_i x b_i \quad x \in X \tag{1.1}$$

where m is an integer and $a_i \in A$, $b_i \in B$ for all i. Mappings of form (1.1) will be called *elementary*, and this paper is concerned with the question of which mappings on X can be approximated by elementary mappings in the point norm topology. (The approximation by elementary mappings in various topologies occurs frequently. Recall, for example, the representation of normal completely bounded maps on von Neumann algebras (see **[13, 14]**).) Denote by $\overline{A_X}^{\mathrm{pn}}$ the closure of A_X in the point norm topology of $B(X)$ (which is defined by the family of seminorms $\mapsto \| (x)\|$ $(x \in X)$). Clearly, the lattice $\mathrm{Lat}(A_X)$ of all closed A_X-invariant subspaces of X is just the lattice of closed A-B-submodules of X. Let $\mathrm{ref}(A_X)$ be the algebra of all bounded operators on X that preserve all elements of $\mathrm{Lat}(A_X)$. Obviously, $\overline{A_X}^{\mathrm{pn}} \subseteq \mathrm{ref}(A_X)$; if the equality holds here, then $\overline{A_X}^{\mathrm{pn}}$ is *reflexive* in the sense of operator theory (see **[2]**). For example, in the special case when $X = B$ and A is the complex field $\mathbb{C}$, A_X is (the set of all right multiplications by elements of) B, $\mathrm{Lat}(A_X)$ is just the lattice of closed right ideals of B, and it follows from a result of Johnson **[6]** (reproved later in **[12]**) that A_X is reflexive. (This is in fact true for general Banach algebras by **[6]**.) As another example, let $X = B$ and let A be B acting by left multiplication on itself; then $\mathrm{Lat}(A_X)$ is the lattice of closed two-sided ideals of B, and, by **[9]**, the algebra $\overline{A_X}^{\mathrm{pn}}$ is reflexive.

In general, the algebra $\overline{A_X}^{\mathrm{pn}}$ is not reflexive (see Example 3.3), since elements of $\overline{A_X}^{\mathrm{pn}}$ also preserve all closed A-B-submodules of X^n (equals the direct sum of n

---

Received 30 June 1995; revised 22 May 1996.

1991 *Mathematics Subject Classification* 46L05.

J. London Math. Soc. (2) 58 (1998) 153–162

copies of X), while this is not necessarily true for elements of $\mathrm{ref}(A_X)$. In this paper, we study the situation in which submodules of X^n are determined by submodules of X in the sense of Definition 1.1.

DEFINITION 1.1. Given a C*-subalgebra A of $L(X)$, let A^c be the commutant of A in $L(X)$. For each closed A-B-submodule Y of X and each $\mathbf{c} \in (A^c)^n$ (where n is an integer), let

$$Y(\mathbf{c}) := \{\mathbf{x} \in X^n : \mathbf{c} \cdot \mathbf{x} \in Y\}$$

where, for two n-tuples $\mathbf{c}$ and $\mathbf{x}$ with components c_j and x_j (respectively), $\mathbf{c} \cdot \mathbf{x}$ of course means $\sum_{j=1}^{n} c_j x_j$. Clearly $Y(\mathbf{c})$ is a closed A-B-submodule of X^n. If, for every n, every closed A-B-submodule of X^n is an intersection of modules of the form $Y(\mathbf{c})$, then we say that A is *X-modular*.

It is not hard to prove that $\overline{A_X}^{\mathrm{pn}}$ is reflexive if A is X-modular (Proposition 3.1), but the converse is not true (see Example 3.2). To see some examples of modular algebras, let X be B with the usual inner product, and assume that B is unital, so that $L(B) = B$. Then it follows from the proof of [**9**, Theorem 2.1] and from [**10**, Proposition 2.1] that the algebras $A = B$ and $A = \mathbb{C}1$ are B-modular. Here we shall prove the following more general result. (Recall that the strict topology on $L(X)$ is defined [**8**] by the family of seminorms $a \mapsto \|ax\| + \|a^*x\|$ $(x \in X)$.)

THEOREM 1.2. *Suppose that A is nondegenerate on X and that $L(X)$ is generated as a* C*-*algebra by $\bar{A} \cup A^c$, where $\bar{A}$ is the closure of A in the strict topology. If A or A^c is nuclear, then A is X-modular.*

The condition that $L(X)$ is generated by $\bar{A} \cup A^c$ implies that A^c is not too small. The author does not know if the nuclearity condition in Theorem 1.2 is indispensable, but a lemma used in the proof of the theorem does not hold without this assumption.

2. *Proof of Theorem* 1.2

We need a variation of the notion of the linking algebra of X introduced in [**3**], which will enable us to reduce the proof of Theorem 1.2 (in some sense) to the special case $X = B$. Recall that, to each pair $x, y \in X$, we can associate an operator $[x, y] \in L(X)$ defined by $[x, y]z = x\langle y, z\rangle$ $(z \in X)$, and the closed linear span $K(X)$ of all such operators is an ideal of $L(X)$. From now on, we write L and K instead of $L(X)$ and $K(X)$ most of the time since X is clear from the context. The *linking algebra* Λ of X (a variation of which was introduced in [**3**]) can be defined as the C*-subalgebra of $L(X \oplus B)$ consisting of all matrices of the form

$$\begin{bmatrix} a & x \\ y^* & b \end{bmatrix}$$

where $a \in L$, $b \in B$ and $x, y \in X$. Here, each $y \in X$ has been identified with the operator $b \mapsto yb$ from B to X, the adjoint of which is given by $y^*(x) = \langle y, x\rangle$ $(x \in X)$. We regard B, X and L as subsets of Λ in the obvious way:

$$B \cong \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}, \quad X \cong \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad L \cong \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}.$$

In this way, the inner product $\langle x, y\rangle$ is just the internal product x^*y in Λ.

DEFINITION AND REMARKS 2.1. For each $\ \in B(X)$, define $\ _\Lambda \in B(\Lambda)$ by

$$\ _\Lambda\left(\begin{bmatrix} a & x \\ y^* & b \end{bmatrix}\right) = \begin{bmatrix} 0 & (x) \\ 0 & 0 \end{bmatrix} \quad a \in L,\ b \in B,\ x, y \in X.$$

(1) If (e_k) and (f_k) are approximate units in A and B, respectively, then $\lim e_k x = x = \lim x f_k$ for each $x \in X$. This implies in particular that, for each $\lambda \in \Lambda$, the (1,2)-entry x of λ is contained in $[A\lambda B]$. It can be proved easily that $\ _\Lambda$ preserves all closed A-Λ-submodules of Λ if preserves all closed A-B-submodules of X. The advantage of the map $\ _\Lambda$ over is that $\ _\Lambda$ acts on a C*-algebra, and we can use the representation theory of C*-algebras to study $\ _\Lambda$.

(2) By a straightforward computation, the commutant of A in Λ is

$$\begin{bmatrix} A^c & 0 \\ 0 & B \end{bmatrix}$$

since A is nondegenerate in $L(X)$.

REMARKS 2.2. In the proof of Theorem 1.2, we shall need certain facts concerning representations of Λ (by a representation, we always mean a *-representation.) Let $\pi : \Lambda \to B(\mathscr{H})$ be a representation, and put $\mathscr{H}_L := [\pi(L)\mathscr{H}]$ and $\mathscr{H}_B := [\pi(B)\mathscr{H}]$. Since $LB = 0$ (the product is computed in Λ), the subspaces $\mathscr{H}_L$ and $\mathscr{H}_B$ of $\mathscr{H}$ are orthogonal. Note that, for each $x \in X$, we have

$$[\pi(x)\mathscr{H}] = [\pi(x)\pi(x)^*\mathscr{H}] = [\pi([x,x])\mathscr{H}] \subseteq [\pi(L)\mathscr{H}] = \mathscr{H}_L$$

and, similarly, $[\pi(x^*)\mathscr{H}] \subseteq \mathscr{H}_B$. In particular, $\pi(x) \mid \mathscr{H}_B^{\perp} = 0$, and $\pi(x)$ can be regarded as an operator from $\mathscr{H}_B$ to $\mathscr{H}_L$. Thus, by restriction, π induces three maps

$$\pi_L : L \to B(\mathscr{H}_L), \quad \pi_B : B \to B(\mathscr{H}_B) \quad \text{and} \quad \pi_X : X \to B(\mathscr{H}_B, \mathscr{H}_L)$$

where π_L and π_B are representations of L and B, respectively, and π_X is a linear contraction satisfying

$$\pi_X(x)^*\pi_X(y) = \pi_B(\langle x, y\rangle), \quad \pi_X(x)\pi_X(y)^* = \pi_L([x,y]) \tag{2.1}$$

and $\pi_X(axb) = \pi_L(a)\pi_X(x)\pi_B(b)$ for all $x, y \in X$ and $a \in L$, $b \in B$. It follows also that $\pi(\Lambda)\mathscr{H} \subseteq \mathscr{H}_L \oplus \mathscr{H}_B$, and hence $\mathscr{H}_L \oplus \mathscr{H}_B = \mathscr{H}$ if π is nondegenerate.

(1) If π is irreducible, then π_L is irreducible. Indeed, if $\mathscr{K} \subseteq \mathscr{H}_L$ is an invariant subspace for $\pi_L(L)$, then (by a straightforward computation) $\mathscr{K} \oplus [\pi(X)^*\mathscr{K}]$ is an invariant subspace for $\pi(\Lambda)$.

(2) If π is irreducible and $\pi_L(K) \neq 0$, then the map π_L is continuous on bounded sets where L and $B(\mathscr{H}_L)$ are equipped with the strict topology. To see this, note that $\pi_L \mid K$ is nondegenerate (since K is an ideal in L (see [**1**])), and L is the multiplier algebra of K, and hence the required continuity of π_L follows from [**8**, Proposition 2.5] (but can also be proved directly).

The following simple lemma, Lemma 2.3, a special case of which has already been used implicitly in [**10**, Proposition 2.1], will be needed. We denote by $\mathrm{M}_{m,n}(\Lambda)$ the set of all $m \times n$ matrices with entries in Λ, and $\mathrm{M}_n(\Lambda) = \mathrm{M}_{n,n}(\Lambda)$.

LEMMA 2.3. *Let A be a* C*-*subalgebra of a unital* C*-*algebra* Λ, *let n be a positive integer, let Z be a closed* Λ-A-*submodule of* Λ^n, *and let*

$$\mathbf{x} = (x_1, \ldots, x_n) \in \Lambda^n.$$

Suppose that $\mathbf{x} \notin Z$. *Then there exists an irreducible representation* $\pi: \Lambda \to B(\mathscr{H})$ *for some Hilbert space* $\mathscr{H}$ *and a projection* $p' \in \mathrm{M}_n(\pi(A)')$ *such that* $\pi_n(\mathbf{x})p' \neq 0$ *and* $\pi_n(Z)p' = 0$, *where* $\pi_n(\mathbf{a}) = (\pi(a_1), \ldots, \pi(a_n))$ *for each* $\mathbf{a} = (a_1, \ldots, a_n)$ *in* Λ^n.

Proof. Identify Λ^n with $\mathrm{M}_{1,n}(\Lambda)$ and with the set of all matrices in $\mathrm{M}_n(\Lambda)$ with the support in the first row; similarly, identify $\mathrm{M}_{n,1}(\Lambda)$ with the matrices in $\mathrm{M}_n(\Lambda)$ with the support in the first column. Then

$$J := [\mathrm{M}_{n,1}(\Lambda)Z]$$

is a closed left ideal in $\mathrm{M}_n(\Lambda)$ and $\mathbf{x} \notin J$. (Namely, $\mathbf{x} \in J$ would imply that $\mathbf{x} \in \mathrm{M}_{1,n}(\Lambda)\mathbf{x} \subseteq \mathrm{M}_{1,n}(\Lambda)J \subseteq [\mathrm{M}_{1,n}(\Lambda)\mathrm{M}_{n,1}(\Lambda)Z] = [\Lambda Z] = Z$.) Hence there exists a pure state ω on $\mathrm{M}_n(\Lambda)$ such that $\omega(J) = 0$ and $\omega(\mathbf{x}^*\mathbf{x}) \neq 0$ (see [**7**, p. 733]). Since each irreducible representation of $\mathrm{M}_n(\Lambda)$ is of the form π_n for some irreducible representation π of Λ, where π_n means π applied to entries of matrices in $\mathrm{M}_n(\Lambda)$ (see [**7**, 11.5.8]), it follows that there exists an irreducible representation $\pi: \Lambda \to B(\mathscr{H})$ and a unit vector $\xi \in \mathscr{H}^n$ such that $\pi_n(J)\xi = 0$ and $\pi_n(\mathbf{x})\xi \neq 0$. Denote by p' the projection with range $[\pi(A + \mathbb{C}1)^{(n)}\xi]$. Then we have $p' \in (\pi(A)^{(n)})' = \mathrm{M}_n(\pi(A)')$ (since the range of p' is invariant under $\pi(A)^{(n)}$), $\pi_n(\mathbf{x})p' \neq 0$ (since the range of p' contains ξ), and $\pi_n(J)p' = 0$ (since $\pi_n(J)p'\mathscr{H} = [\pi_n(J(A + \mathbb{C}_1)^{(n)})\xi] \subseteq [\pi_n(J)\xi] = 0$). Since $Z^*Z \subseteq J$, we have, in particular, $\pi_n(Z^*Z)p' = 0$, and hence $\pi_n(Z)p' = 0$.

Note that, with $a'_1, \ldots, a'_n$ being the elements of a suitable column of p' (where p' is as in the proof of Lemma 2.3), we have $a'_j \in \pi(A)'$ for all j and

$$\sum_{j=1}^{n} \pi(x_j)a'_j \neq 0 \quad \text{and} \quad \sum_{j=1}^{n} \pi(z_j)a'_j = 0 \quad \text{for all} \quad \mathbf{z} = (z_1, \ldots, z_n) \in Z. \tag{2.2}$$

We denote by $A \otimes B$ the minimal (equals spatial) tensor product of two C*-algebras, while the algebraic tensor product is denoted by $A \odot B$. Recall that a factor $N \subseteq B(\mathscr{H})$ is semidiscrete if the correspondence $a \otimes a' \mapsto aa'$ extends to an (isometric) *-isomorphism from $N \otimes N'$ to the C*-subalgebra of $B(\mathscr{H})$ generated by $N \cup N'$ (see [**5**]). A C*-algebra A is called nuclear if there is a unique C* norm on $A \odot B$ for each C*-algebra B. If A is nuclear, then $\overline{\pi(A)}$ is injective for each (factor) representation π of A, and hence it is semidiscrete by [**4**]. We use this fact in the proof of Theorem 1.2 through the application of Lemma 2.4.

LEMMA 2.4. *Let* $M \subseteq N \subseteq B(\mathscr{H})$ *be factors such that M or N is semidiscrete. If* $S \subseteq \overline{\overline{MN'}}$ (*equals the* C*-*algebra generated by* $M \cup N'$) *and* $p' \in M'$ *is a projection such that* $Sp' = 0$, *then the projection* $q' \in N'$ *with range* $[Np'\mathscr{H}]$ *also satisfies* $Sq' = 0$.

Proof. By minimality of the spatial tensor norm, there is a natural *-epimorphism $\vartheta: \overline{\overline{MM'}} \to M \otimes M'$ (given on elementary tensors by $\vartheta(aa') = a \otimes a'$). For each normal functional ρ on M, let $\Phi_\rho: M \otimes M' \to M'$ be the slice map (determined by $\Phi_\rho(a \otimes a') = \rho(a)a'$ (see [7, 12.4.36])), and let ${}_\rho = \Phi_\rho\vartheta$. Then ${}_\rho(xa') = {}_\rho(x)a'$ for all $x \in \overline{\overline{MM'}}$ and $a' \in M'$, and hence, for $x \in S$, the relation $xp' = 0$ implies that ${}_{\rho}(x)p' = 0$. However, for $x \in \overline{\overline{MN'}}$, we have ${}_{\rho}(x) \in N'$, and hence ${}_{\rho}(x)Np'\mathscr{H} = N\,{}_{\rho}(x)p'\mathscr{H} = 0$, and hence ${}_{\rho}(x)q' = 0$ for all $x \in S$. Thus ${}_{\rho}(xq') = 0$ for all $x \in S$, and, since this holds for all normal functionals ρ on M, it follows that $\vartheta(xq') = 0$. Since M or N is semidiscrete, the restriction of ϑ to $\overline{\overline{MN'}}$ is one-to-one (the commutant of a semidiscrete algebra is known to be semidiscrete), and hence $Sq' = 0$.

Remark 2.5. The hypothesis of semidiscreteness in Lemma 2.4 is not redundant. To see this, consider two different subfactors $M \subseteq N \subseteq B(\mathscr{H})$ such that $\overline{\overline{MN'}}$ contains $K(\mathscr{H})$, let $p' \in M' \backslash N'$ be a projection, and choose $x \in K(\mathscr{H})$ so that p' is the projection onto the kernel of x. Then $xp' = 0$, but $xq' \neq 0$ for every projection $q' > p'$. For a concrete example of such a pair of factors, let M be the left group von Neumann algebra of the free group G on two generators a_1, a_2, and let N' be the right group von Neumann algebra of the index 2 subgroup H of G generated by $\{a_1, a_2^2\}$ acting naturally on $L^2(G)$ (so that $M \subseteq N$ (see [7, 6.9.41])). A simple calculation shows that $M' \cap N = \mathbb{C}$, and hence the C*-algebra $\overline{\overline{MN'}}$ is irreducible, and, by arguments similar to those in [**7**, p. 862], one can show that $\overline{\overline{MN'}}$ contains all compact operators, and hence $x \in \overline{\overline{MN'}}$.

Proof of Theorem 1.2. Let Z be any closed A-B-submodule of X^n and let $\mathbf{x} \in X^n$. We must prove that, if $\mathbf{x} \notin Z$, then there exist $\mathbf{c} \in (A^c)^n$ and a closed A-B-submodule Y of X such that $\mathbf{c} \cdot \mathbf{x} \notin Y$ and $\mathbf{c} \cdot \mathbf{z} \in Y$ for all $\mathbf{z} \in Z$. We identify L, B and X with the corresponding subsets in the linking algebra Λ of X, as explained above. Since $\mathbf{x} \notin Z$, the element $\mathbf{x}^* := (x_1^*, \ldots, x_n^*) \in \Lambda^n$ is not contained in the closed Λ-A-submodule $[\Lambda Z^*]$ of Λ^n. (To see this, note that, for each $\mathbf{v} = (v_1, \ldots, v_n) \in [\Lambda Z^*] \subseteq \Lambda^n$, the n-tuple $(w_1, \ldots, w_n)$, where w_j is the (2,1) entry of the matrix $v_j \in \Lambda$, is in Z^*). Therefore, by Lemma 2.3 (applied to $[\Lambda Z^*]$ and $\mathbf{x}^*$ instead of Z and $\mathbf{x}$), there exists an irreducible representation $\pi : \Lambda \to B(\mathscr{H})$ and a projection $p' \in \mathrm{M}_n(\pi(A)')$ such that

$$\pi_n(\mathbf{x}^*)p' \neq 0 \quad \text{and} \quad \pi_n(ZZ^*)p' = 0. \tag{2.3}$$

By Remarks 2.2, π induces three representations

$$\pi_L : L \to B(\mathscr{H}_L), \quad \pi_B : B \to B(\mathscr{H}_B) \quad \text{and} \quad \pi_X : X \to B(\mathscr{H}_B, \mathscr{H}_L)$$

where $\mathscr{H}_L \oplus \mathscr{H}_B = \mathscr{H}$. By Remark 2.2(1), π_L is irreducible. Moreover, $\pi_L(K) \neq 0$, where $K = K(X)$; otherwise, the second identity in (2.1) would imply that $\pi_X(X) = 0$, but this is impossible, since $\pi_n(\mathbf{x}^*)p' \neq 0$. Therefore, by Remark 2.2(2), the representation π_L is strictly continuous on the unit ball of L. By hypothesis, L is generated by $\bar{A} \cup A^c$, and hence, by irreducibility of π_L, it follows that the C*-algebra generated by $\pi_L(\bar{A}) \cup \pi_L(A^c)$ is strongly dense in $B(\mathscr{H}_L)$. By [**8**, p. 12], the unit ball A_1 of A is strictly dense in the unit ball $\bar{A}_1$ of $\bar{A}$, and hence $\pi_L(A_1)$ is strongly dense in $\pi_L(\bar{A}_1)$ by continuity, and, consequently, $\pi_L(A)$ is strongly dense in $\pi_L(\bar{A})$. We conclude that the C*-algebra generated by $\pi_L(A) \cup \pi_L(A^c)$ is strongly dense in $B(\mathscr{H}_L)$, and hence

$$\pi_L(A)' \cap \pi_L(A^c)' = \mathbb{C}.$$

Since $\pi_L(A^c)$ and $\pi_L(A)$ commute, it follows now that the von Neumann algebras

$$M := \overline{\pi_L(A)} \quad \text{and} \quad N := \pi_L(A^c)'$$

satisfy $M \subseteq N$ and $M' \cap N = \mathbb{C}$; in particular, M and N are factors. By hypothesis, A or A^c is nuclear, and hence at least one of the factors M, N is semidiscrete.

From $\pi(A) = \pi_L(A)\oplus 0 \subseteq B(\mathscr{H}_L\oplus\mathscr{H}_B)$, we have $\pi(A)' = M'\oplus B(\mathscr{H}_B)$, and hence the projection p' can be decomposed as $p' = p'_1\oplus p'_2$, where $p'_1\in \mathrm{M}_n(M')$ and $p'_2\in \mathrm{M}_n(B(\mathscr{H}_B))$. From (2.3), it follows then by an easy calculation that

$$\pi_n(\mathbf{x}^*)p'_1 \neq 0 \quad \text{and} \quad (\pi_L)_n(ZZ^*)p'_1 = 0.$$

Note that we regard $S := (\pi_L)_n(ZZ^*)$ as a subset of $\mathrm{M}_n(\pi_L(L))$ (elements of Z are matrices supported in the first column), and hence S is contained in the C*-algebra $\overline{\overline{M^{(n)}N^{(n)\prime}}}$, where $M^{(n)} = M\otimes 1$ (since $\overline{\overline{MN'}} \supseteq \overline{\overline{\pi_L(A)\pi_L(A^c)}} = \pi_L(L)$). Applying Lemma 2.4, we find a projection $q'\in \mathrm{M}_n(N')$ such that $q' \geqslant p'$ and $(\pi_L)_n(ZZ^*)q' = 0$. It follows that

$$\pi_n(\mathbf{x}^*)q' \neq 0 \quad \text{and} \quad \pi_n(Z^*)q' = 0.$$

Denoting by $d'_1, \ldots, d'_n$ the elements of a suitable column of the matrix q', we have

$$\sum_{j=1}^{n} \pi(x_j)^* d'_j \neq 0 \quad \text{and} \quad \sum_{j=1}^{n} \pi(z_j)^* d'_j = 0$$

for all $\mathbf{z} = (z_1, \ldots, z_n)\in Z$. Choose a unit vector $\eta\in\mathscr{H}_L$ such that $\sum_{j=1}^{n} \pi(x_j)^* d'_j\eta \neq 0$, and let e' be the projection with range $[N\eta]$. Then $e'\in N' = \overline{\pi_L(A^c)}$, and

$$\sum_{j=1}^{n} \pi(x_j)^* d'_j e' \neq 0 \quad \text{and} \quad \sum_{j=1}^{n} \pi(z_j)^* d'_j e' = 0$$

for all $\mathbf{z}\in Z$. Using the noncommutative Lusin theorem (see [**11**] or [**15**]), it follows that there exists a subprojection $f' \leqslant e'$ in N' that is so close to e' that

$$\sum_{j=1}^{n} \pi(x_j)^* d'_j f' \neq 0$$

and there exist elements $c_j^*\in A^c$ such that $\pi_L(c_j^*)f' = d'_j f'$ ($j = 1, \ldots, n$). Then we have

$$\pi\left(\sum_{j=1}^{n} c_j x_j\right)^* f' \neq 0 \quad \text{and} \quad \pi\left(\sum_{j=1}^{n} c_j z_j\right)^* f' = 0 \tag{2.4}$$

for all $\mathbf{z}\in Z$. Since $f'\in\overline{\pi_L(A^c)}$, f' commutes with $\pi(A)$, and hence the set

$$Y := \{v\in X : \pi(v)^* f' = 0\}$$

is a closed A-B-submodule of X. Finally, (2.4) shows that $\mathbf{c}\cdot\mathbf{x}\notin Y$ and $\mathbf{c}\cdot\mathbf{z}\in Y$ for all $\mathbf{z}\in Z$, where $\mathbf{c} := (c_1, \ldots, c_n)\in (A^c)^n$.

Problem 2.6. Is the condition of nuclearity in Theorem 1.2 necessary?

3. *The reflexivity of* $\overline{A_X}^{\mathrm{pn}}$

Now we shall consider the connection between X-modularity and mappings that preserve submodules, which is the motivation for introducing the notion of X-modularity.

Proposition 3.1. *If $A \subseteq L(X)$ is X-modular, then $\overline{A_X}^{\mathrm{pn}}$ is reflexive.*

Proof. Suppose that $\in B(X)$ preserves all closed A-B-submodules of X. To prove that $\in \overline{A_X}^{\mathrm{pn}}$, it suffices to show that, for each n, the map ${}^{(n)} : X^n \to X^n$ (defined by applying on components) preserves all closed A-B-submodules of X^n. Since A is X-modular, it suffices to show that $^{(n)}$ preserves all A-B-submodules of the form $Y(\mathbf{c})$, where $\mathbf{c} \in (A^c)^n$ and Y is a closed A-B-submodule of X, and hence it suffices to prove that commutes with A^c, or (equivalently) that the induced map $_\Lambda$ on Λ (see Definition and Remarks 2.1) commutes with the commutant $C := A^c \oplus B$ of A in Λ.

Identify Λ with its image under the universal representation, and extend $_\Lambda$ to the weak-operator continuous map ψ on the weak* closure $\bar{\Lambda}$ of Λ. Choose any $c = c^* \in C$, and let p be the spectral projection of c corresponding to some open subset of the spectrum of c. Since, by the spectral theory, p is a strong-operator limit of a sequence dominated by p in the C*-algebra generated by c, and hence a sequence in $p\Lambda \cap \Lambda$, we have $p\bar{\Lambda} = \overline{p\Lambda \cap \Lambda}$. Since p commutes with A, $p\Lambda \cap \Lambda$ is a norm closed A-Λ-submodule of Λ, and hence $_\Lambda(p\Lambda \cap \Lambda) \subseteq p\Lambda \cap \Lambda$, and it follows that $\psi(p\bar{\Lambda}) \subseteq p\bar{\Lambda}$. Since a spectral projection corresponding to a closed set can be approximated by larger spectral projections corresponding to open sets, we also have $\psi(p^\perp\bar{\Lambda}) \subseteq p^\perp\bar{\Lambda}$, and it follows that ψ commutes with p. Thus ψ commutes with spectral projections of c (corresponding to open sets), and hence ψ (and therefore also $_\Lambda$) commutes with c. This proves that $_\Lambda$ commutes with C.

We show by an example that the converse of Proposition 3.1 is not true: the reflexivity of $\overline{A_X}^{\mathrm{pn}}$ does not necessarily imply that A is X-modular.

EXAMPLE 3.2. Let $X = B = K(\mathscr{H}^2) + \mathbb{C}1_{\mathscr{H}^2}$ and $A = K(\mathscr{H})^{(2)} + \mathbb{C}1_{\mathscr{H}^2}$, where $\mathscr{H}$ is an infinite dimensional Hilbert space, $1_{\mathscr{H}^2}$ is the identity operator on $\mathscr{H}^2$, $K(\mathscr{H})$ is the ideal of compact operators on $\mathscr{H}$, and $K(\mathscr{H})^{(2)}$ is the algebra of diagonal 2×2 matrices with the same element from $K(\mathscr{H})$ on the diagonal. Then $L(X) = B$ and $A^c = \mathbb{C}1_{\mathscr{H}^2}$. Choose any two linearly independent operators $a, b \in K(\mathscr{H})$, and let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in X^2$ be defined by

$$x_1 = y_2 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \quad \text{and} \quad x_2 = y_1 = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}.$$

Let $Z = [A\mathbf{x}B]$. We shall prove that $\mathbf{c} \cdot \mathbf{y} \in [\mathbf{c} \cdot Z]$ for each $\mathbf{c} \in (A^c)^2$ (hence $\mathbf{y}$ is in the intersection of all closed A-B-submodules of the form $Y(\mathbf{c})$ containing Z, where $\mathbf{c} \in (A^c)^2$ and Y is a closed A-B-submodule of X), but that nevertheless $\mathbf{y} \notin Z$. Thus Z is not an intersection of modules of the form $Y(\mathbf{c})$. This means that A is not X-modular.

In fact, we claim that $[\mathbf{c} \cdot Z] = K(\mathscr{H}^2)$ for each nonzero $\mathbf{c} = (c_1, c_2) \in (A^c)^2 \cong \mathbb{C}^2$. To see this, note that $[\mathbf{c} \cdot Z]$ is a closed right ideal in $K(\mathscr{H}^2)$ and is invariant for the left multiplication by A, and hence $[\mathbf{c} \cdot Z] = pK(\mathscr{H}^2)$ for some projection $p \in A'$, where $A' = \mathrm{M}_2(\mathbb{C}1_{\mathscr{H}})$ is the commutant of A in $B(\mathscr{H}^2)$. Suppose that $[\mathbf{c} \cdot Z] \neq K(\mathscr{H}^2)$. Then $p^\perp \neq 0$, and, from $p^\perp \mathbf{c} \cdot \mathbf{x} = 0$, we see that the rows of the matrix

$$\mathbf{c} \cdot \mathbf{x} = c_1 x_1 + c_2 x_2 = \begin{bmatrix} c_1 a & c_2 a \\ c_2 b & c_1 a \end{bmatrix}$$

are linearly dependent. Thus, for suitable $\lambda_1, \lambda_2 \in \mathbb{C}$, not both 0, we have

$$\lambda_1 c_1 a + \lambda_2 c_2 b = 0 \quad \text{and} \quad \lambda_1 c_2 a + \lambda_2 c_1 a = 0.$$

Since a and b are linearly independent, these identities imply that $\lambda_1 c_1 = 0$, $\lambda_2 c_2 = 0$ and $\lambda_1 c_2 + \lambda_2 c_1 = 0$, and hence $c_1 = c_2 = 0$. This proves that $[\mathbf{c} \cdot Z] = K(\mathscr{H}^2)$ for each nonzero $\mathbf{c} \in (A^c)^2$.

If $\mathbf{y} \in Z$, then we would have $\mathbf{c} \cdot \mathbf{y} \in [\mathbf{c} \cdot Z]$ for each $\mathbf{c} = (c_1, c_2) \in A'^2$, but $A' \cong \mathrm{M}_2(\mathbb{C})$, and, with

$$c_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad c_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix},$$

we have $\mathbf{c} \cdot Z = 0$ (since $\mathbf{c} \cdot \mathbf{x} = 0$) and $\mathbf{c} \cdot \mathbf{y} \neq 0$. Thus $\mathbf{y} \notin Z$, and therefore A is not B-modular.

We now show that, nevertheless, each $\ \in B(B)$ that preserves all closed A-B-submodules of B is in $\overline{A_X}^{\mathrm{pn}}$. Suppose, on the contrary, that, for some n and some $\mathbf{x} = (x_1, \ldots, x_n) \in B^n$, the element $\ ^{(n)}(\mathbf{x})$ is not in $[A\mathbf{x}B]$. Then there exists an irreducible representation $\pi: B \to B(\mathscr{H})$ and elements $a_j' \in \pi(A)'$ such that

$$\sum_{j=1}^{n} a_j' \pi(x_j) = 0 \quad \text{and} \quad \sum_{j=1}^{n} a_j' \pi(\ (x_j)) \neq 0. \tag{3.1}$$

(This can be deduced in the same way as formulas (2.2) following the proof of Lemma 2.3.) Since $B = K(\mathscr{H}^2) + \mathbb{C}1_{\mathscr{H}^2}$, π is either the representation on the one-dimensional space (hence $\ker \pi = K(\mathscr{H}^2)$), or π is (equivalent to) the identity representation. In the first case, $\pi(A)' = \mathbb{C}$, and hence (3.1) can be rewritten as

$$\pi\left(\sum_{j=1}^{n} a_j' x_j\right) = 0 \quad \text{and} \quad \pi\ \left(\sum_{j=1}^{n} a_j' x_j\right) \neq 0.$$

With $y = \sum_{j=1}^{n} a_j' x_j$, we then have $y \in \ker \pi$ and $\ (y) \notin \ker \pi$, but this is a contradiction, since preserves all closed two-sided ideals of B. Hence π must be the identity representation. If we decompose $x_j = y_j + \lambda_j 1_{\mathscr{H}^2}$, where $y_j \in K(\mathscr{H}^2)$ and $\lambda_j \in \mathbb{C}$, then (3.1) can be rewritten as

$$\sum_{j=1}^{n} a_j' y_j + \sum_{j=1}^{n} \lambda_j a_j' = 0 \quad \text{and} \quad \sum_{j=1}^{n} a_j' \ (y_j) + \sum_{j=1}^{n} \lambda_j a_j' \ (1_{\mathscr{H}^2}) \neq 0. \tag{3.2}$$

Since $A' = \mathrm{M}_2(\mathbb{C}1_{\mathscr{H}})$, $\sum_{j=1}^{n} \lambda_j a_j'$ is compact if and only if $\sum_{j=1}^{n} \lambda_j a_j' = 0$; hence it follows from (3.2) that

$$\sum_{j=1}^{n} a_j' y_j = 0 \quad \text{and} \quad \sum_{j=1}^{n} a_j' \ (y_j) \neq 0. \tag{3.3}$$

However, since preserves closed A-B-submodules of $K(\mathscr{H}^2)$, commutes with all (projections, and hence with all) elements of A'. The two relations (3.3) are therefore in contradiction.

Example 3.3 shows that there exists a Hilbert module X and a unital C*-algebra A in $L(X)$ such that $\overline{A_X}^{\mathrm{pn}}$ is not reflexive.

EXAMPLE 3.3. Let s be the unilateral shift on a separable Hilbert space $\mathscr{H}$, put

$$\tilde{s} = \begin{bmatrix} 0 & s \\ 1_{\mathscr{H}} & 0 \end{bmatrix} \in B(\mathscr{H}^2),$$

let B be the C*-subalgebra of $B(\mathscr{H}^2)$ generated by $\tilde{s}$, let A be the C*-subalgebra of B generated by $\tilde{s}^2$, and regard B as a Hilbert B-module in the usual way.

Since $\tilde{s}^2 = s \oplus s$, A consists of all matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

where $a \in \mathrm{C}^*(s)$. Since s is well known to be an irreducible operator, $A' = \mathrm{M}_2(\mathscr{C}1_{\hat{\mathscr{H}}})$. Note that B is irreducible and $B \supset K(\mathscr{H}^2)$. Since $\mathrm{C}^*(s)/K(\mathscr{H})$ is commutative (in fact, isomorphic to the continuous functions $C(T)$, where T is the unit circle), and s is a unitary modulo $K(\mathscr{H})$, the set of all compact perturbations of matrices of the form

$$t = \begin{bmatrix} a & sb \\ b & a \end{bmatrix} \quad a, b \in \mathrm{C}^*(s)$$

is a C*-algebra, which is easily seen to be equal to B.

Claim 3.4. If a closed A-B-submodule Y of B contains some non-compact operator, then $Y \supseteq K(\mathscr{H}^2)$.

Proof. Since $Y \cap K(\mathscr{H}^2)$ is a right ideal of $K(\mathscr{H}^2)$ invariant for the left multiplication by A, $Y \cap K(\mathscr{H}^2) = pK(\mathscr{H}^2)$ for some projection $p \in A'$, and we must prove that $p^\perp = 0$. Suppose that $p^\perp \neq 0$, and let

$$t = \begin{bmatrix} a & sb \\ b & a \end{bmatrix} + c$$

be a non-compact operator in Y, where $a, b \in \mathrm{C}^*(s)$ and $c \in K(\mathscr{H}^2)$. From $tK(\mathscr{H}^2) \subseteq Y \cap K(\mathscr{H}^2)$, we have $p^\perp t K(\mathscr{H}^2) = 0$, and hence $p^\perp t = 0$. Since $p^\perp \in A' = \mathrm{M}_2(\mathbb{C}1_{\mathscr{H}})$, it follows that, for some $\alpha, \beta \in \mathbb{C}$, not both 0, we have

$$\alpha a + \beta b \in K(\mathscr{H}) \quad \text{and} \quad \alpha s b + \beta a \in K(\mathscr{H}). \tag{3.4}$$

Denoting by $\dot{x} \in \mathrm{C}^*(s)/K(\mathscr{H}) = C(T)$ the coset of an element $x \in \mathrm{C}^*(s)$, expressions (3.4) imply that

$$(\alpha^2 \dot{s} - \beta^2)\dot{b} = 0 \quad \text{and} \quad (\alpha^2 \dot{s} - \beta^2)\dot{a} = 0.$$

Since $\dot{s}$ is just the identity function on T, the last two identities imply that $\dot{a} = 0$ and $\dot{b} = 0$. However, this is in contradiction with the fact that t is not compact. Hence $p^\perp$ must be 0. This proves Claim 3.4.

Let $\pi : B \to B/K(\mathscr{H}^2)$ be the natural map, let $\Phi : B/K(\mathscr{H}^2) \to K(\mathscr{H}^2)$ be any bounded linear map, and let $= \Phi\pi$. It follows easily from Claim 3.4 that preserves all closed A-B-submodules of B. We now choose Φ so that will not be in $\overline{A_B}^{\mathrm{pn}}$.

Choose any non-zero $e \in K(\mathscr{H})$, and put

$$x_1 = \begin{bmatrix} 1_{\mathscr{H}} & 0 \\ 0 & 1_{\mathscr{H}} \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 & 1_{\mathscr{H}} \\ s^* & 0 \end{bmatrix}, \quad z_1 = \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}, \quad z_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Choose Φ so that $\Phi(\pi(x_j)) = z_j$ $(j = 1, 2)$, and put

$$a_1' = \begin{bmatrix} 0 & 1_{\mathscr{H}} \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad a_2' = \begin{bmatrix} -1_{\mathscr{H}} & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $a_j' \in A'$, $(x_j) = z_j$ $(j = 1, 2)$, $a_1' x_1 + a_2' x_2 = 0$ and $a_1' z_1 + a_2' z_2 \neq 0$, and hence is not in $\overline{A_B}^{\mathrm{pn}}$.

References

1. W. B. ARVESON, *An invitation to* C*-*algebras*, Graduate Texts in Mathematics 39 (Springer, Berlin, 1976).
2. W. B. ARVESON, *Ten lectures on operator algebras*, CBMS Regional Conference Series in Mathematics 55 (American Mathematical Society, Providence, RI, 1984).
3. L. G. BROWN, P. GREEN and M. A. RIEFFEL, 'Stable isomorphism and strong Morita equivalence of C*-algebras', *Pacific J. Math.* 71 (1977) 349–363.
4. A. CONNES, 'Classification of injective factors', *Ann. Math.* 104 (1976) 73–115.
5. E. G. EFFROS and E. C. LANCE, 'Tensor products of operator algebras', *Adv. Math.* 25 (1977) 1–34.
6. B. E. JOHNSON, 'Centralizers and operators reduced by maximal ideals', *J. London Math. Soc.* 43 (1968) 231–233.
7. R. V. KADISON and J. R. RINGROSE, *Fundamentals of the theory of operator algebras* – Vol. 1 & Vol. 2 (Academic Press, London, 1983 & 1986).
8. E. C. LANCE, *Hilbert* C*-*modules*, London Mathematical Society Lecture Notes 210 (Cambridge University Press, Cambridge, 1995).
9. B. MAGAJNA, 'A transitivity theorem for algebras of elementary operators', *Proc. Amer. Math. Soc.* 118 (1993) 119–127.
10. B. MAGAJNA, 'Interpolation by elementary operators', *Studia Math.* 105 (1993) 77–92.
11. G. K. PEDERSEN, C*-*algebras and their automorphism groups* (Academic Press, London, 1979).
12. V. S. SHUL'MAN, 'Operator algebras with strongly cyclic vectors', *Mat. Zametki* 16 (1974) 253–257 (in Russian).
13. A. SINCLAIR and R. SMITH, *Hochschild cohomology of von Neumann algebras*, London Mathematical Society Lecture Notes 203 (Cambridge University Press, Cambridge, 1995).
14. R. R. SMITH, 'Completely bounded module maps and the Haagerup tensor product', *J. Funct. Anal.* 102 (1991) 156–175.
15. M. TAKESAKI, *Theory of operator algebras I* (Springer, New York, 1979).

Department of Mathematics
University of Ljubljana
Jadranska 19
Ljubljana 1000
Slovenia

E-mail: Bojan.Magajna@fmf.uni-lj.si