

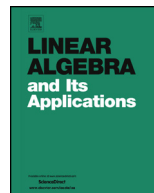


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Orthonormal pairs of operators

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ABSTRACT

We consider pairs of operators $A, B \in B(\mathcal{H})$, where \mathcal{H} is a Hilbert space, such that there exist a linear isometry f from the span of $\{A, B\}$ into \mathbb{C}^2 mapping A, B into orthonormal vectors. We prove some necessary conditions for the existence of such an f and determine all such pairs among commuting normal operators. Then we characterize all such pairs A, B (in fact, we consider general sets instead of just pairs) under the additional requirement that f is a complete isometry, when \mathcal{H} carries the column (or row) operator space structure. We also metrically characterize elements in a C*-algebra with orthogonal ranges.

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1. Introduction

Various notions of orthogonality for vectors in a Banach space were introduced already by Birkhoff [3] and James [10] (recent surveys by Bottazzi, Conde and Sain and by Grover and Sushil are [5] and [9]), which were investigated even in the context of Hilbert C*-modules by Arambašić and Rajić [1]. One possible natural definition of orthogonality,

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investigated by Eskandari, Moslehian and Popovici in [7] and called *Pythagoras orthogonality*, is the following: two vectors x, y in a normed space are orthogonal, which is denoted as $x \perp_P y$, if there exists a linear isometry f from the linear span of x and y into a Hilbert space such that the vectors $f(x)$ and $f(y)$ are orthogonal in the usual sense. This is a much more restricted notion than Pythagorean orthogonality and other types of orthogonality introduced in [10].

For the Banach space $B(\mathcal{H})$ of all bounded linear operators on a Hilbert space \mathcal{H} , perhaps the most studied kind of orthogonality, the Birkhoff-James orthogonality, has been characterized in terms of numerical ranges [18], [12, 2.2] (although the term “Birkhoff-James orthogonality” was not used there) and by Bhatia and Šemrl [2]. Here we will study Pythagoras orthogonality in $B(\mathcal{H})$, especially in the case when \mathcal{H} is finite-dimensional, and an appropriate variant of it in the context of operator spaces.

In Section 2 we will present some necessary conditions for Pythagoras orthogonality. For example, for each such pair $A, B \in M_n(\mathbb{C})$ and each $\lambda \in \mathbb{C}$ the operator $A + \lambda B$ is not invertible. This is in sharp contrast with the situation in Hilbert spaces over \mathbb{R} , where the Pauli spin matrices are an example of mutually orthogonal hermitean unitary operators. In Section 3 we characterize Pythagoras orthogonality for commuting normal operators in terms of their joint spectrum. Then, following the paradigm that the natural maps between operator spaces are the completely bounded maps (instead of all bounded ones [4], [6], [15], [16]), we consider the central topic of this article. Namely, the question, which sets of operators in $B(\mathcal{K})$ can be mapped completely isometrically into orthonormal sets in a Hilbert space \mathcal{H} , where \mathcal{H} carries the column operator space structure as defined e.g. in [6], [16]. We call such sets of operators *column orthonormal*. It turns out in Section 4 that, in contrast to the usual Pythagoras orthogonality, column orthogonal sets of operators have a more definitive simple characterization. A set of norm 1 operators $C_j \in B(\mathcal{H})$ is shown to be column orthogonal if and only if $\|\sum_j C_j C_j^*\| \leq 1$ and there exists a representation π of $B(\mathcal{H})$ on some Hilbert space \mathcal{K} and a cyclic vector $\xi \in \mathcal{K}$ for π such that $\|\pi(C_j)\xi\| = 1$ simultaneously for all j . In Section 5 we study what happens if in the condition $\|A\alpha + B\beta\|^2 = \|A\|^2|\alpha|^2 + \|B\|^2|\beta|^2$ for Pythagoras orthogonality we replace scalars α and β with elements X, Y of a C^* -algebra \mathcal{A} and replace $\|A\|^2, \|B\|^2$ with positive elements $P, Q \in \mathcal{A}$. That is, for fixed positive $P, Q \in \mathcal{A}$ we characterize all pairs of elements $A, B \in \mathcal{A}$ that satisfy the identity

$$\|X^*A^*AX + Y^*B^*BY\| = \|X^*PX + Y^*QY\|$$

for all $X, Y \in \mathcal{A}$. We show that in this case A and B have orthogonal ranges (that is, $A^*B = 0$) and, in the special case when $Q = P$ is a projection, A and B are necessarily partial isometries with the same initial projection P , so that A and B are column orthonormal. Finally, in Section 6 we return to Pythagoras orthogonality and describe all operators A that are Pythagoras orthogonal to a projection of rank 1. This indicates that such orthogonality depends on the action of A on the entire Hilbert space.

Note that if $A \perp_P B$ for operators $A, B \in B(\mathcal{H})$, then also $f(A) \perp_P f(B)$ for each linear or conjugate-linear isometry f of $B(\mathcal{H})$. Particular examples of linear isometries on $B(\mathcal{H})$ are: (i) maps of the form $X \mapsto UXV$, where $U, V^* \in B(\mathcal{H})$ are isometries, and (ii) $X \mapsto X^t$. (All linear surjective isometries are known to be compositions of this to types [11, 10.5.26, 10.5.32], [17].) The map $X \mapsto X^*$ is a conjugate linear isometry. Thus, $A \perp_P B$ if and only if $A^* \perp_P B^*$. Recall that for $B \in B(\mathcal{H})$ at least one of the operators B, B^* has a polar decomposition in which the partial isometric part is an isometry. Suppose that $B = U|B|$, where U^* is an isometry. Then $A \perp_P B$ implies that $U^*A \perp_P |B|$. The reverse implication also holds, since U is isometric on the range of U^* , which contains the ranges of U^*A and $|B|$. Also, multiplying A and B by nonzero scalars does not change orthogonality. Therefore, for shorter formulation of results we will often assume that

A and B are linear operators on a Hilbert space \mathcal{H} with $\|A\| = 1 = \|B\|$.

2. Some necessary conditions for Pythagoras orthogonality

By definition, for $A, B \in B(\mathcal{H}, \mathcal{K})$ with $\|A\| = 1 = \|B\|$, the condition $A \perp_P B$ means that $\|A + \lambda B\|^2 = 1 + |\lambda|^2$. This can be written as

$$\|(A + \lambda B)^*(A + \lambda B)\| = 1 + |\lambda|^2.$$

Since for a positive operator T with the spectrum $\sigma(T)$ the relations $\|T\| \in \sigma(T)$ and $T \leq \|T\|I$ hold, where I is the identity operator, we can reformulate the orthogonality condition in the following way, stated as a lemma for easier reference:

Lemma 2.1. *Suppose that $\|A\| = 1 = \|B\|$, where $A, B \in B(\mathcal{H}, \mathcal{K})$. Then $A \perp_P B$ if and only if for all $\lambda \in \mathbb{C}$ the operator*

$$(1 + |\lambda|^2)I - (A + \lambda B)^*(A + \lambda B)$$

is positive and not invertible.

Proposition 2.2. *Suppose that $A, B \in M_n(\mathbb{C})$ and $\|A\| = 1 = \|B\|$. If $A \perp_P B$ then for each $\lambda \in \mathbb{C}$ the operator $A + \lambda B$ is not invertible.*

Proof. By Lemma 2.1 we have

$$\det[(|\lambda|^2 + 1)I - (A + \lambda B)^*(A + \lambda B)] = 0$$

for each $\lambda \in \mathbb{C}$. For $\lambda \in \mathbb{R}$ this can be rewritten as

$$\det[\lambda^2(I - B^*B) - \lambda(B^*A + A^*B) + I - A^*A] = 0.$$

Since the left side of this equality is a polynomial in λ , it must be identically 0 for all $\lambda \in \mathbb{C}$. In particular for $\lambda = i$ we get $\det[B^*B - A^*A - i(B^*A + A^*B)] = 0$, which we may rewrite as

$$\det[(B^* - iA^*)(B - iA)] = 0.$$

We may replace in this argument A by ωA for any $\omega \in \mathbb{C}$ with $|\omega| = 1$, since $\omega A \perp_P B$ and $\|\omega A\| = 1$. Thus $\det(B + i\omega A) = \det(B^* - i\overline{\omega}A^*) = 0$ or $\det(B - i\omega A) = 0$, and at least one of these two possibilities holds for infinitely many values of ω . Since these determinants are polynomials in ω , it follows that at least one of them is identically 0 for all $\omega \in \mathbb{C}$ and this clearly implies that $A + \lambda B$ is not invertible for $\lambda \in \mathbb{C}$. \square

Remark 2.3. Since Pythagoras orthogonality is symmetric relation, it follows that in Proposition 2.2 also the operator B is not invertible.

Problem. Can Proposition 2.2 be generalized to operators on infinite dimensional Hilbert spaces?

Suppose that there exists a unit vector $\xi \in \mathcal{H}$ so that $\|B\xi\| = \|B\| = 1$. If $B \geq 0$, this means that $B\xi = \xi$ (since $0 \leq \|(I - B)\xi\|^2 = 2 - 2\|\sqrt{B}\xi\|^2 \leq 0$, where the inequality follows from $1 = \|\sqrt{B}\sqrt{B}\xi\| \leq \|\sqrt{B}\xi\|$). If $A \perp_P B$, then from the inequality

$$\|A\xi\|^2 + 2\operatorname{Re}(\lambda\langle\xi, A\xi\rangle) + |\lambda|^2 = \|(A + \lambda B)\xi\|^2 \leq \|A + \lambda B\|^2 = 1 + |\lambda|^2,$$

which can be written as $2\operatorname{Re}(\lambda\langle\xi, A\xi\rangle) \leq 1 - \|A\xi\|^2$, we conclude (by considering $|\lambda| \rightarrow \infty$) the following lemma:

Lemma 2.4. *Suppose that $\|A\| = 1 = \|B\|$, $B \geq 0$ and that $A \perp_P B$. If $\xi \in \mathcal{H}$ is such that $\|\xi\| = 1 = \|B\xi\|$, then $\langle A\xi, \xi \rangle = 0$.*

Now we can easily classify all pairs of orthogonal operators A, B in $M_2(\mathbb{C})$.

Proposition 2.5. *The only pair $A, B \in M_2(\mathbb{C})$ with $\|A\| = 1 = \|B\|$ that satisfies $A \perp_P B$ is, up to an isometry of $M_2(\mathbb{C})$,*

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.1}$$

Proof. We may assume $B \geq 0$. Since B is not invertible by Proposition 2.2, one of the eigenvalues of B is 0, hence we may assume that B is as stated in (2.1). Then it follows from Lemma 2.4 that A is of the form

$$A = \begin{bmatrix} 0 & \beta \\ \gamma & \delta \end{bmatrix} \quad (\beta, \gamma, \delta \in \mathbb{C}).$$

But by Proposition 2.2 $A + \lambda B$ is not invertible, hence $0 \equiv \det(A + \lambda B) = \delta\lambda - \beta\gamma$. Thus $\delta = 0$ and $\beta\gamma = 0$. If $\gamma = 0$, we apply the transposition, so in any case we can achieve that A is of the form

$$A = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}, \text{ where } |\gamma| = 1.$$

Multiplying A and B from the left by the unitary matrix $U = \begin{bmatrix} 1 & 0 \\ 0 & \bar{\gamma} \end{bmatrix}$ we arrive to the pair (A, B) as stated in the proposition. \square

If $A, B \in B(\mathcal{H})$ and $A \perp_P B$, then

$$H := \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix} \text{ and } K := \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

are selfadjoint and $H \perp_P K$. However, two nonzero selfadjoint operators on \mathbb{C}^n can not be Pythagoras orthogonal if one of them is positive. This is a consequence of the following lemma.

Lemma 2.6. *If $A, B \in B(\mathbb{C}^n)$ are selfadjoint, $B \geq 0$ and $\det(A + \lambda B) = 0$ for all $\lambda \in \mathbb{C}$, then $\ker A \cap \ker B \neq 0$. Hence A and B are simultaneously unitarily similar to matrices that both have the last column and the last row equal to 0.*

Proof. Replacing A and B by S^*AS and S^*BS , where $S \in M_n(\mathbb{C})$ is invertible, does not change the problem, hence we may assume that B is a projection, so that relative to the decomposition $\mathbb{C}^n = B(\mathbb{C}^n) \oplus \ker B$ the two operators are

$$A = \begin{bmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

We must prove that there exists a nonzero vector in

$$\ker \begin{bmatrix} A_2 \\ A_3 \end{bmatrix},$$

or equivalently, that the rank of this matrix is less than the number of its columns. If we multiply A and B from the left by an invertible block-diagonal matrix of the form $U = I \oplus V$ and from the right by U^* , this does not change the problem since

$$\begin{bmatrix} A_2 \\ A_3 \end{bmatrix} \text{ transforms into } \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \end{bmatrix} V^*,$$

which does not change the rank. Thus we may assume (using an appropriate choice of V) that A_3 is diagonal, of the form $A_3 = I \oplus 0$, so that A and B have now the form

$$A = \begin{bmatrix} A_1 & C_1 & C_2 \\ C_1^* & I & 0 \\ C_2^* & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Multiplying the matrix

$$A + \lambda B = \begin{bmatrix} \lambda I + A_1 & C_1 & C_2 \\ C_1^* & I & 0 \\ C_2^* & 0 & 0 \end{bmatrix}$$

from the left by the matrix

$$P = \begin{bmatrix} I & -C_1 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and from the right by P^* , transforms $A + \lambda B$ into

$$A(\lambda) := \begin{bmatrix} \lambda I + A_1 - C_1 C_1^* & 0 & C_2 \\ 0 & I & 0 \\ C_2^* & 0 & 0 \end{bmatrix}.$$

By this operation the sub-matrix consisting of the last two block-columns of $A + \lambda B$ has been simply multiplied from the left by P , which can not change the rank. Now we have

$$0 = \det A(\lambda) = \det \begin{bmatrix} \lambda I + A_1 - C_1 C_1^* & C_2 \\ C_2^* & 0 \end{bmatrix}. \tag{2.2}$$

For large enough λ the matrix $\lambda I + A_1 - C_1 C_1^*$ is invertible, hence (2.2) implies that

$$\begin{aligned} 0 &= \det \begin{bmatrix} I & 0 \\ -C_2^*(\lambda I + A_1 - C_1 C_1^*)^{-1} & I \end{bmatrix} \det \begin{bmatrix} \lambda I + A_1 - C_1 C_1^* & C_2 \\ C_2^* & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} \lambda I + A_1 - C_1 C_1^* & C_2 \\ 0 & -C_2^*(\lambda I + A_1 - C_1 C_1^*)^{-1} C_2 \end{bmatrix} \\ &= -\det(\lambda I + A_1 - C_1 C_1^*) \det(C_2^*(\lambda I + A_1 - C_1 C_1^*)^{-1} C_2). \end{aligned}$$

Thus the square matrix $C_2^*(\lambda I + A_1 - C_1 C_1^*)^{-1} C_2$ is not invertible, hence not injective. Since $\lambda I + A_1 - C_1 C_1^* \geq 0$ if $\lambda \in \mathbb{R}$ is large enough, we can take the square root and it

follows that $(\lambda I + A_1 - C_1 C_1^*)^{-1/2} C_2$ is not injective, hence C_2 is not injective and so the columns of C_2 must be linearly dependent.

We have shown that there exists a unit vector in $\ker A \cap \ker B$. Choosing this vector as the last vector of an orthonormal basis of \mathbb{C}^n , we can represent operators A and B by matrices that have the last row and the last column identically 0. \square

Proposition 2.7. *If $A, B \in M_n(\mathbb{C})$ are nonzero, selfadjoint and $B \geq 0$, then $A \not\perp_P B$.*

Proof. Assume the contrary, that $A \perp_P B$. We may suppose that $\|A\| = 1 = \|B\|$. Then $\det(A + \lambda B) = 0$ for all $\lambda \in \mathbb{C}$ by Proposition 2.2, hence by Lemma 2.6 we may suppose that A and B have the last row and the last column equal to 0. In other words, $A = A_1 \oplus 0$ and $B = B_1 \oplus 0$, where A_1 and B_1 are self-adjoint matrices of size $(n - 1) \times (n - 1)$, with $B_1 \geq 0$ and $\|A_1\| = 1 = \|B_1\|$. It is easy to verify that $A_1 \perp_P B_1$, hence this reduces the problem to one dimension smaller. Continuing in this way we arrive after finite number of steps at 1×1 matrices α, β , that is $\alpha, \beta \in \mathbb{R}$, such that $\beta = 1 = |\alpha|$ and $\alpha \perp_P \beta$. This means that $|\alpha + \lambda|^2 = |\alpha + \lambda\beta|^2 = 1 + |\lambda|^2$ for all $\lambda \in \mathbb{C}$, which is clearly a contradiction. \square

If $P \in B(\mathcal{H})$ is a projection and $U \in B(\mathcal{H})$ is a partial isometry such that $U^*U = P$ and $UU^* = P^\perp = I - P$, then it is easy to verify that $P \perp_P U$. Proposition 2.5 shows that this is essentially the only example of a Pythagoras orthonormal pair on $\mathcal{H} = \mathbb{C}^2$. The previous two propositions could lead to an impression that orthogonal pairs of operators are rare, but in fact, already on \mathbb{C}^3 there are many such pairs. Below we present two classes of examples that were found with the help of Proposition 2.2 and Lemma 2.1. Additional example, namely, the description of all operators that are Pythagoras orthogonal to a projection of rank 1, is postponed to Section 6, since it requires a lot of computation.

Example 2.8. (i) Here is an example, where the kernels of A and B have only 0 in their intersection and the same holds for the intersection of kernels of A^* and B^* .

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \alpha & 0 \\ 0 & \beta & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $0 < b < 1$, $\beta \neq 0$ and $|\alpha|^2 \leq (1 - |\beta|^2)(1 - b^2)$ (hence also $|\alpha|^2 + |\beta|^2 \leq 1$). Let us verify that this pair of operators satisfies the condition for orthogonality in Lemma 2.1. Indeed, $\|A\| = 1 = \|B\|$ and the eigenvalues of the matrix

$$(|\lambda|^2 + 1)I - (A + \lambda B)^*(A + \lambda B) = \begin{bmatrix} 1 & 0 & -\bar{\lambda} \\ 0 & (1 - b^2)|\lambda|^2 - \bar{\alpha}b\lambda - \alpha b\bar{\lambda} + 1 - |\alpha|^2 - |\beta|^2 & 0 \\ -\lambda & 0 & |\lambda|^2 \end{bmatrix}$$

are 0, $|\lambda|^2 + 1$ and $(1 - b^2)|\lambda|^2 - \bar{\alpha}b\lambda - \alpha b\bar{\lambda} + 1 - |\alpha|^2 - |\beta|^2$. Thus we have only to show that the last eigenvalue is non-negative for all $\lambda \in \mathbb{C}$. This is equivalent to the fact that the matrix

$$\begin{bmatrix} 1 - b^2 & -\bar{\alpha}b \\ -\alpha b & 1 - |\alpha|^2 - |\beta|^2 \end{bmatrix}$$

is positive, since its diagonal terms and the determinant are non-negative.

(ii) An example of a Pythagorean orthogonal pair in $M_{3,2}(\mathbb{C})$ is

$$A = \begin{bmatrix} 0 & 0 \\ u \sin \phi \sin \psi & w \sin \phi \cos \psi \\ v \cos \psi & -\bar{u}vw \sin \psi \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & \cos \phi \\ 0 & 0 \end{bmatrix},$$

where $u, v, w \in \mathbb{C}$ have absolute value 1 and $\phi, \psi \in \mathbb{R}$. Namely, it can be shown by a routine (although somewhat lengthy) computation, which we will omit, that this pair satisfies the criterion of Lemma 2.1. Here both A and B can have rank 2.

3. Pythagoras orthogonal pairs of commuting normal operators

If A and B are commuting normal operators then there is a polar decomposition $B = U|B|$ of B , where U is a unitary in the abelian W^* -algebra generated by A , B and the identity. Then the operators U^*A and $|B|$ commute and are normal, hence in studying the Pythagoras orthogonality for such operators there is no loss of generality in assuming that one of the operators is positive.

Recall [14, p. 22] that the (joint) spectrum $\sigma(A, B)$ of two commuting normal operators is defined as

$$\sigma(A, B) = \{(\omega(A), \omega(B)) : \omega \in \Delta\},$$

where Δ denotes the set of all multiplicative functionals on the C^* -algebra \mathcal{A} generated by A and B .

Proposition 3.1. *Let $A, B \in B(\mathcal{H})$ be commuting normal operators with $\|A\| = 1 = \|B\|$ and $B \geq 0$. Then $A \perp_P B$ if and only if the spectrum $\sigma(A, B)$ is contained in the unit half-ball $\{(\zeta, t) \in \mathbb{C} \times \mathbb{R} : |\zeta|^2 + t^2 \leq 1, t \geq 0\}$ and contains the hemisphere $\{(\zeta, t) \in \mathbb{C} \times \mathbb{R} : |\zeta|^2 + t^2 = 1, t \geq 0\}$.*

Proof. Since the norm of a normal operator is equal to its spectral radius the condition $A \perp_P B$ is equivalent to

$$\max_{\omega \in \Delta} |\omega(A + \lambda B)|^2 = 1 + |\lambda|^2 \quad \forall \lambda \in \mathbb{C}.$$

This condition means (using also $B \geq 0$) that

$$(1 - \omega(B)^2)|\lambda|^2 - 2\operatorname{Re}(\lambda\omega(A^*B)) + 1 - |\omega(A)|^2 \geq 0 \quad \forall \lambda \in \mathbb{C}, \forall \omega \in \Delta \tag{3.1}$$

and that for each $\lambda \in \mathbb{C}$ there exist $\omega_\lambda \in \Delta$ such that equality holds in (3.1) when $\omega = \omega_\lambda$. If $\omega(B) \neq 1$, we can write (3.1) as

$$|\sqrt{1 - \omega(B)^2}\lambda - \frac{\omega(B^*A)}{\sqrt{1 - \omega(B)^2}}|^2 + 1 - |\omega(A)|^2 - \frac{|\omega(B^*A)|^2}{1 - \omega(B)^2} \geq 0 \quad \forall \lambda \in \mathbb{C}, \tag{3.2}$$

which means that $1 - |\omega(A)|^2 - \frac{|\omega(B^*A)|^2}{1 - \omega(B)^2} \geq 0$, that is

$$|\omega(A)|^2 + \omega(B)^2 \leq 1. \tag{3.3}$$

This holds even if $\omega(B) = 1$, for in this case (3.1) implies (by considering $|\lambda| \rightarrow \infty$) that $\omega(A) = 0$. Further, since for $\omega = \omega_\lambda$ equality holds in (3.1), $\omega_\lambda(B) \neq 1$, for otherwise (3.3) would imply that $\omega_\lambda(A) = 0$ and then the equality could not hold in (3.1). Hence equality must hold also in (3.2). Thus for $\omega = \omega_\lambda$ equality holds in (3.3) and in (3.2). Using $|\omega_\lambda(A)|^2 + \omega_\lambda(B)^2 = 1$ we can simplify the equality case of (3.2) when $\omega = \omega_\lambda$ to $\omega_\lambda(B) = \bar{\lambda}\omega_\lambda(A)$, hence we have now

$$\forall \lambda \in \mathbb{C} \exists \omega_\lambda \in \Delta \text{ such that } |\omega_\lambda(A)|^2 + |\omega_\lambda(B)|^2 = 1 \text{ and } \omega_\lambda(B) = \bar{\lambda}\omega_\lambda(A). \tag{3.4}$$

The inequality (3.3), together with $B \geq 0$, means that $\sigma(A, B)$ is contained in the half-ball as stated in the theorem, while (3.4) means that $\sigma(A, B)$ contains a point (ζ, t) in the intersection of the hemisphere with the ray $t = \bar{\lambda}\zeta$. Each point in the hemisphere $\{(\zeta, t) \in \mathbb{C} \times \mathbb{R} : |\zeta|^2 + t^2 = 1, t \geq 0\}$ is on such a ray, except the north pole $(0, 1)$, but $\sigma(A, B)$ is closed, hence it must contain the whole hemisphere. \square

Corollary 3.2. *Two non-zero commuting normal operators A, B on a finite dimensional Hilbert space can not be Pythagoras orthogonal.*

Proof. The fact that $C^*(A, B)$ is finite dimensional (and abelian) implies that $C^*(A, B)$ has only finitely many multiplicative functionals, consequently $\sigma(A, B)$ is a finite set. Hence $\sigma(A, B)$ can not contain the hemisphere $\{(\zeta, t) \in \mathbb{C} \times \mathbb{R} : |\zeta|^2 + t^2 = 1, t \geq 0\}$, so by Proposition 3.1 A and B can not be Pythagoras orthogonal. \square

The joint (algebraic) numerical range of an n -tuple (A_1, \dots, A_n) of elements of a C^* -algebra \mathcal{A} is defined as

$$V(A_1, \dots, A_n) = \{(\omega(A_1), \dots, \omega(A_n)) : \omega \in S(\mathcal{A})\},$$

where $S(\mathcal{A})$ is the set of all states on \mathcal{A} (= positive functionals of norm 1). Pythagoras orthogonality of general operators can be characterized as follows:

Proposition 3.3. *Let $A, B \in B(\mathcal{H})$ and $\|A\| = 1 = \|B\|$. Then $A \perp_P B$ if and only if the set $V := V(I - A^*A, I - B^*B, B^*A)$ is contained in the “cone”*

$$C = \{(x, y, z) \in \mathbb{R}^2 \times \mathbb{C} : 0 \leq x \leq 1, 0 \leq y \leq 1, |z| \leq \sqrt{xy}\}$$

and for all $w \in \mathbb{C}$ with $|w| = 1$ and $s, t \in \mathbb{R}_+ = (0, \infty)$ the set V intersects the closed ray from 0 in the direction of vector $(s, t, w\sqrt{st})$.

The idea of the proof. Since $\|T\|^2 = \|T^*T\| = \max\{\omega(T^*T) : \omega \in S(B(\mathcal{H}))\}$ for each $T \in B(\mathcal{H})$, the condition

$$\|A + \lambda B\|^2 = \|(A + \lambda B)^*(A + \lambda B)\| = 1 + |\lambda|^2$$

is equivalent to the simultaneously validity of the following two conditions:

$$\begin{aligned} \omega((A + \lambda B)^*(A + \lambda B)) &\leq 1 + |\lambda|^2 \quad \forall \lambda \in \mathbb{C}, \quad \forall \omega \in S(B(\mathcal{H})) \quad \text{and} \\ \forall \lambda \in \mathbb{C} \quad \exists \omega_\lambda \in S(B(\mathcal{H})) \quad \text{such that} \quad \omega_\lambda((A + \lambda B)^*(A + \lambda B)) &= 1 + |\lambda|^2. \end{aligned}$$

Using these two conditions the proof can be accomplished by an elementary computation, analogous to the one in the proof of Proposition 3.1. Since the proposition will not be used later in the paper we will omit this details. \square

4. Column orthogonal operators

The tensor product $M_n(\mathbb{C}) \otimes B(\mathcal{H})$ used below is the usual tensor product with the (unique C^* -tensor) norm that comes from the natural isomorphisms $M_n(\mathbb{C}) \otimes B(\mathcal{H}) \cong M_n(B(\mathcal{H})) \cong B(\mathcal{H}^n)$.

Definition 4.1. A finite set of operators $B_j \in B(\mathcal{H})$ is called *column orthogonal* if at least one of the operators is 0 or the operators $C_j := \|B_j\|^{-1}B_j$ satisfy

$$\left\| \sum \alpha_j \otimes C_j \right\|^2 = \left\| \sum \alpha_j^* \alpha_j \right\| \tag{4.1}$$

for all $\alpha_j \in M_n(\mathbb{C})$ and all $n \in \mathbb{N}$. A general set of operators in $B(\mathcal{H})$ is column orthogonal if all of its finite subsets are column orthogonal.

For a finite set (C_j) the condition (4.1) means that there is a completely isometric isomorphism φ from the linear span of (C_j) into a column Hilbert space \mathcal{H} with an orthonormal set (ϵ_j) such that $\varphi(C_j) = \epsilon_j$ for all j . (For a formal definition of a column Hilbert space see e.g. [6], [16], [13].) Thus, in particular, the norm of the row $\left[C_0 \quad C_1 \quad \dots \right] = \sum_j E_{1,j} \otimes C_j$ (where $E_{i,j} := \epsilon_i \otimes \epsilon_j^*$ are matrix units) is equal to $\left\| \left[\epsilon_0 \quad \epsilon_1 \quad \dots \right] \right\| = 1$, hence

$$\left\| \sum C_j C_j^* \right\| = 1.$$

Similarly we could define *row orthonormal* set by declaring that for any finite subset C_0, \dots, C_m there is a complete isometry from $\text{span}\{C_0, \dots, C_m\}$ into some row Hilbert space \mathcal{H}^* , mapping the C_j 's onto orthonormal vectors in \mathcal{H}^* . In this case the identity (4.1) is replaced by $\left\| \sum \alpha_j \otimes C_j \right\|^2 = \left\| \sum \alpha_j \alpha_j^* \right\|$ ($\alpha_j \in M_n(\mathbb{C})$) and such operators necessary satisfy $\left\| \sum C_j^* C_j \right\| = 1$. The theorem below characterizes column orthonormal sets and the characterization of row orthonormal sets can be obtained then by taking adjoints, but first we need a simple lemma.

Lemma 4.2. *Let $C_j \in B(\mathcal{K})$ satisfy $\sum_{j=0}^m C_j C_j^* \leq I$ and let ω be a state on $B(\mathcal{K})$ such that $\omega(C_j^* C_j) = 1$ for all j . Then $\omega(C_k^* C_j) = 0$ if $k \neq j$.*

Proof. Let π be the cyclic representation of $B(\mathcal{K})$ on a Hilbert space \mathcal{H} that corresponds to ω by the GNS construction, $\xi \in \mathcal{H}$ the corresponding cyclic unit vector and $A_j = \pi(C_j)$. Then by hypothesis $\sum_{j=0}^m A_j A_j^* = \pi(\sum_{j=0}^m C_j C_j^*) \leq I$ and $\|A_j \xi\|^2 = \langle \pi(C_j)^* \pi(C_j) \xi, \xi \rangle = \omega(C_j^* C_j) = 1$ for all j . Since $\langle A_j \xi, A_k \xi \rangle = \omega(C_k^* C_j)$, we have to prove that $\langle A_j \xi, A_k \xi \rangle = 0$ if $j \neq k$. Let $E := \xi \otimes \xi^*$ be the projection onto $\mathbb{C}\xi$ and $P_j = A_j E A_j^* = A_j \xi \otimes (A_j \xi)^*$. Since $\|A_j \xi\| = 1$, each P_j is a projection onto $\mathbb{C}A_j \xi$. We have $\sum_{j=0}^m P_j = \sum_{j=0}^m A_j E A_j^* \leq \sum_{j=0}^m A_j A_j^* \leq I$, which implies that the ranges of P_j are mutually orthogonal. (Indeed, $\sum_{j=0}^m P_k P_j P_k \leq P_k$ implies that $\sum_{j \neq k} P_k P_j P_k \leq 0$. Since $P_k P_j P_k \geq 0$, this means that $P_k P_j (P_k P_j)^* = P_k P_j P_k = 0$, hence $P_k P_j = 0$.) Thus $A_j \xi \perp A_k \xi$ if $k \neq j$. \square

Theorem 4.3. *A set of norm 1 operators $C_j \in B(\mathcal{K})$ acting on a Hilbert space \mathcal{K} is column orthogonal if and only if $\sum C_j C_j^* \leq I$ and there exists a state ω on $B(\mathcal{K})$ (or on the C^* -algebra generated by all C_j) such that $\omega(C_j^* C_j) = 1$ for all j . (Note that the last condition just means that in the cyclic representation π arising from ω the operators $\pi(C_j)$ all attain their norms 1 at the same vector η , namely at a cyclic vector for π , so that $\omega(T) = \langle \pi(T)\eta, \eta \rangle$ for $T \in B(\mathcal{K})$.)*

Proof. Suppose that $\sum_j C_j C_j^* \leq I$ and that there exists a state ω satisfying $\omega(C_j^* C_j) = 1$ for all j . For any matrices $\alpha_j, \beta_j \in M_n(\mathbb{C})$ and finite subset F of indexes we then have

$$\begin{aligned} \left\| \sum_{j \in F} \alpha_j \otimes C_j \right\|^2 &= \left\| \sum_{j \in F} C_j \otimes \alpha_j \right\|^2 = \left\| \sum_{j \in F} (C_j \otimes I)(I \otimes \alpha_j) \right\|^2 \\ &= \left\| \begin{bmatrix} C_1 \otimes I & C_2 \otimes I & \dots \end{bmatrix} \begin{bmatrix} I \otimes \alpha_1 \\ I \otimes \alpha_2 \\ \vdots \end{bmatrix} \right\|^2 \\ &\leq \left\| \begin{bmatrix} C_1 \otimes I & C_2 \otimes I & \dots \end{bmatrix} \right\|^2 \left\| \begin{bmatrix} I \otimes \alpha_1 \\ I \otimes \alpha_2 \\ \vdots \end{bmatrix} \right\|^2 \end{aligned}$$

$$\leq \left\| \sum_{j \in F} C_j C_j^* \right\| \left\| \sum_{j \in F} \alpha_j^* \alpha_j \right\| \leq \left\| \sum_{j \in F} \alpha_j^* \alpha_j \right\|.$$

Further, by Lemma 4.2 $\omega(C_k^* C_j) = \delta_{k,j}$, hence

$$\begin{aligned} & \left\| \sum_{j \in F} \alpha_j \otimes C_j \right\|^2 = \left\| \left(\sum_{j \in F} \alpha_j \otimes C_j \right)^* \left(\sum_{j \in F} \alpha_j \otimes C_j \right) \right\| \\ & \geq \left\| (\text{id} \otimes \omega) \left[\left(\sum_{j \in F} \alpha_j \otimes C_j \right)^* \left(\sum_{j \in F} \alpha_j \otimes C_j \right) \right] \right\| = \left\| \sum_{j,k \in F} \alpha_k^* \alpha_j \otimes \omega(C_k^* C_j) \right\| = \left\| \sum_{j \in F} \alpha_j^* \alpha_j \right\|. \end{aligned}$$

Thus (4.1) holds and the C_j are column orthonormal.

Suppose now conversely, that the operators C_j are column orthonormal. Assume first that the set is finite, say C_0, \dots, C_m . We have already observed after the Definition 4.1 that $\left\| \sum_j C_j C_j^* \right\| = 1$, thus we have only to prove the existence of an appropriate state ω . Let B_j ($j = 0, \dots, m$) be rank one operators on ℓ^2 given in the usual orthonormal basis of ℓ^2 by matrices

$$B_j = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

where 1 is in the j -th row. Since the operator sets $(B_j)_{j=0}^m$ and $(C_j)_{j=0}^m$ are both column orthonormal, there exists a completely isometric isomorphism $\varphi : \text{span}(C_j) \rightarrow \text{span}(B_j)$ such that $\varphi(C_j) = B_j$. As a complete contraction, φ is necessarily of the form $\varphi(T) = X\pi(T)Y$ for suitable contractions X and Y and a representation π of the C^* -algebra \mathcal{A} generated by $(C_j)_{j=0}^m$ on a Hilbert space \mathcal{H} [15, pp. 99 and 102]. Let $A_j := \pi(C_j)$, so that

$$B_j = X A_j Y \quad (j = 0, \dots, m). \tag{4.2}$$

Denote $P := B_0$ (a projection). Since $B_j = B_j P$, we may replace in (4.2) Y by $Y P$, so the operators $X : \mathcal{H} \rightarrow \ell^2$ and $Y : \ell^2 \rightarrow \mathcal{H}$ are of the form

$$X = \begin{bmatrix} \xi_0^* \\ \xi_1^* \\ \xi_2^* \\ \vdots \end{bmatrix}, \quad Y = \begin{bmatrix} \eta & 0 & 0 & \dots \end{bmatrix},$$

where $\eta : \mathbb{C} \rightarrow \mathcal{H}$ is essentially a vector in \mathcal{H} and the $\xi_j^* : \mathcal{H} \rightarrow \mathbb{C}$ are contractive linear functionals on \mathcal{H} , hence given by $\xi_j^*(\zeta) = \langle \zeta, \xi_j \rangle$ ($\zeta \in \mathcal{H}$) for some vectors $\xi_j \in \mathcal{H}$ with $\|\xi_j\| \leq 1$. Comparing the entries of matrices in (4.2) we have now

$$\langle A_j \eta, \xi_j \rangle = 1 \quad \text{and} \quad \langle A_j \eta, \xi_k \rangle = 0, \quad \text{if } k \neq j. \tag{4.3}$$

Since $\|A_j \eta\| \leq 1$ and $\|\xi_j\| \leq 1$, the first equality in (4.3) implies that $\|\xi_j - A_j \eta\|^2 \leq 2 - 2\operatorname{Re}(\langle A_j \eta, \xi_j \rangle) = 0$, hence $\xi_j = A_j \eta$. Thus it follows from (4.3) that $\langle A_j \eta, A_k \eta \rangle = \delta_{k,j}$. Hence the state ω on $B(\mathcal{K})$, defined by $\omega(T) = \langle \pi(T)\eta, \eta \rangle$ satisfies $\omega(C_k^* C_j) = \delta_{k,j}$. This completes the proof in the case when the set (C_j) is finite. If the set of operators (C_j) is infinite, we can apply the argument just given to each of its finite subsets F to obtain a state ω_F satisfying $\omega_F(C_k^* C_j) = \delta_{j,k}$ for all $j, k \in F$, and then we take a weak* limit point ω of the net of states ω_F . \square

Recall that each state on the C^* -algebra of compact operators $K(\mathcal{K})$ is of the form $T \mapsto \sum_{j=1}^\infty \langle T \xi_j, \xi_j \rangle$, where $\xi_j \in \mathcal{K}$ and $\sum_j \|\xi_j\|^2 = 1$. (If $\dim \mathcal{K} < \infty$ the sum can be taken to have only finitely many terms, so each state is a convex combination of vector states.) Further, each state on $B(\mathcal{K})$ can be approximated by vector states, hence Theorem 4.3 implies the following corollary.

Corollary 4.4. *A set of norm 1 operators C_j ($j = 0, \dots, m$) on $\mathcal{K} = \mathbb{C}^r$ ($r \in \mathbb{N}$) is column orthogonal if and only if $\|\sum_{j=0}^m C_j C_j^*\| = 1$ and all the operators C_j achieve their norms at the same unit vector in \mathcal{K} . The same conclusion holds for compact operators on an infinite dimensional Hilbert space \mathcal{K} . For general operators a similar conclusion holds, but the norm attaining condition must be replaced by: for each $\varepsilon > 0$ there exists a unit vector $\xi \in \mathcal{K}$ such that $\|C_j \xi\| > 1 - \varepsilon$ for all j .*

For norm 1 operators C_j and unit vector ξ the norming condition $\|C_j \xi\| \approx 1$ in Corollary 4.4 is equivalent to the requirement that the norm of the column $(C_0, \dots, C_m)^T$ is \sqrt{m} . Since in the proof of Theorem 4.3 we have used only the row and the column structure, it follows in particular that the operator space structure of the column Hilbert space is determined already by the norms on spaces of rows and columns (columns with orthonormal entries are sufficient), which, however, has been proved already by Mathes [13].

5. A metric characterization of operators with orthogonal ranges

In this section we will study a more restrictive form of Pythagoras orthogonality, which turns out to be also a special case of column orthogonality, in which scalars are replaced by elements of a C^* -algebra \mathcal{A} . For this, we will need a metric characterization of pairs $A, B \in \mathcal{A}$ satisfying $A^* B = 0$. First a lemma is needed, which (as pointed to me by an anonymous referee) follows from [8, Lemma 2.3], but we will present a short direct proof.

Lemma 5.1. *Let $A, B \in \mathcal{A}$, where \mathcal{A} is a C^* -algebra. Then $B^* B \leq A^* A$ if and only if $\|BX\| \leq \|AX\|$ for all (positive) $X \in \mathcal{A}$.*

Proof. If $B^*B \leq A^*A$, then $X^*B^*BX \leq X^*A^*AX$ for all $X \in \mathcal{A}$, which implies that $\|BX\|^2 = \|X^*B^*BX\| \leq \|X^*A^*AX\| = \|AX\|^2$. To prove the converse, suppose that $B^*B \not\leq A^*A$. Then there exists $t \in (0, 1)$ such that the positive part X of the operator $tB^*B - A^*A$ is not zero, that is

$$X := (tB^*B - A^*A)_+ \neq 0. \tag{5.1}$$

(Otherwise $tB^*B - A^*A \leq 0$ for all $t \in (0, 1)$ and letting $t \rightarrow 1$ it would follow that $B^*B \leq A^*A$). Then $X \geq 0$ and

$$0 \neq X(tB^*B - A^*A)X \geq 0. \tag{5.2}$$

Hence $XA^*AX \leq tXB^*BX$ and therefore

$$\|AX\|^2 = \|XA^*AX\| \leq t\|XB^*BX\| = t\|BX\|^2 < \|BX\|^2,$$

since $BX \neq 0$. (Namely, if $BX = 0$, then (5.2) would be a contradiction.) \square

Theorem 5.2. For elements A, B in any C^* -algebra \mathcal{A} the equality $A^*B = 0$ holds if and only if

$$\|AX\| \leq \|AX + BY\| \quad \forall X, Y \in \mathcal{A}. \tag{5.3}$$

Proof. We may replace X and Y in (5.3) by XC and YC for any $C \in \mathcal{A}$ and then apply Lemma 5.1 to AX and $AX + BY$ (instead of A and B). In this way we see that (5.3) holds if and only if

$$C^*X^*A^*AXC \leq C^*(AX + BY)^*(AX + BY)C \quad \forall X, Y, C \in \mathcal{A},$$

which can be rewritten as

$$2\operatorname{Re}(C^*X^*A^*BYC) + C^*Y^*B^*BYC \geq 0.$$

Replacing Y by twY , where $w \in \mathbb{C}$ with $|w| = 1$ and $t \in (0, \infty)$, we obtain equivalent condition

$$2\operatorname{Re}(wC^*X^*A^*BYC) + tC^*Y^*B^*BYC \geq 0.$$

Considering $t \rightarrow 0$, we see that

$$\operatorname{Re}(w(C^*X^*A^*BYC)) \geq 0 \quad \forall w \in \mathbb{C} \text{ with } |w| = 1.$$

This means that $C^*X^*A^*BYC = 0$ (to see this, apply states of \mathcal{A} to $C^*X^*A^*BYC$) and, since C, X and Y are arbitrary, $A^*B = 0$. The verification of converse is easy. \square

Corollary 5.3. Let P_j ($j = 0, \dots, m$) be fixed positive elements in a C^* -algebra \mathcal{A} . For elements A_j in \mathcal{A} the equality

$$\left\| \sum_j A_j X_j \right\|^2 = \left\| \sum_j X_j^* P_j X_j \right\| \tag{5.4}$$

holds for all $X_j \in \mathcal{A}$ if and only if $A_j^* A_j = P_j$ and $A_k^* A_j = 0$ if $k \neq j$. In particular, if all P_j are equal to a projection P , then the A_j are partial isometries with orthogonal ranges and the same initial projection P .

Proof. If A_j satisfy (5.4) for all $X_j \in \mathcal{A}$, then (taking $X_j = 0$ for $j \neq 0$) we get

$$\|X_0^* A_0^* A_0 X_0\| = \|A_0 X_0\|^2 = \|X_0^* P_0 X_0\|$$

for all $X_0 \in \mathcal{A}$, hence by Lemma 5.1 (applied to $A = A_0$ and $B = \sqrt{P_0}$) $A_0^* A_0 = P_0$. Similarly $A_j^* A_j = P_j$ for all j . Further, from (5.4) we now have

$$\begin{aligned} \|A_j X_j + A_k X_k\|^2 &= \|X_j^* P_j X_j + X_k^* P_k X_k\| \\ &\geq \|X_k^* P_k X_k\| = \|X_k^* A_k^* A_k X_k\| = \|A_k X_k\|^2 \end{aligned}$$

for all $X_k, Y_k \in \mathcal{A}$ and all $k \neq j$, hence by Theorem 5.2 $A_k^* A_j = 0$. This proves the corollary in one direction, while the proof in the reverse direction is straightforward: if $A_j^* A_j = P_j$ and $A_k^* A_j = 0$ for $k \neq j$, then

$$\left\| \sum_j A_j X_j \right\|^2 = \left\| \left(\sum_j A_j X_j \right)^* \left(\sum_k A_k X_k \right) \right\| = \left\| \sum_j X_j^* P_j X_j \right\|. \quad \square$$

Proposition 5.4. Suppose that $A, B \in \mathcal{A}$ satisfy $A^* B = 0$. Then $A \perp_P B$ if and only if there exists a state ω on \mathcal{A} such that $\omega(A^* A) = \|A\|^2$ and $\omega(B^* B) = \|B\|^2$. In this case A and B are column orthogonal.

Proof. Since $A^* B = 0 = B^* A$,

$$\|\alpha \otimes A + \beta \otimes B\|^2 = \|\alpha^* \alpha \otimes A^* A + \beta^* \beta \otimes B^* B\| \quad \forall \alpha, \beta \in M_n(\mathbb{C}). \tag{5.5}$$

If $A \perp_P B$, then $\|\alpha A + \beta B\|^2 = |\alpha|^2 \|A\|^2 + |\beta|^2 \|B\|^2$ for all $\alpha, \beta \in \mathbb{C} = M_1(\mathbb{C})$, hence it follows from (5.5) that $\|\alpha^2 A^* A + \beta^2 B^* B\| = |\alpha|^2 \|A\|^2 + |\beta|^2 \|B\|^2$ for all $\alpha, \beta \in \mathbb{C}$. Denoting $s = |\alpha|^2, t = |\beta|^2, P = A^* A$ and $Q = B^* B$, this is equivalent to

$$\|sP + tQ\| = s\|P\| + t\|Q\| \quad \forall s, t \in \mathbb{R}_+. \tag{5.6}$$

Let $s > 0, t > 0$ be fixed. Since for positive operators norm is equal to the numerical radius, we can choose a state ω on \mathcal{A} such that $\omega(sP + tQ) = \|sP + tQ\|$, and then we have

$$\|sP + tQ\| = s\omega(P) + t\omega(Q) \leq s\|P\| + t\|Q\|.$$

Here equality holds by (5.6), hence it follows that $\omega(P) = \|P\|$ and $\omega(Q) = \|Q\|$, that is, $\omega(A^*A) = \|A\|^2$ and $\omega(B^*B) = \|B\|^2$.

Conversely, if there exists a state ω satisfying $\omega(A^*A) = \|A\|^2$ and $\omega(B^*B) = \|B\|^2$ and $A \neq 0 \neq B$, then by Theorem 4.3 $A_0 := \|A\|^{-1}A$ and $B_0 := \|B\|^{-1}B$ are column orthonormal, since $(A_0A_0^* + B_0B_0^*)^2 = A_0(A_0^*A_0)A_0^* + B_0(B_0^*B_0)B_0^* \leq A_0A_0^* + B_0B_0^*$ implies that $A_0A_0^* + B_0B_0^* \leq I$. \square

6. Operators orthogonal to a projection of rank one

In this section we will determine (up to equivalence) all operators A with $\|A\| = 1$ that are Pythagoras orthogonal to a projection B of rank one. We will see that Pythagoras orthogonality can depend on the action of A on the entire Hilbert space. If $A \perp_P B$, then by Lemma 2.4, relative to the decomposition $\mathcal{H} = B\mathcal{H} \oplus (1 - B)\mathcal{H}$, $A \in \mathcal{B}(\mathcal{H})$ is represented by a matrix of the form

$$A = \begin{bmatrix} 0 & a^* \\ b & C \end{bmatrix}, \text{ hence } A + \lambda B = \begin{bmatrix} \lambda & a^* \\ b & C \end{bmatrix},$$

where $a, b \in (I - B)\mathcal{H} = \mathcal{B}(\mathbb{C}, (I - B)\mathcal{H})$. (Here $a^* : (I - B)\mathcal{H} \rightarrow \mathbb{C}$ acts as $a^*(\zeta) = \langle \zeta, a \rangle$). Thus

$$(|\lambda|^2 + 1)I - (A + \lambda B)^*(A + \lambda B) = \begin{bmatrix} 1 - \|b\|^2 & -\bar{\lambda}a^* - b^*C \\ -\lambda a - C^*b & (|\lambda|^2 + 1)I - aa^* - C^*C \end{bmatrix}. \tag{6.1}$$

By Lemma 2.1 $A \perp_P B$ if and only if this matrix is positive and singular. If $\|b\| = 1$, then the off-diagonal terms of the matrix (6.1) must be 0 for all λ by positivity of the matrix, hence $a = 0$ and $C^*b = 0$. If $\|b\| < 1$, we may multiply the matrix (6.1) from the left by the matrix

$$S := \begin{bmatrix} 1 & 0 \\ \gamma(\lambda a + C^*b) & I \end{bmatrix}, \text{ where } \gamma = \frac{1}{1 - \|b\|^2},$$

and form the right by S^* to obtain

$$\begin{bmatrix} \gamma^{-1} & 0 \\ 0 & F(\lambda) \end{bmatrix}, \text{ where } F(\lambda) := (|\lambda|^2 + 1)I - aa^* - C^*C - \gamma(\lambda a + C^*b)(\bar{\lambda}a^* + b^*C).$$

This does not change the non-invertibility and positivity, hence $A \perp_P B$ if and only if $F(\lambda)$ is positive and not invertible. Since $\|A\| = 1$, the last column of A is a contraction, that is,

$$aa^* + C^*C \leq I, \tag{6.2}$$

so that the operator

$$E(\lambda) := (|\lambda|^2 + 1)I - aa^* - C^*C$$

is invertible and positive if $\lambda \neq 0$. We may write

$$F(\lambda) = E(\lambda)^{1/2}[I - D(\lambda)]E(\lambda)^{1/2}, \tag{6.3}$$

where

$$D(\lambda) = \gamma E(\lambda)^{-1/2}(\lambda a + C^*b)(\bar{\lambda}a^* + b^*C)E(\lambda)^{-1/2}.$$

Observe that $D(\lambda)$ is a rank 1 operator of the form cc^* (where $c \in \mathcal{H}$) and each such operator has only two eigenvalues, namely 0 and $c^*c = \|c\|^2$ (since $(cc^*)c = \|c\|^2c$). From (6.3) we see that $F(\lambda)$ is not invertible and positive if and only if 1 is an eigenvalue of $D(\lambda)$ (this implies that $D(\lambda) \leq 1$ since the only other eigenvalue of $D(\lambda)$ is 0). Hence it follows that $A \perp_P B$ if and only if

$$\gamma \|E(\lambda)^{-1/2}(\lambda a + C^*b)\|^2 = 1.$$

This can be written as

$$(\lambda a + C^*b)^*E(\lambda)^{-1}(\lambda a + C^*b) = \frac{1}{\gamma} = 1 - \|b\|^2. \tag{6.4}$$

Now observe (by considering $F(\lambda)/|\lambda|^2$ as $|\lambda| \rightarrow \infty$) that the non-invertibility of $F(\lambda)$ implies that $I - \gamma aa^*$ is not invertible, hence $\gamma \|a\|^2 = 1$, that is

$$\|a\|^2 + \|b\|^2 = 1.$$

We can now write (6.4) as

$$(\bar{\lambda}a^* + b^*C)E(\lambda)^{-1}(\lambda a + C^*b) = \|a\|^2 \quad (|\lambda| \neq 0),$$

or, replacing λ by $\frac{1}{\lambda}$,

$$(a^* + \bar{\lambda}b^*C)(I + |\lambda|^2T)^{-1}(a + \lambda C^*b) = \|a\|^2, \text{ where } T = I - aa^* - C^*C. \tag{6.5}$$

For $\lambda \in \mathbb{C}$ satisfying $|\lambda|||T|| < 1$ we can expand $(I + |\lambda|^2T)^{-1}$ and rewrite (6.5) as

$$(a^* + \bar{\lambda}b^*C)(I - |\lambda|^2T + |\lambda|^4T^2 - |\lambda|^6T^3 + \dots)(a + \lambda C^*b) = \|a\|^2. \tag{6.6}$$

Looking at coefficients of various powers of λ and $\bar{\lambda}$ we see that

$$a^*T^nC^*b = 0 \quad (n = 0, 1, 2, \dots) \quad \text{and} \tag{6.7}$$

$$b^*CT^nC^*b = a^*T^{n+1}a \quad (n = 0, 1, 2, \dots). \tag{6.8}$$

If $\|b\| = 1$, then we may apply the above arguments to A^* instead of A , which shows that in this case the identities (6.6) and (6.7) holds with the roles of a and b interchanged. (Recall also that $a = 0$ if $\|b\| = 1$.)

Now assume that $\|b\| \neq 1$ and, to simplify further arguments, observe that we may initially replace A and B by equivalent operators of the form SAT and $SBT = B$, where $S = 1 \oplus U$ and $T = 1 \oplus V$ are unitary, hence we may assume that C is positive (and diagonal if $\dim \mathcal{H} < \infty$). Then the identity (6.7) can also be written as $\langle T^n a, Cb \rangle = 0$, which means that the cyclic subspaces $[C^*(T)a]$ and $[C^*(T)Cb]$ are orthogonal. (Here $C^*(T)$ is the C^* -algebra generated by T , which is just the closure of polynomials in T since $T^* = T$.) Further, (6.8) can be written as

$$\langle T^k Cb, T^l Cb \rangle = \langle T^k T^{1/2} a, T^l T^{1/2} a \rangle \quad (k, l \in \mathbb{N}),$$

which implies that there is a unique surjective isometry $V : [C^*(T)T^{1/2}a] \rightarrow [C^*(T)Cb]$ satisfying $VT^k T^{1/2} a = T^k Cb$. In other words, $VT^{1/2} a = Cb$ and $VT = TV$. Decomposing \mathcal{H} as $\mathcal{H} = [C^*(T)a] \oplus [C^*(T)Cb] \oplus \mathcal{K}$ (where by definition \mathcal{K} is the orthogonal complement of the first two summands), T is represented by a block diagonal matrix of the form $T_1 \oplus T_2 \oplus T_3$. If we define the unitary operator U on \mathcal{H} by

$$U = \begin{bmatrix} 0 & V^* & 0 \\ V & 0 & 0 \\ 0 & 0 & I \end{bmatrix},$$

then $U = U^*$ commutes with T and $UT^{1/2} a = Cb$. This is true even if $\|b\| = 1$, since we have already established (from the positivity of the matrix (6.1)) that in this case $a = 0$ and $Cb = 0$. This proves in one direction the following proposition.

Proposition 6.1. *Up to isometries of $B(\mathcal{H})$ all operators $A \in B(\mathcal{H})$ with $\|A\| = 1$ that satisfy $A \perp_P B$, where $B \in B(\mathcal{H})$ is a projection of rank one, are, relative to the decomposition $\mathcal{H} = B\mathcal{H} \oplus \ker B$, of the form*

$$A = \begin{bmatrix} 0 & a^* \\ b & C \end{bmatrix}, \tag{6.9}$$

where $\|a\|^2 + \|b\|^2 = 1$, $C \geq 0$ and $Cb = UT^{1/2} a$ for a self-adjoint unitary U satisfying $UT = TU$ and $U[C^*(T)T^{1/2}a] \perp [C^*(T)a]$, where $T = I - aa^* - C^2$. (Thus in particular (6.7) and (6.8) together are equivalent to (6.5).)

Proof. By the above arguments we only need to verify that the identity (6.5) holds if $Cb = UT^{1/2}a$ and $C^* = C$, where U and T are as in the proposition. Using the definition of T , the identity (6.5), which we need to verify, can be rewritten as

$$\langle (I + |\lambda|^2 T)^{-1}(I + \lambda UT^{1/2})a, (I + \lambda UT^{1/2})a \rangle = \|a\|^2. \tag{6.10}$$

Since $U[C^*(T)T^{1/2}a] \perp [C^*(T)a]$, $U = U^*$ and $UT = TU$, we have in particular

$$\langle (I + |\lambda|^2 T)^{-1}a, UT^{1/2}a \rangle = 0 = \langle (I + |\lambda|^2 T)^{-1}a, T^{1/2}Ua \rangle \quad (\text{when } \lambda \neq 0),$$

hence, the left side of (6.10) is equal to

$$\begin{aligned} &\langle (I + |\lambda|^2 T)^{-1}a, (I + \bar{\lambda}T^{1/2}U)(I + \lambda UT^{1/2})a \rangle \\ &= \langle (I + |\lambda|^2 T)^{-1}a, (I + |\lambda|^2 T)a \rangle = \|a\|^2. \quad \square \end{aligned}$$

Now we would like to reformulate (6.7) and (6.8) so that a and b would appear symmetrically. Using the definition of T , it can easily be proved by induction that, assuming $C^* = C$, (6.7) is equivalent to

$$\langle (C^{2n}a, Cb) \rangle = 0 \quad (n = 0, 1, 2, \dots). \tag{6.11}$$

Suppose now that $\dim \mathcal{H} < \infty$. Then it follows from Proposition 2.2 that $\det C = 0$ (hence 0 must be an eigenvalue of C) since $\det C$ is the coefficient of λ in the development of $\det(A + \lambda B)$. Observe that we need to verify (6.7) and (6.8) only for n smaller than the degree m of the minimal polynomial of T , since T^m, T^{m+1}, \dots can all be expressed as linear combinations of T^k for $k < m$. Let $\gamma_1, \dots, \gamma_m$ be the nonzero eigenvalues of C . Let a_j and b_j be the components of a and b in the eigenspace $\ker(C - \gamma_j I)$. Then (6.11) can be written as

$$\sum_{j=1}^m \gamma_j^{2n+1} \langle a_j, b_j \rangle = 0 \quad (n = 0, 1, 2, \dots).$$

Since $\det[\gamma_j^{2n+1}] \neq 0$ ($j = 1, \dots, m, n = 0, \dots, m - 1$), it follows that (6.11) is equivalent to

$$\langle a_j, b_j \rangle = 0 \quad (j = 1, \dots, m). \tag{6.12}$$

Note that $Ta = a - \langle a, a \rangle a - C^2 a = \|b\|^2 a - C^2 a$, hence in the case $n = 0$ the identity (6.8) says that $\|Cb\|^2 = \langle Ta, a \rangle = \|b\|^2 \|a\|^2 - \|Ca\|^2$, so that

$$\|Ca\|^2 + \|Cb\|^2 = \|a\|^2 \|b\|^2. \tag{6.13}$$

Let us now consider the case $n = 1$ of (6.8). Using (6.11) and that $C^* = C$ we compute $TCb = (I - aa^* - C^2)Cb = Cb - C^3b$. Hence $\langle TCb, Cb \rangle = \|Cb\|^2 - \|C^2b\|^2$ and therefore by (6.8) in the case $n = 1$ and using (6.11) again we have

$$\begin{aligned} \|Cb\|^2 - \|C^2b\|^2 &= \langle TCb, Cb \rangle = \langle T^2a, a \rangle \\ &= \|Ta\|^2 = \|\|b\|^2a - C^2a\|^2 = \|b\|^4\|a\|^2 - 2\|b\|^2\|Ca\|^2 + \|C^2a\|^2. \end{aligned}$$

This can be rewritten as

$$\|C^2a\|^2 + \|C^2b\|^2 = \|Cb\|^2 - \|b\|^2(\|a\|^2\|b\|^2 - 2\|Ca\|^2).$$

By using (6.13) and the identity $\|a\|^2 + \|b\|^2 = 1$ the right side simplifies to $\|Cb\|^2 - \|b\|^2(\|Cb\|^2 - \|Ca\|^2) = \|a\|^2\|Cb\|^2 + \|b^2\|Ca\|^2$, hence

$$\|C^2a\|^2 + \|C^2b\|^2 = \|a\|^2\|Cb\|^2 + \|b\|^2\|Ca\|^2. \tag{6.14}$$

If $n \geq 3$, the computation, required to rewrite (6.8) in a way in which a and b appear symmetrically, seems to be so long that the author is not able to accomplish it. In the example below we will need only the cases $n = 0, 1$.

Example 6.2. Let us determine (up to equivalence) all $A \in M_3(\mathbb{C})$ that satisfy $A \perp_P B$, where $B \in M_3(\mathbb{C})$ is a projection of rank 1. By what we have established above we may suppose that A and B are of the form

$$\begin{bmatrix} 0 & \bar{a}_1 & \bar{a}_2 \\ b_1 & \alpha & 0 \\ b_2 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $0 \leq \alpha \leq 1$,

$$|a_1|^2 + |a_2|^2 + |b_1|^2 + |b_2|^2 = \|a\|^2 + \|b\|^2 = 1 \tag{6.15}$$

and from (6.12), (6.13) and (6.14)

$$\alpha a_1 \bar{b}_1 = 0, \tag{6.16}$$

$$\alpha^2(|a_1|^2 + |b_1|^2) = (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2), \tag{6.17}$$

$$\alpha^4(|a_1|^2 + |b_1|^2) = \alpha^2[(|a_1|^2 + |a_2|^2)|b_1|^2 + (|b_1|^2 + |b_2|^2)|a_1|^2]. \tag{6.18}$$

It is not hard to solve this system of equations to obtain for A the matrices of the following forms or their transposes:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ b_1 & 0 & 0 \\ b_2 & 0 & 0 \end{bmatrix},$$

where $|b_1|^2 + |b_2|^2 = 1$, and

$$A = \begin{bmatrix} 0 & \bar{a}_1 & 0 \\ 0 & \alpha & 0 \\ b_2 & 0 & 0 \end{bmatrix},$$

where $|a_1|^2 + |b_2|^2 = 1$, $0 < \alpha \leq 1$ and $|b_2| = \alpha$ or $|b_2| = 1$.

Declaration of competing interest

None declared.

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