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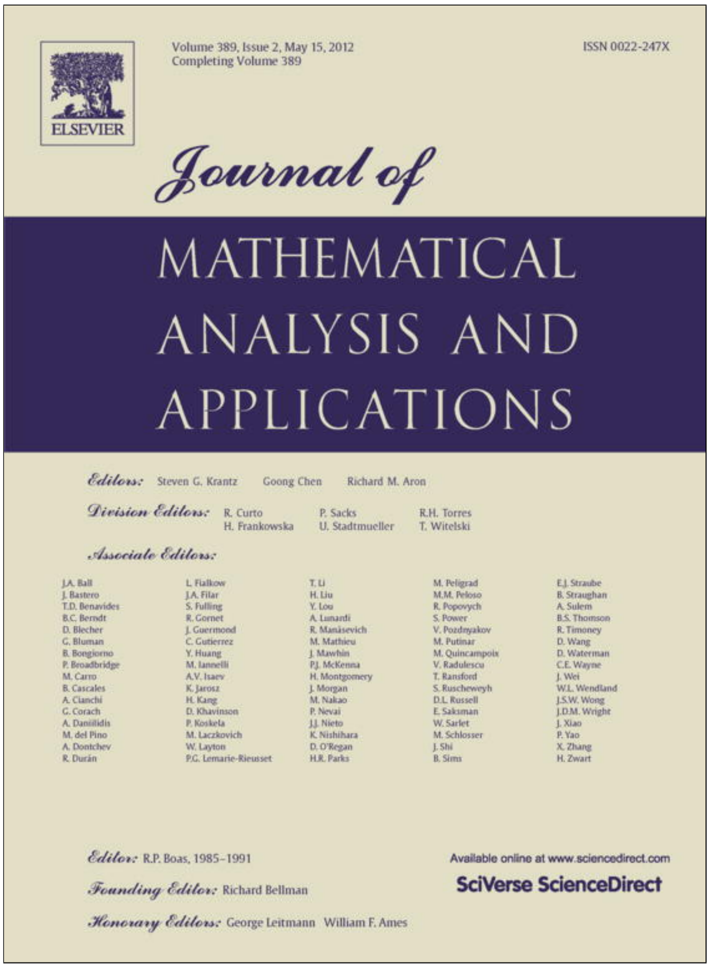

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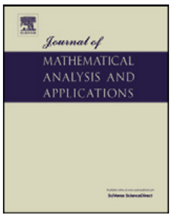

Fixed points of normal completely positive maps on $\mathrm{B}(\mathcal{H})$

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ABSTRACT

Given a sequence of bounded operators a_j on a Hilbert space $\mathcal{H}$ with $\sum_{j=1}^{\infty} a_j^* a_j = 1 = \sum_{j=1}^{\infty} a_j a_j^*$, we study the map Ψ defined on $\mathrm{B}(\mathcal{H})$ by $\Psi(x) = \sum_{j=1}^{\infty} a_j^* x a_j$ and its restriction Φ to the Hilbert–Schmidt class $\mathrm{C}^2(\mathcal{H})$. In the case when the sum $\sum_{j=1}^{\infty} a_j^* a_j$ is norm-convergent we show in particular that the operator $\Phi - 1$ is not invertible if and only if the C*-algebra A generated by $\{a_j\}_{j=1}^{\infty}$ has an amenable trace. This is used to show that Ψ may have fixed points in $\mathrm{B}(\mathcal{H})$ which are not in the commutant A' of A even in the case when the weak* closure of A is injective. However, if A is abelian, then all fixed points of Ψ are in A' even if the operators a_j are not positive.

1. Introduction and notation

It is well known that all normal (= weak* continuous) completely positive maps on $\mathrm{B}(\mathcal{H})$ (the algebra of all bounded operators on a separable Hilbert space $\mathcal{H}$) are of the form

$$\Psi_a(x) = \sum_{j=1}^{\infty} a_j^* x a_j = a^* x^{(\infty)} a, \tag{1.1}$$

where $a_j \in \mathrm{B}(\mathcal{H})$ are such that the column $a := (a_j)$ represents a bounded operator from $\mathcal{H}$ to $\mathcal{H}^{\infty}$, and $x^{(\infty)}$ denotes the block-diagonal operator matrix with x along the diagonal. The sum $a^* a = \sum_{j=1}^{\infty} a_j^* a_j$ is convergent in the strong (weak, weak*, …) operator topology. If $a^* a = 1$ (the identity operator on $\mathcal{H}$), then the map Ψ_a is unital. Ψ_a is dual to the map Ψ_{*a} defined on the trace class $\mathrm{T}(\mathcal{H})$ by

$$\Psi_{*a}(t) = \sum_{j=1}^{\infty} a_j t a_j^*. \tag{1.2}$$

So, if we assume in addition that the sum $\sum_{j=1}^{\infty} a_j a_j^*$ is convergent in the strong operator topology, then the map Ψ_a itself preserves $\mathrm{T}(\mathcal{H})$. If moreover $\sum_{j=1}^{\infty} a_j a_j^* = 1$, then the map $\Psi_a \mid \mathrm{T}(\mathcal{H})$ preserves the trace (that is, $Tr(\Psi_a(t)) = Tr(t)$ for all $t \in \mathrm{T}(\mathcal{H})$). Such maps are called unital quantum channels in quantum computation theory [21]. A selfadjoint operator $x \in \mathrm{B}(\mathcal{H})$ which is fixed by Ψ_a (that is, $\Psi_a(x) = x$) represents a physical quantity that passes unchanged through the quantum channel, so it is important to know the set $\mathcal{F}_a$ of all fixed points of Ψ_a. The structure of the set $\mathcal{F}_a$ is studied in several papers (see e.g. [3,6,23,30,36] and references there). Obviously $\mathcal{F}_a$ is a unital weak operator closed selfadjoint subspace of $\mathrm{B}(\mathcal{H})$ (in particular, it is spanned by positive elements) and $\mathcal{F}_a$ contains the commutant A' of the C*-algebra A generated by the operators a_j. If it happens that the positive part $\mathcal{F}_a^+$ of $\mathcal{F}_a$ is closed under the operation $x \mapsto x^2$, then it is well

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known that $\mathcal{F}_a = A'$ [6,3]. (For a proof, just note that $(ax - x^{(\infty)}a)^*(ax - x^{(\infty)}a) = x^2 + \Psi_a(x^2) - \Psi_a(x)x - x\Psi_a(x) = 0$, hence $ax = x^{(\infty)}a$. The assumption that $\sum_{j=1}^{\infty} a_j a_j^* = 1$ is not needed for this conclusion.) It is proved in [3] that each $x \in \mathcal{F}_a^+$ which can be diagonalized and the sequence of eigenvalues arranged in a decreasing order is in fact in A'. But in general $\mathcal{F}_a$ is not equal to A'. Namely, since the map Ψ_a is a complete contraction and $\Psi_a \mid A'$ is the identity, Ψ_a must be the identity also on the injective envelope of A'. Hence, if A' is not injective then $\mathcal{F}_a \neq A'$. It is proved in [3] that $\mathcal{F}_a$ is always an injective operator space. If all a_j are positive operators the operator Ψ_a is called a (generalized) Lüders operator. In [3] an example of Lüders operator is given where $\mathcal{F}_a \neq A'$. It is asked in [3] if the injectivity of A' (or equivalently, the injectivity of the weak* closure $\overline{A}$ of A) implies the equality $\mathcal{F}_a = A'$ for Lüders operators. In physics the von Neumann algebras usually appear as direct limits of finite dimensional C*-algebras and are therefore injective, so the question seems interesting also from the viewpoint of physics. We shall show that the answer is negative even in the case when A is an irreducible subalgebra of $\mathrm{B}(\mathcal{H})$ (so that $A' = \mathbb{C}1$) and only finitely many a_j's are nonzero. We remark that without positivity requirement $a_j \geqslant 0$ the question is much easier, one can construct counterexamples by using direct sums of suitable Toeplitz operators.

The basic idea for a counterexample is to consider the action of Lüders operators (where for simplicity we assume that only finitely many a_j's are nonzero or at least that the sum in (1.1) is norm-convergent) on the quotient $\mathrm{B}(\mathcal{H})/K$, where K is a twosided ideal in $\mathrm{B}(\mathcal{H})$. We will exploit the fact that the commutant $\dot{A}^c$ of the image $\dot{A}$ of A in $\mathrm{B}(\mathcal{H})/K$ can be very large so that not all of its elements can be lifted to A'. For example, if K is the (unique) closed ideal $\mathrm{K}(\mathcal{H})$ of all compact operators, then it is a well-known consequence of Voiculescu's theorem [11] that $\dot{A}^c$ is so large that $\dot{A}^{cc} = \dot{A}$ (note that A is separable), while A' consists of scalars only if A is irreducible. Now let $\dot{\Psi}$ be the map induced by $\Psi := \Psi_a$ on $\mathrm{B}(\mathcal{H})/K$ and let $\dot{x} \in \dot{A}^c$ be such that $\dot{x}$ cannot be lifted to an element in A'. Then $\dot{\Psi}(\dot{x}) = \dot{x}$, hence, denoting by x any lift in $\mathrm{B}(\mathcal{H})$ of $\dot{x}$,

$$y := \Psi(x) - x \in K. \tag{1.3}$$

Since $\dot{x}$ cannot be lifted to A', it follows that $x + z \notin A'$ for all $z \in K$. So, if we can find $z \in K$ such that $\Psi(x + z) = x + z$, then we will have $x + z \in \mathcal{F}_a \setminus A'$. Using (1.3), the condition for z is that

$$(1 - \Psi)(z) = y.$$

We could then find such a z if we knew that the map $(1 - \Psi) \mid K$ is invertible. But in the case $K = \mathrm{K}(\mathcal{H})$ the operator $(1 - \Psi) \mid K$ cannot be invertible since its second adjoint on $\mathrm{B}(\mathcal{H})$ (the bidual of $\mathrm{K}(\mathcal{H})$) is just $1 - \Psi$, which has nontrivial kernel (containing A'). Similarly $(1 - \Psi) \mid \mathrm{T}(\mathcal{H})$ is not invertible. So we have to consider other (non-closed) ideals, the simplest of which is the Hilbert–Schmidt class $\mathrm{C}^2(\mathcal{H})$. But in this case every operator that commutes with a C*-algebra A modulo $\mathrm{C}^2(\mathcal{H})$ is a perturbation of an element of A' by an element of $\mathrm{C}^2(\mathcal{H})$ (see [20]). So we will have to consider operators that commute with all a_j modulo $\mathrm{C}^2(\mathcal{H})$, but do not commute module $\mathrm{C}^2(\mathcal{H})$ with the whole C*-algebra A generated by the operators a_j. (This is possible since the space $\mathrm{C}^2(\mathcal{H})$ is not closed in the usual operator norm.)

In Section 2 we shall see that an operator Ψ_a of the form (1.1) (with the sums $\sum_{j=1}^{\infty} a_j^* a_j = 1 = \sum_{j=1}^{\infty} a_j a_j^*$ weak* converging) always preserves $\mathrm{C}^2(\mathcal{H})$, so we may consider the restriction $\Phi_a := \Psi_a \mid \mathrm{C}^2(\mathcal{H})$. We shall prove that if the operator $\Phi_a - 1$ is not invertible then there exists a state ρ on $\mathrm{B}(\mathcal{H})$ such that

$$\rho\left(\sum_{j=1}^{\infty} b_j a_j\right) = \rho\left(\sum_{j=1}^{\infty} a_j b_j\right)$$

for all operators $b_j \in \mathrm{B}(\mathcal{H})$ such that the two series $\sum_{j=1}^{\infty} b_j b_j^*$ and $\sum_{j=1}^{\infty} b_j^* b_j$ are weak* convergent. Conversely, if there exists a state ρ on $\mathrm{B}(\mathcal{H})$ such that $\rho(cd) = \rho(dc)$ for all $d \in \mathrm{B}(\mathcal{H})$ and all c in the C*-algebra A generated by $\{a_j\}_{j=1}^{\infty} \cup \{1\}$ and

$$\sum_{j=1}^{\infty} \rho\big(a_j^* a_j\big) = 1,$$

then the map $\Phi_a - 1$ is not invertible. Thus, in the case when the series $\sum_{j=1}^{\infty} a_j^* a_j$ is norm-convergent, $\Phi_a - 1$ is not invertible if and only if A has an amenable trace in the sense of [7], [8]. This result is then used in Section 3 to study fixed points of Ψ_a on $\mathrm{B}(\mathcal{H})$.

In the beginning of Section 4 we will present some general observations on the spectra of maps on $\mathrm{B}(\mathcal{H})$ of the form $\Theta : x \mapsto \sum_{j=1}^{\infty} a_j x b_j$, where (a_j) and (b_j) are two commutative sequences of normal operators such that the sums $\sum a_j a_j^*$ and $\sum b_j^* b_j$ are weak* convergent. We observe that the spectrum of Θ in the Banach algebra $\mathrm{CB}(\mathrm{B}(\mathcal{H}))$ of all completely bounded maps on $\mathrm{B}(\mathcal{H})$ is the same as the spectrum of Θ in certain natural subalgebras of $\mathrm{CB}(\mathrm{B}(\mathcal{H}))$. (Here some facts from the theory of operator spaces will be needed, but these results are not used in the rest of the paper.) The spectrum of such a map can be much larger than the closure of the set σ of all sums $\sum_{j=1}^{\infty} \phi(a_j)\psi(b_j)$, where ϕ and ψ are characters on the C*-algebras generated by (a_j) and (b_j), respectively, but all eigenvalues of Θ are contained in σ.

At the end of Section 4 we will provide a short proof of the fact that if the C*-algebra A generated by the operators (a_j) is abelian, then the fixed points of Φ_a are contained in A'. For positive operators a_j this was proved in [36] and also in [23], but our proof is different even in this case.

2. Amenable traces and the spectrum of Φ_a

Throughout the section $a=(a_j)$ is a bounded operator from a separable Hilbert space $\mathcal{H}$ to the direct sum $\mathcal{H}^\infty$ of countably many copies of $\mathcal{H}$, such that the components $a_j\in \mathrm{B}(\mathcal{H})$ satisfy

$$a^*a=\sum_{j=1}^{\infty}a_j^*a_j=1=\sum_{j=1}^{\infty}a_ja_j^*. \tag{2.1}$$

(The first equality is by the definition of a.) As in the Introduction, $\Psi=\Psi_a$ denotes the map on $\mathrm{B}(\mathcal{H})$ defined by (1.1). By $\mathrm{C}^2(\mathcal{H})$ we denote the ideal of all Hilbert–Schmidt operators on $\mathcal{H}$, and $\|x\|_2$ denotes the Hilbert–Schmidt norm of an element $x\in \mathrm{C}^2(\mathcal{H})$, which is defined by $\|x\|_2=\sqrt{Tr(x^*x)}$.

Proposition 2.1. (i) *$\Psi(\mathrm{C}^2(\mathcal{H}))\subseteq \mathrm{C}^2(\mathcal{H})$ and the restriction $\Phi:=\Psi\mid \mathrm{C}^2(\mathcal{H})$ is a contraction, that is $\|\Phi(x)\|_2\leqslant \|x\|_2$ for all $x\in \mathrm{C}^2(\mathcal{H})$.*

(ii) *For all $x\in \mathrm{C}^2(\mathcal{H})$ the inequalities*

$$\left\|ax-x^{(\infty)}a\right\|_2^2\leqslant 2\left\|x-\Phi(x)\right\|_2\|x\|_2 \quad \text{and} \quad \left\|\Phi(x)-x\right\|_2\leqslant \left\|ax-x^{(\infty)}a\right\|_2$$

hold.

(iii) *The operator $\Phi-1$ is not invertible if and only if there exists a sequence of selfadjoint elements $x_k\in \mathrm{C}^2(\mathcal{H})$ with $\|x_k\|_2=1$ such that*

$$\lim_{k\to\infty}\left\|\Phi(x_k)-x_k\right\|_2=0.$$

Proof. (i) Since $\|a\|=1$, we have that $aa^*\leqslant 1$ (the identity operator on $\mathcal{H}^\infty$). Using this and the equality $\sum_{j=1}^{\infty}a_ja_j^*=1$, we compute that for each $x\in \mathrm{C}^2(\mathcal{H})$

$$\left\|\Phi(x)\right\|_2^2=Tr\big(a^*x^{*(\infty)}aa^*x^{(\infty)}a\big)\leqslant Tr\big(a^*\big(x^*x\big)^{(\infty)}a\big)=\sum_{j=1}^{\infty}Tr\big(a_j^*x^*xa_j\big)=\sum_{j=1}^{\infty}Tr\big(xa_ja_j^*x^*\big)=Tr\big(xx^*\big)=\|x\|_2^2.$$

(ii) Using the relations $a^*a=1$, $aa^*\leqslant 1$, $Tr(\Phi(x^*x))=Tr(x^*x)$ and the well-known properties of the trace we have

$$\begin{aligned}\left\|ax-x^{(\infty)}a\right\|_2^2&=Tr\big(\big(ax-x^{(\infty)}a\big)^*\big(ax-x^{(\infty)}a\big)\big)\\&=Tr\big(x^*x+\Phi\big(x^*x\big)-\Phi(x)^*x-x^*\Phi(x)\big)\\&=Tr\big(\big(x-\Phi(x)\big)^*x+x^*\big(x-\Phi(x)\big)\big)\\&\leqslant 2\|x\|_2\left\|x-\Phi(x)\right\|_2.\end{aligned}$$

Similarly

$$\left\|\Phi(x)-x\right\|_2=\left\|a^*\big(x^{(\infty)}a-ax\big)\right\|_2\leqslant \left\|a^*\right\|\left\|x^{(\infty)}a-ax\right\|_2=\left\|ax-x^{(\infty)}a\right\|_2.$$

(iii) The existence of a sequence (x_k) as in (iii) clearly implies that the map $\Phi-1$ is not invertible (in $\mathrm{B}(\mathrm{C}^2(\mathcal{H}))$). Conversely, if $\Phi-1$ is not invertible, then 1 is a boundary point of the spectrum of Φ since $\|\Phi\|\leqslant 1$. But all boundary points of the spectrum are approximate eigenvalues [10, p. 215], so there exists a sequence of elements $x_k\in \mathrm{C}^2(\mathcal{H})$ such that $\|x_2\|_2=1$ and $\lim\|\Phi(x_k)-x_k\|_2=0$. By passing to an appropriate subsequence of real or imaginary parts of x_k and normalizing we can obtain a sequence of selfadjoint elements in $\mathrm{C}^2(\mathcal{H})$ satisfying the condition in (iii). □

In the proof of the main result of this section we will need two simple facts stated in the following remark.

Remark 2.2. If $x=(x_j)$ and $y=(y_j)$ are two operators from $\mathcal{H}$ to $\mathcal{H}^\infty$ of the Hilbert–Schmidt class (so that in particular $x_j,y_j\in \mathrm{C}^2(\mathcal{H})$) then:

(i) $\|x\|_2=\|x^T\|_2$, where x^T is the row $[x_j]$ regarded as the Hilbert–Schmidt operator from $\mathcal{H}^\infty$ to $\mathcal{H}$.
(ii) $Tr(x^*y)=\sum_{j=1}^{\infty}Tr(x_j^*y_j)$, where the series converges absolutely.

Part (i) is immediate. To prove (ii), we choose an orthonormal basis (ξ_k) of $\mathcal{H}$ and compute that

$$Tr\big(x^*y\big)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\big\langle x_j^*y_j\xi_k,\xi_k\big\rangle=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\big\langle x_j^*y_j\xi_k,\xi_k\big\rangle=\sum_{j=1}^{\infty}Tr\big(x_j^*y_j\big),$$

where the change of the order of summation is permissible since

$$\sum_{j,k=1}^{\infty} \left|\langle x_j^* y_j \xi_k, \xi_k\rangle\right| \leqslant \left(\sum_{j,k=1}^{\infty} \|y_j\xi_k\|^2\right)^{1/2} \left(\sum_{j,k=1}^{\infty} \|x_j\xi_k\|^2\right)^{1/2} = \|x\|_2\|y\|_2 < \infty.$$

Recall that a trace on a C*-subalgebra $A \subseteq \mathrm{B}(\mathcal{H})$ is called amenable if it can be extended to a state ρ on $\mathrm{B}(\mathcal{H})$ such that $\rho(cd) = \rho(dc)$ for all $c \in A$ and $d \in \mathrm{B}(\mathcal{H})$ [7,8]. We also recall the Powers–Störmer inequality: $\|x - y\|_2^2 \leqslant \|x^2 - y^2\|_1$ for all positive $x, y \in \mathrm{C}^2(\mathcal{H})$. (A proof can be found for example in [8]. Usually the inequality is used in the form $\|xu - ux\|_2^2 \leqslant \|x^2u - ux^2\|_1$ for positive $x \in \mathrm{C}^2(\mathcal{H})$ and a unitary u.)

Theorem 2.3. *Let A be the* C**-algebra generated by the identity and the operators $a_j \in \mathrm{B}(\mathcal{H})$ satisfying* (2.1) *and let $\Phi = \Phi_a$ be the restriction to $\mathrm{C}^2(\mathcal{H})$ of the map Ψ defined by* (1.1). *If $\Phi - 1$ is not invertible then there exists a state ρ on $\mathrm{B}(\mathcal{H})$ such that*

$$\rho\left(b^T a\right) = \rho\left(a^T b\right) \quad \text{for all } b = (b_j) \in \mathrm{B}\left(\mathcal{H}, \mathcal{H}^{\infty}\right) \text{ such that } b^T \in \mathrm{B}\left(\mathcal{H}^{\infty}, \mathcal{H}\right). \tag{2.2}$$

Conversely, if ρ is a state on $\mathrm{B}(\mathcal{H})$ such that $\rho(cd) = \rho(dc)$ for all $c \in A$ and $d \in \mathrm{B}(\mathcal{H})$ and

$$\sum_{j=1}^{\infty} \rho\left(a_j^* a_j\right) = 1, \tag{2.3}$$

then the map $\Phi - 1$ is not invertible.

Thus, if at least one of the series in (2.1) *is norm-convergent, then the map $\Phi - 1$ is not invertible if and only if A has an amenable trace.*

Proof. If $\Phi - 1$ is not invertible then by Proposition 2.1 there exists a sequence of selfadjoint elements x_k in $\mathrm{C}^2(\mathcal{H})$ with $\|x_k\|_2 = 1$ and

$$\lim_{k\to\infty} \left\|ax_k - x_k^{(\infty)}a\right\|_2 = 0.$$

Let ρ_k be the state on $\mathrm{B}(\mathcal{H})$ defined by $\rho_k(d) = Tr(dx_k^2)$ and let ρ be a weak* limit point of the sequence (ρ_k). Note that for each $x \in \mathrm{C}^2(\mathcal{H})$ and $b = (b_j) \in \mathrm{B}(\mathcal{H}, \mathcal{H}^{\infty})$ we have $Tr(a^T bx^2) = Tr(xa^T bx)$ and (by Remark 2.2(ii) since ax and $(x^* b_j^*)$ are in $\mathrm{C}^2(\mathcal{H}, \mathcal{H}^{\infty})$)

$$Tr\left(b^T x^{(\infty)} ax\right) = \sum_{j=1}^{\infty} Tr(b_j x a_j x) = \sum_{j=1}^{\infty} Tr(a_j x b_j x) = Tr\left(a^T x^{(\infty)} bx\right).$$

Using this and Remark 2.2(i) we now compute that

$$\begin{aligned} \left|Tr\left(b^T ax_k^2\right) - Tr\left(a^T bx_k^2\right)\right| &= \left|Tr\left(b^T\left(ax_k - x_k^{(\infty)}a\right)x_k\right) + Tr\left(\left(a^T x_k^{(\infty)} - x_k a^T\right)bx_k\right)\right| \\ &\leqslant \left\|b^T\right\| \left\|ax_k - x_k^{(\infty)}a\right\|_2 + \left\|a^T x_k^{(\infty)} - x_k a^T\right\|_2 \|b\| \\ &= \left(\|b\| + \left\|b^T\right\|\right)\left\|ax_k - x_k^{(\infty)}a\right\|_2 \xrightarrow{k\to\infty} 0. \end{aligned}$$

Since ρ is a weak* limit point of (ρ_k), this implies that $\rho(b^T a) = \rho(a^T b)$. In particular $\rho(a_j d) = \rho(da_j)$ for all a_j and all $d \in \mathrm{B}(\mathcal{H})$, which implies that $\rho \mid A$ is an amenable trace.

Suppose now conversely, that ρ is a state on $\mathrm{B}(\mathcal{H})$ satisfying (2.3) and $\rho(cd) = \rho(dc)$ for all $c \in A$ and $d \in \mathrm{B}(\mathcal{H})$. Since the series in (2.3) is convergent, given $\varepsilon > 0$, there exists $m \in \mathbb{N}$ such that

$$\sum_{j=m+1}^{\infty} \rho\left(a_j a_j^*\right) = \sum_{j=m+1}^{\infty} \rho\left(a_j^* a_j\right) < \frac{\varepsilon}{8}. \tag{2.4}$$

Since normal states are weak* dense in the state space of $\mathrm{B}(\mathcal{H})$, there exists a net of positive operators $y_k \in \mathrm{T}(\mathcal{H})$ with the trace norm $\|y_k\|_1 = 1$ such that the states $\rho_k(d) := Tr(dy_k)$ $(d \in \mathrm{B}(\mathcal{H}))$ weak* converge to ρ. By passing to a subnet we may assume that

$$\left|(\rho_k - \rho)\left(\sum_{j=m+1}^{\infty} \left(a_j a_j^* + a_j^* a_j\right)\right)\right| < \frac{\varepsilon}{4}. \tag{2.5}$$

Let $a_{(m)} = (a_1, \dots, a_m) \in \mathrm{B}(\mathcal{H}, \mathcal{H}^m)$. Observe that the trace class operators $a_{(m)} y_k - y_k^{(m)} a_{(m)} \in \mathrm{T}(\mathcal{H}, \mathcal{H}^m)$ converge weakly to 0 since for all $d = [d_1, \dots, d_m] \in \mathrm{B}(\mathcal{H}^m, \mathcal{H})$ we have (denoting by $y^{(m)}$ the direct sum of m copies of an operator y)

$$\begin{aligned} Tr\bigl(d\bigl(a_{(m)}y_k - y_k^{(m)}a_{(m)}\bigr)\bigr) &= \sum_{j=1}^{m} Tr\bigl(d_j(a_jy_k - y_ka_j)\bigr) \\ &= \sum_{j=1}^{m} Tr\bigl((d_ja_j - a_jd_j)y_k\bigr) \xrightarrow{k} \sum_{j=1}^{m} \rho(d_ja_j - a_jd_j) = 0. \end{aligned}$$

Therefore suitable convex combinations of operators $a_{(m)}y_k - y_k^{(m)}a_{(m)}$ must converge to 0 in norm; thus, replacing the y_k's by suitable convex combinations, we may assume that

$$\bigl\|a_{(m)}y_k - y_k^{(m)}a_{(m)}\bigr\|_1 \xrightarrow{k} 0.$$

Let $x_k = y_k^{1/2}$. It follows from the Powers–Störmer inequality (by expressing the components a_j of $a_{(m)}$ as linear combinations of unitaries) that

$$\sum_{j=1}^{m} \|a_jx_k - x_ka_j\|_2 \xrightarrow{k} 0. \tag{2.6}$$

Now we can estimate

$$\begin{aligned} \bigl\|ax_k - x_k^{(\infty)}a\bigr\|_2^2 &= \sum_{j=1}^{m} \|a_jx_k - x_ka_j\|_2^2 + \sum_{j=m+1}^{\infty} \|a_jx_k - x_ka_j\|_2^2 \\ &\leqslant \sum_{j=1}^{m} \|a_jx_k - x_ka_j\|_2^2 + 2\sum_{j=m+1}^{\infty} \bigl(\|a_jx_k\|_2^2 + \|x_ka_j\|_2^2\bigr) \\ &= \sum_{j=1}^{m} \|a_jx_k - x_ka_j\|_2^2 + 2\sum_{j=m+1}^{\infty} \bigl(Tr\bigl(a_jx_k^2a_j^* + a_j^*x_k^2a_j\bigr)\bigr) \\ &= \sum_{j=1}^{m} \|a_jx_k - x_ka_j\|_2^2 + 2\rho_k\Biggl(\sum_{j=m+1}^{\infty} a_j^*a_j + a_ja_j^*\Biggr). \end{aligned}$$

Using (2.4) and (2.5) it follows now that

$$\bigl\|ax_k - x_k^{(\infty)}a\bigr\|_2^2 < \sum_{j=1}^{m} \|a_jx_k - x_ka_j\|_2^2 + \varepsilon,$$

hence (2.6) implies that $\|ax_k - x_k^{(\infty)}a\|_2^2 < \varepsilon$ for some k. Since $\varepsilon > 0$ was arbitrary, Proposition 2.1 tells us that the map $\Phi - 1$ is not invertible.

If the series $\sum_{j=1}^{\infty} a_j^*a_j$ is norm-convergent to 1 then the condition (2.3) is automatically satisfied for any state ρ. If $\rho \mid A$ is tracial then the same conclusion holds if we assume the norm convergence of the series $\sum_{j=1}^{\infty} a_ja_j^* = 1$. Finally, observe that for any state ρ on $\mathrm{B}(\mathcal{H})$ satisfying $\rho(cd) = \rho(dc)$ for all $c \in A$ and $d \in \mathrm{B}(\mathcal{H})$ the condition (2.3) implies (2.2) since

$$\Biggl|\rho\Biggl(\sum_{j=m}^{\infty} b_ja_j\Biggr)\Biggr| \leqslant \sum_{j=m}^{\infty} \bigl|\rho(b_ja_j\bigr| \leqslant \sum_{j=m}^{\infty} \rho\bigl(b_jb_j^*\bigr)^{1/2}\rho\bigl(a_j^*a_j\bigr)^{1/2} \leqslant \|b\|\Biggl(\sum_{j=m}^{\infty} \rho\bigl(a_j^*a_j\bigr)\Biggr)^{1/2} \xrightarrow{m\to\infty} 0$$

and similarly $|\rho(\sum_{j=m}^{\infty} a_jb_j)| \xrightarrow{m\to\infty} 0$. □

Corollary 2.4. *If the von Neumann algebra $\overline{A}$ generated by the operators a_j (satisfying (2.1)) is finite and injective then the operator $\Phi_a - 1$ is not invertible.*

Proof. Let $E : \mathrm{B}(\mathcal{H}) \to \overline{A}$ be a conditional expectation, τ any normal tracial state on $\overline{A}$ and $\rho = \tau E$. The state $\rho = \tau E$ satisfies the condition (2.3) and $\rho(cd) = \rho(dc)$ for all $c \in \overline{A}$ and $d \in \mathrm{B}(\mathcal{H})$ (since E is an $\overline{A}$-bimodule map), hence the map $\Phi_a - 1$ is not invertible. □

Given an arbitrary von Neumann algebra $R \subseteq \mathrm{B}(\mathcal{H})$, it is known that if the norm of every elementary operator on $\mathrm{C}^2(\mathcal{H})$ of the form $x \mapsto \frac{1}{n}\sum_{j=1}^{n} u_jxu_j^*$, where the coefficients $u_j \in R$ are unitary, is equal to 1, then R is finite and injective; in the case when R is a factor this was proved by Connes [9, Remark 5.29], for general von Neumann algebras see [18].

Problem. Suppose that R is a von Neumann algebra such that the norm of every elementary operator on $C^2(\mathcal{H})$ of the form $x \mapsto \sum_{j=1}^{n} a_j x a_j$, where $a_j \in R$ are positive with $\sum_j a_j^2 = 1$, is equal to 1. Is then R necessarily injective and finite?

Remark 2.5. We have seen at the end of the proof of Theorem 2.3 that if ρ is a state on $B(\mathcal{H})$ such that $\rho(cd) = \rho(dc)$ for all $c \in A$ and $d \in B(\mathcal{H})$ and the condition (2.3) holds for an $a = (a_j)_j$ satisfying (2.1), then (2.2) also holds. But the converse is not true: (2.2) does not imply (2.3). Indeed, let $R \subseteq B(\mathcal{H})$ be an abelian infinite dimensional von Neumann algebra, ω any non-normal state on R and $E : B(\mathcal{H}) \to R$ a conditional expectation. Since ω is not normal there exists in R a sequence (a_j) of mutually orthogonal projections with the sum 1 such that $\sum_{j=1}^{\infty} \omega(a_j) < 1$. Let $\rho = \omega E$, a state on $B(\mathcal{H})$. Even though E is not necessarily weak* continuous the equalities

$$E(b^T a) = \sum_{j=1}^{\infty} E(b_j) a_j = \sum_{j=1}^{\infty} a_j E(b_j) = E(a^T b)$$

hold for all $b = (b_j)$ ($b_j \in B(\mathcal{H})$) such that the two sums $\sum_{j=1}^{\infty} b_j^* b_j$ and $\sum_{j=1}^{\infty} b_j b_j^*$ are weak* convergent. (This is so because E is a completely positive $\overline{A}$-bimodule map; see [15] or [19].) Hence $\rho(b^T a) = \omega(E(b^T a)) = \omega(E(a^T b)) = \rho(a^T b)$. But $\sum_{j=1}^{\infty} \rho(a_j^* a_j) = \sum_{j=1}^{\infty} \omega(a_j) < 1$.

We show now by an example that the condition (2.2) is not automatically fulfilled by states satisfying $\rho(cd) = \rho(dc)$ for all $c \in A$ and $d \in B(\mathcal{H})$.

Example 2.6. Choose an orthonormal basis (ξ_j) ($j = 1, 2 \ldots$) of $\mathcal{H}$ and let a_j be the rank 1 orthogonal projection onto $\mathbb{C}\xi_j$. Then the C*-algebra A, generated by (a_j) and 1, is the C*-algebra of all convergent sequences acting as diagonal operators. For each j let b_j be a rank 1 partial isometry such that $b_j b_j^* = a_{2j}$ and $b_j^* b_j = a_j$. Then

$$a^T b = \sum_{j=1}^{\infty} a_j b_j = 0,$$

while

$$b^T a = \sum_{j=1}^{\infty} b_j a_j = \sum_{j=1}^{\infty} b_j =: v$$

is an isometry with the range projection $p = vv^* = \sum_{j=1}^{\infty} a_{2j}$ of infinite rank. Let $q : B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H}) = C(H)$ be the quotient map, θ a state on $C(H)$ such that $\theta(q(v)) \neq 0$, and $\rho := \theta q$. Then $q(c)$ is a scalar for each $c \in A$, hence for each $d \in B(\mathcal{H})$

$$\rho(cd) = \theta\big(q(c)q(d)\big) = \theta\big(q(c)\big)\theta\big(q(d)\big) = \rho(dc).$$

But nevertheless $\rho(b^T a) = \rho(v) \neq 0 = \rho(a^T b)$.

Problem. Is the necessary condition (2.2) also sufficient for the conclusion of Theorem 2.3? In other words, may the stronger condition (2.3) be replaced by (2.2)?

The answer is affirmative at least when $a = (a_j)$ is such that the operator $x^{(\infty)} a \in B(\mathcal{H}, \mathcal{H}^{\infty})$ is of trace class for a dense set of trace class operators $x \in T(\mathcal{H})$. Namely, in this case we can modify the proof of Theorem 2.3 as follows. First we approximate the state ρ in Theorem 2.3 by normal states coming from operators $y_k \in T(\mathcal{H})$ such that the operators $y_k^{(\infty)} a \in B(\mathcal{H}, \mathcal{H}^{\infty})$ are of trace class. Then we verify that the sequence $(y_k^{(\infty)} a - a y_k)$ converges weakly to 0. Finally we show that $\|\sqrt{y_k}^{(\infty)} a - a\sqrt{y_k}\|_2 \xrightarrow{k\to\infty} 0$. For the last step wee need the following consequence of the Powers–Störmer inequality.

Proposition 2.7. *For all operators $b \in B(\mathcal{K}, \mathcal{H})$ and positive operators $x \in T(\mathcal{H})$, $y \in T(\mathcal{K})$ the inequality*

$$\|by - xb\|_2^2 \leqslant \gamma \left\|by^2 - x^2 b\right\|_1 \|b\| \tag{2.7}$$

holds, where $\gamma = \frac{8}{9}\sqrt{3}$.

Proof. By considering the operator

$$\begin{bmatrix} 0 & b \\ b^* & 0 \end{bmatrix}$$

instead of b and

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$$

instead of both x and y, the proof can be reduced immediately to the case when $b = b^*$ and $y = x$. (Further, in this case we may replace b by $b + s1$ for a suitable scalar s so that we may assume that both $\|b\|$ and $-\|b\|$ are in the spectrum of b.) Denote $\beta = \|b\|$ and for $t \in \mathbb{R} \setminus \{0\}$ let

$$u_t = (b - ti)(b + ti)^{-1}, \quad \text{so that } b = ti(1 + u_t)(1 - u_t)^{-1}.$$

Since u_t is unitary, we have by the Powers–Störmer inequality

$$\|u_t x - x u_t\|_2^2 \leqslant \|u_t x^2 - x^2 u_t\|_1,$$

which can be rewritten as

$$\big\|(b + ti)^{-1} z_t (b + ti)^{-1}\big\|_2^2 \leqslant \big\|2ti(b + ti)^{-1}\big(bx^2 - x^2 b\big)(b + ti)^{-1}\big\|_1, \tag{2.8}$$

where $z_t := 2ti(bx - xb)$. Since $\|z_t\|_2 \leqslant \|b + ti\|^2 \|(b + ti)^{-1} z_t (b + ti)^{-1}\|_2$, (2.8) implies that

$$\|z_t\|_2^2 \leqslant 2t\|b + ti\|^4 \big\|(b + ti)^{-1}\big\|^2 \big\|bx^2 - x^2 b\big\|_1.$$

Thus (since $\|(b + ti)^{-1}\|^2 \leqslant t^{-2}$ and $\|b + ti\|^2 \leqslant \beta^2 + t^2$)

$$\|bx - xb\|_2^2 \leqslant \frac{(\beta^2 + t^2)^2}{2t^3} \big\|bx^2 - x^2 b\big\|_1.$$

Taking the minimum over t of the right-hand side of this inequality, we obtain the desired estimate (2.7). □

3. On the fixed points of the map Ψ_a

As we indicated already in the Introduction, Theorem 2.3 implies the following corollary.

Corollary 3.1. *With the notation as in Theorem* 2.3, *suppose that the* C*-*algebra A has no amenable traces and that the two series* $\sum_{j=1}^{\infty} a_j^* a_j = 1 = \sum_{j=1}^{\infty} a_j a_j^*$ *are norm-convergent. If there exists an operator* $y \in \mathrm{B}(\mathcal{H})$ *such that the operator* $y^{(\infty)}a - ay$ *is in the Hilbert–Schmidt class and* y *is not in* $A' + \mathrm{C}^2(\mathcal{H})$, *then the operator* $\Psi = \Psi_a$ *defined on* $\mathrm{B}(\mathcal{H})$ *by* $\Psi_a(x) = \sum_{j=1}^{\infty} a_j^* x a_j$ *has fixed points which are not in* A'.

Proof. Observe that $y - \Psi(y) \in \mathrm{C}^2(\mathcal{H})$ since

$$y - \Psi(y) = a^*\big(ay - y^{(\infty)}a\big)$$

and $y^{(\infty)}a - ya$ is in the Hilbert–Schmidt class by the hypothesis. By Theorem 2.3 the map $(\Psi - 1)|\mathrm{C}^2(\mathcal{H})$ is invertible, hence there exists a $z \in \mathrm{C}^2(\mathcal{H})$ such that $(\Psi - 1)(z) = y - \Psi(y)$. This means that $\Psi(y + z) = y + z$. Hence $x := y + z$ is a fixed point of Ψ, and x is not in A' since $y \notin A' + \mathrm{C}^2(\mathcal{H})$ and $z \in \mathrm{C}^2(\mathcal{H})$. □

Now we give an example which satisfies the conditions of Corollary 3.1 and solves a problem left open in [3].

Example 3.2. Let v_i $(i = 1, 2)$ be the isometries defined on $\mathcal{H} = \ell^2(\mathbb{N})$ by

$$v_1 e_j = e_{2j} \quad \text{and} \quad v_2 e_j = e_{2j+1} \quad (j = 0, 1, 2, \ldots),$$

where (e_j) is an orthonormal basis of $\mathcal{H}$. Then $v_1 v_1^* + v_2 v_2^* = 1$ and the C*-algebra A generated by $\{v_1, v_2\}$ is the Cuntz algebra $O(2)$ (defined in [12] or [17]), which has no tracial states (and is nuclear).

To show that A is irreducible, choose any $d \in A'$ and let

$$de_0 = \sum_{j=0}^{\infty} \alpha_j e_j \quad (\alpha_j \in \mathbb{C}).$$

Then

$$\sum_{j=0}^{\infty} \alpha_j e_j = de_0 = dv_1^* e_0 = v_1^* de_0 = \sum_{j=0}^{\infty} \alpha_{2j} e_j,$$

which implies that $\alpha_j = \alpha_{2j}$ for all j. Similarly, from $0 = dv_2^* e_0 = v_2^* de_0 = \sum_{j=0}^{\infty} \alpha_{2j+1} e_j$ we see that $\alpha_{2j+1} = 0$ for all j. It follows that $\alpha_j = 0$ for all $j > 0$. Thus $de_0 = \alpha_0 e_0$ and consequently $d(v_1^{k_1} v_2^{k_2} v_1^{k_3} \ldots)e_0 = (v_1^{k_1} v_2^{k_2} v_1^{k_3} \ldots)de_0 = \alpha_0 (v_1^{k_1} v_2^{k_2} \ldots)e_0$

for any sequence $k_1, k_2, \ldots$ in $\mathbb{N}$. Since the linear span of vectors of the form $(v_1^{k_1} v_2^{k_2} \ldots)e_0$ is dense in $\mathcal{H}$, it follows that $d = \alpha_0 1$.

We will show that there exists a positive diagonal operator $y \in \mathrm{B}(\mathcal{H})$ such that

$$yv_2 = v_2 y, \qquad yv_1 - v_1 y \in \mathrm{C}^2(\mathcal{H}), \quad \text{but } y \notin \mathbb{C}1 + \mathrm{C}^2(\mathcal{H}) = A' + \mathrm{C}^2(\mathcal{H}).$$

Let $ye_j = t_j e_j$, where t_j are nonnegative scalars to be specified. The condition $yv_2 = v_2 y$ means that

$$t_{2j+1} = t_j \quad (j = 0, 1, 2, \ldots). \tag{3.1}$$

On the other hand, the condition $yv_1 - v_1 y \in \mathrm{C}^2(\mathcal{H})$ means that

$$\sum_{j=0}^{\infty} (t_{2j} - t_j)^2 < \infty. \tag{3.2}$$

To satisfy these two conditions, choose t_j, for example, as follows. If j is of the form $j = 2^k$ $(k \in \mathbb{N})$ let $t_j = (k+1)^{-1/2}$. If j is not a power of 2 define t_j recursively by

$$t_j = \begin{cases} t_{\frac{j}{2}}, & \text{if } j \text{ is even;} \\ t_{\frac{j-1}{2}}, & \text{if } j \text{ is odd.} \end{cases}$$

Then $t_{2j+1} = t_j$ for all $j \in \mathbb{N}$, so (3.1) holds. Further, $t_{2j} = t_j$ for all j which are not powers of 2, hence the sum in (3.2) reduces to

$$\sum_{k=0}^{\infty} \left(\frac{1}{\sqrt{k+1}} - \frac{1}{\sqrt{k+2}} \right)^2 < \infty.$$

The so defined operator y is not in $\mathbb{C}1 + \mathrm{C}^2(\mathcal{H})$ since the series $\sum_{j=0}^{\infty} (t_j + \alpha)^2$ diverges for all $\alpha \in \mathbb{C}$.

Finally, we write v_1 and v_2 as linear combinations of positive elements $a_j \in \operatorname{span}\{1, v_1, v_1^*, v_2, v_2^*\}$ $(j = 1, \ldots, 8)$ such that $b := \sum_{j=1}^{8} a_j^2 < 1$; so y commutes modulo $\mathrm{C}^2(\mathcal{H})$ with all a_j, but does not commute with all a_j. Define $a_0 = (1-b)^{1/2}$ and $a = (a_0, \ldots, a_8)$. Then Ψ_a is a Lüders operator for which not all fixed points are in A' $(= \mathbb{C}1)$ by Corollary 3.1, since y commutes modulo $\mathrm{C}^2(\mathcal{H})$ with all a_j and $y \notin A' + \mathrm{C}^2(\mathcal{H})$. To show that y commutes with a_0 modulo $\mathrm{C}^2(\mathcal{H})$, we can use the holomorphic functional calculus or expand $(1-b)^{1/2}$ in the Taylor series $\sum_{k=0}^{\infty} c_k b^k$. Using the identity $b^k y - y b^k = \sum_{j=0}^{k-1} b^{k-j-1}(by - yb)b^j$, we estimate

$$\|a_0 y - y a_0\|_2 \leqslant \sum_{k=1}^{\infty} |c_k| \left\| b^k y - y b^k \right\|_2 \leqslant \sum_{k=0}^{\infty} k |c_k| \|b\|^{k-1} \|by - yb\|_2 < \infty,$$

hence $a_0 y - y a_0 \in \mathrm{C}^2(\mathcal{H})$.

4. The case of commuting operators

In this section we study the spectrum and fixed points of normal completely bounded maps on $\mathrm{B}(\mathcal{K}, \mathcal{H})$, where $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces. We denote by $\mathrm{CB}(\mathrm{B}(\mathcal{K}, \mathcal{H}))$ the space of all completely bounded maps on $\mathrm{B}(\mathcal{K}, \mathcal{H})$. Given C*-subalgebras $A \subseteq \mathrm{B}(\mathcal{H})$ and $B \subseteq \mathrm{B}(\mathcal{K})$, we let $A \overset{eh}{\otimes} B$ be the Banach subalgebra of $\mathrm{CB}(\mathrm{B}(\mathcal{K}, \mathcal{H}))$ consisting of all maps Θ that can be represented in the form

$$\Theta(x) := \sum_{j=1}^{\infty} c_j x d_j, \tag{4.1}$$

where $c_j \in A$ and $d_j \in B$ are such that the row $c = [c_j]$ and the column $d = (d_j)$ represent bounded operators in $\mathrm{B}(\mathcal{H}^{\infty}, \mathcal{H})$ and $\mathrm{B}(\mathcal{K}, \mathcal{K}^{\infty})$, respectively. Thus the sums

$$\sum_{j=1}^{\infty} c_j c_j^* \quad \text{and} \quad \sum_{j=1}^{\infty} d_j^* d_j \tag{4.2}$$

converge in the strong operator topology. We will write such a map Θ simply as

$$\Theta = c \odot d = \sum_{j=1}^{\infty} c_j \otimes d_j.$$

The space $A \overset{eh}{\otimes} B$ coincides with the extended Haagerup tensor product (defined in [5,16,25]), but we shall not need this fact. The subspace $A \overset{h}{\otimes} B$ of $A \overset{eh}{\otimes} B$, consisting of elements $c \odot d \in A \overset{eh}{\otimes} B$ for which the two sums in (4.2) are norm-convergent, is a Banach subalgebra of $A \overset{eh}{\otimes} B$, and can be identified with the Haagerup tensor product, but again we shall not need this last fact. If M and N are von Neumann algebras then $M \overset{eh}{\otimes} N$ coincides with the space $\mathrm{NCB}_{M',N'}(\mathrm{B}(\mathcal{K},\mathcal{H}))$ of all normal completely bounded M', N'-bimodule endomorphisms of $\mathrm{B}(\mathcal{K},\mathcal{H})$ (see [34] or [24, 1.2]; here M' denotes the commutant of M). It is well known that a weak* continuous map Θ between Banach spaces is invertible if and only if its preadjoint map Θ_* is invertible [10]. Thus, if $\Theta \in M \overset{eh}{\otimes} N$ is invertible, then so is Θ_* (as a bounded map on $\mathrm{T}(\mathcal{H},\mathcal{K})$), hence $\Theta^{-1} = ((\Theta_*)^{-1})^*$ is weak* continuous. Since Θ^{-1} is also an M', N'-bimodule map, it follows that $M \overset{eh}{\otimes} N$ is an inverse-closed subalgebra of $\mathrm{CB}(\mathrm{B}(\mathcal{K},\mathcal{H}))$. The spectrum of an element c in a Banach algebra A is denoted by $\sigma_A(c)$. We summarize the above discussion in the following proposition.

Proposition 4.1. *If M and N are von Neumann subalgebras of $\mathrm{B}(\mathcal{H})$ and $\mathrm{B}(\mathcal{K})$ (respectively) then*

$$\sigma_{M \overset{eh}{\otimes} N}(\Theta) = \sigma_{\mathrm{CB}(\mathrm{B}(\mathcal{K},\mathcal{H}))}(\Theta)$$

for each $\Theta \in M \overset{eh}{\otimes} N$.

In many cases Proposition 4.1 can be sharpened to the identity $(M \overset{eh}{\otimes} N)^{cc} = M \overset{eh}{\otimes} N$. Namely, it is known (see [15] or [19]) that the commutant $(M \overset{eh}{\otimes} N)^c$ of $M \overset{eh}{\otimes} N$ inside $\mathrm{CB}(\mathrm{B}(\mathcal{K},\mathcal{H}))$ is the algebra $\mathrm{CB}_{M,N}(\mathrm{B}(\mathcal{K},\mathcal{H}))$ of all completely bounded M, N-bimodule endomorphisms of $\mathrm{B}(\mathcal{K},\mathcal{H})$, which we will denote simply by $M' \overset{\sigma}{\otimes} N'$, thus

$$\bigl(M \overset{eh}{\otimes} N\bigr)^c = M' \overset{\sigma}{\otimes} N'. \tag{4.3}$$

(We remark that the notation $M' \overset{\sigma}{\otimes} N'$ usually means the normal Haagerup tensor product as defined in [14,16], [4, p. 41], but the two algebras $M' \overset{\sigma}{\otimes} N'$ and $\mathrm{CB}_{M,N}(\mathrm{B}(\mathcal{K},\mathcal{H}))$ are naturally completely isometrically and weak* homeomorphically isomorphic by [14] (a simpler proof of a more general fact is in [26, 4.4]).) By a surprising result of Hofmeier and Wittstock [19] the commutant of $M' \overset{\sigma}{\otimes} N'$ in $\mathrm{CB}(\mathrm{B}(\mathcal{K},\mathcal{H}))$ consists only of weak* continuous maps, if M and N do not have central parts of type $I_{\infty,n}$ for $n \in \mathbb{N}$, that is

$$\bigl(M' \overset{\sigma}{\otimes} N'\bigr)^c = M \overset{eh}{\otimes} N. \tag{4.4}$$

(In [19] only the case $N = M$ is considered, but the usual argument with the direct sum $M \oplus N$ reduces the general situation to this case.) This holds in particular when M and N are abelian, thus, in this case we deduce from (4.3) and (4.4) that $(M \overset{eh}{\otimes} N)^{cc} = M \overset{eh}{\otimes} N$.

For noncommuting sequences (c_j) and (d_j) not much is known about the spectrum of the operator $\Theta = c \odot d$ defined by (4.1). For example, if $d_j = c_j$ are positive, it was not known even if the spectrum of Θ is contained in $\mathbb{R}^+$ [28]. We mention here the following consequence of results of Shulman and Turovskii [33], which extends [28, Corollary 6].

Proposition 4.2. *Suppose that $c_j \in \mathrm{B}(\mathcal{H})$, $d_j \in \mathrm{B}(\mathcal{K})$ are positive and such that $\sum_{j=1}^{\infty} \|c_j\| \|d_j\| < \infty$. If for each j at least one of the operators c_j, d_j is compact then all eigenvalues of the operator $\Theta = c \odot d$ defined by (4.1) on $\mathrm{B}(\mathcal{K},\mathcal{H})$ are in $\mathbb{R}^+$.*

Proof. By [33, 6.10] each eigenvector corresponding to a nonzero eigenvalue λ of Θ is nuclear, hence in particular in the Hilbert–Schmidt class $\mathrm{C}^2(\mathcal{K},\mathcal{H})$. Since the restriction $\Theta|\mathrm{C}^2(\mathcal{K},\mathcal{H})$ is a positive operator on a Hilbert space, its spectrum is contained in $\mathbb{R}^+$, hence $\lambda \in \mathbb{R}^+$. □

We denote by $\Delta(A)$ the spectrum (that is, the space of all multiplicative linear functionals) of a commutative Banach algebra A. If A and B are commutative operator algebras then it is easy to see that

$$\Delta\bigl(A \overset{h}{\otimes} B\bigr) = \Delta(A) \times \Delta(B). \tag{4.5}$$

For the spectrum of $A \overset{eh}{\otimes} B$, however, there is no such simple formula. In the case when M and N are (abelian) von Neumann algebras there is an injective contraction from $M \overset{eh}{\otimes} N$ into $M \overline{\otimes} N$ (which will be regarded as inclusion and is dual to the natural contraction $M_* \overset{\wedge}{\otimes} N_* \to M_* \overset{h}{\otimes} N_*$ [4, 1.5.13], [16, 6.1]), and one might conjecture that the spectrum of an element of $M \overset{eh}{\otimes} N$ is the same as the spectrum of its image in $M\overline{\otimes}N$, but this is not always true even in the special case $M = \ell^\infty(\mathbb{N}) = N$. In this case $C := M\overline{\otimes}N$ is the von Neumann algebra $\ell^\infty(\mathbb{N}\times\mathbb{N})$ of all bounded sequences on $\mathbb{N}\times\mathbb{N}$. Further, $D := M \overset{eh}{\otimes} N$ is the algebra of all Schur multipliers on $\mathrm{B}(\ell^2(\mathbb{N}))$ (see [29, Theorem 5.1]), which consists of all sequences $d \in \ell^\infty(\mathbb{N}\times\mathbb{N})$ such that the double sequence $[d_{i,j}x_{i,j}]$ is a matrix of a bounded operator on $\ell^2(\mathbb{N})$ for every $[x_{i,j}]$ representing a bounded operator on $\ell^2(\mathbb{N})$. Such an element $d \in D$ is invertible in C if and only if the closure of the set $\{d_{i,j}\}$ in $\mathbb{C}$ does not contain 0, but this does not guarantee invertibility of d in D. To see this, we consider the following example suggested to us by Milan Hladnik and Victor Shulman.

Example 4.3. Let D_0 be the subalgebra of D consisting of Toeplitz–Schur multipliers, that is, Schur multipliers $d = [d_{i-j}]$ that are constant along the diagonals. If d is invertible in D, then d^{-1} is also the inverse of d in C, hence d^{-1} consists of the double sequence $[d^{-1}_{i-j}]$, which is in D_0; so D_0 is inverse-closed in D. On the other hand, it is known that the entries of each Toeplitz–Schur multiplier $[d_{i-j}]$ are the Fourier coefficients of a complex regular Borel measure μ on the unit circle $\mathbb{T}$ (that is, $d_k = \int_{\mathbb{T}} \bar{z}^k\, d\mu$) and conversely; that is, D_0 is isomorphic to the measure algebra $M(\mathbb{T})$ for the convolution. (A proof of this can be found in [1].) But by [32, 5.3.4] there exists a noninvertible measure $\mu \in M(\mathbb{T})$ such that the Fourier coefficients of μ are all real and $\geqslant 1$, so the corresponding Schur multiplier is invertible in C but not in D. Moreover, by [32, Theorem 6.4.1] the spectrum of such a multiplier $[d_{i-j}]$ can contain any point in $\mathbb{C}$ even if $d_k \in [-1,1]$ for all k.

We remark that the spectra of elementary operators $x \mapsto \sum_{j=1}^m c_j x d_j$, where m is finite and $c = (c_j), d = (d_j) \subseteq \mathrm{B}(\mathcal{H})$ are two commutative families, have been intensively studied in the past (see [13] and the references in [13] and in [2]), but the results do not apply to the case of infinite m, where the two series $\sum_{j=1}^\infty c_j c_j^*$ and $\sum_{j=1}^\infty d_j^* d_j$ converge in the weak* topology. Even if we assume that all the components c_j and d_j are normal operators, the above example suggests that the spectrum of $c \odot d$ cannot be described in terms of spectra of c_j and d_j in the same way as for finite m-tuples.

If A is an abelian Banach algebra and $c = (c_j)$ is a sequence of elements in A we set

$$\sigma_A(c) = \bigl\{\bigl(\rho(c_1), \rho(c_2), \ldots\bigr)\colon\ \rho \in \Delta(A)\bigr\}.$$

Lemma 4.4. *If $c = (c_j)$ is a sequence in a commutative unital C^* algebra $A \subseteq \mathrm{B}(\mathcal{H})$ such that the series $\sum_{j=1}^\infty c_j^* c_j$ is norm-convergent, then $\sigma_A(c)$ is a norm compact subset of ℓ^2. If this sum is merely weak* convergent, then $\sigma_A(c)$ is a weakly compact subset of ℓ^2.*

Proof. For any character $\rho \in \Delta(A)$ and any finite n we have

$$\sum_{j=1}^n \bigl|\rho(c_j)\bigr|^2 = \rho\left(\sum_{j=1}^n c_j^* c_j\right) \leqslant \|c\|^2,$$

which implies that $(\rho(c_j)) \in \ell^2$ with $\|(\rho(c_j))\| \leqslant \|c\|$. It is easy to prove that the map $\rho \mapsto (\rho(c_j))$ from $\Delta(A)$ to ℓ^2 is weak* to weak continuous, so its range $\sigma_A(c)$ is a weakly compact set since $\Delta(A)$ is weak* compact. If the series $\sum_{j=1}^\infty c_j^* c_j$ is norm-convergent, then the same map is weak* to norm continuous, hence $\sigma_A(c)$ is a norm compact set in this case. □

Given two elements $\lambda = (\lambda_j)$ and $\mu = (\mu_j)$ in ℓ_2 we denote

$$\lambda \cdot \mu := \sum_{j=1}^\infty \lambda_j \mu_j.$$

Further, for two subsets $\sigma_j \subseteq \ell^2$, we denote

$$\sigma_1 \cdot \sigma_2 := \{\lambda \cdot \mu\colon\ \lambda \in \sigma_1,\ \mu \in \sigma_2\}.$$

Since the map $(\lambda, \mu) \mapsto \lambda \cdot \mu$ is continuous, $\sigma_1 \cdot \sigma_2$ is a compact subset of $\mathbb{C}$ if σ_1 and σ_2 are norm compact subsets of ℓ^2.

Proposition 4.5. *Let (c_j) and (d_j) be two commutative families of normal operators in $\mathrm{B}(\mathcal{H})$ and $\mathrm{B}(\mathcal{K})$ (respectively) such that the two series (4.2) are weak* convergent. Let A and B be the C^* algebras generated by $\{1\} \cup (c_j)$ and $\{1\} \cup (d_j)$, respectively, and $\overline{A}$, $\overline{B}$ their weak* closures, so that the map $\Theta = c \odot d$ is an element of $\overline{A} \overset{eh}{\otimes} \overline{B}$.*

(i) *If the two series (4.2) are norm-convergent (that is, if $\Theta \in A \overset{h}{\otimes} B$) then $\sigma_{\mathrm{CB}(\mathrm{B}(\mathcal{H}))}(\Theta) = \sigma_A(c) \cdot \sigma_B(d)$.*
(ii) *In general the point spectrum of Θ is contained in $\sigma_A(c) \cdot \sigma_B(d)$.*

Proof. The spectrum of an element Θ in a unital commutative Banach algebra D is always equal to $\{\rho(\Theta)\colon\ \rho \in \Delta(D)\}$. This applies to our element $\Theta = c \odot d$ in $D = \overline{A} \overset{eh}{\otimes} \overline{B}$. Given $\rho \in \Delta(D)$, denote $\phi = \rho \mid (\overline{A} \otimes 1)$ and $\psi = \rho \mid (1 \otimes B)$. Then $\phi \in \Delta(A)$,

$\psi \in \Delta(B)$ and (by the norm continuity) $\rho \mid (\overline{A} \overset{h}{\otimes} \overline{B}) = \phi \otimes \psi$. Conversely, any two characters $\phi \in \Delta(\overline{A})$ and $\psi \in \Delta(\overline{B})$ define the character $\phi \otimes \psi$ on D by $(\phi \otimes \psi)(\sum_{j=1}^{\infty} x_j \otimes y_j) = \sum_{j=1}^{\infty} \phi(x_j)\psi(y_j)$. So, if the two series (4.2) are norm-convergent then $\sigma_D(\Theta) = \sigma_{\overline{A}}(c) \cdot \sigma_{\overline{B}}(d)$. Since all characters on A and B extend to characters on $\overline{A}$ and $\overline{B}$, respectively, it also follows that $\sigma_{A \overset{h}{\otimes} B}(\Theta) = \sigma_A(c) \cdot \sigma_B(d) = \sigma_D(\Theta)$. By Proposition 4.1 we have that $\sigma_{\mathrm{CB}(\mathrm{B}(\mathcal{K},\mathcal{H}))}(\Theta) = \sigma_D(\Theta)$ for each $\Theta \in D$. This concludes the proof of (i).

To prove (ii), let λ be an eigenvalue of Θ and $x \in \mathrm{B}(\mathcal{K},\mathcal{H})$ a corresponding nonzero eigenvector, so that $\Theta(x) = \lambda x$. Using a variant of Egoroff's theorem [35, p. 85] it follows that in any neighborhoods of the identity 1 (in the strong operator topology) there exist projections $e \in \overline{A}$ and $f \in \overline{B}$ such that the two series $\sum_{j=1}^{\infty} c_j c_j^* e$ and $\sum_{j=1}^{\infty} d_j^* d_j f$ converge uniformly. We may choose e and f so that $exf \neq 0$. Since

$$(ec)(exf)^{\infty}(df) = \lambda exf,$$

λ is an eigenvalue of $(ec) \odot (df)$, hence $\lambda \in \sigma_{\overline{A}}(ec) \cdot \sigma_{\overline{B}}(df)$ by (i). Since $\phi(e) \in \{0,1\}$ for each $\phi \in \Delta(\overline{A})$ and similarly for f, it follows that $\lambda \in \sigma_{\overline{A}}(c) \cdot \sigma_{\overline{B}}(d) = \sigma_A(c) \cdot \sigma_B(d)$ if $\lambda \neq 0$. If $\lambda = 0$, we apply the result just obtained to the map $\Theta + 1 = \tilde{c} \odot \tilde{d}$, where $\tilde{c} = [1, c_1, c_2, \ldots]$ and $\tilde{d} = (1, d_1, d_2, \ldots)$ and the eigenvalue 1 of this map. □

Theorem 4.6. *Let $a = (a_j)$ and $b = (b_j)$ be two commutative sequences of normal operators on (separable) Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ (respectively) such that $\sum_{j=1}^{\infty} a_j a_j^* = 1$ and $\sum_{j=1}^{\infty} b_j^* b_j = 1$, where the sums are weak* convergent. Then the fixed points of the map $\Theta = a \odot b = \sum_{j=1}^{\infty} a_j \otimes b_j$ on $\mathrm{B}(\mathcal{H})$ are precisely the operators $x \in \mathrm{B}(\mathcal{H})$ that intertwine a and b^* (that is, $a_j x = x b_j^*$ for all j).*

Proof. Clearly the intertwiners of a and b^* are fixed points of Θ since $\sum_{j=1}^{\infty} b_j^* b_j = 1$, so only the converse needs a proof. By considering

$$\begin{bmatrix} a_j & 0 \\ 0 & b_j^* \end{bmatrix}, \quad \begin{bmatrix} a_j^* & 0 \\ 0 & b_j \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$$

instead of a_j, b_j and x (respectively), the proof can easily be reduced to the case where $b_j = a_j^*$. So we assume that $b_j = a_j^*$ for all j and we have to prove that each fixed point x of Θ commutes with all a_j. Let A be the C*-algebra generated by 1 and (a_j) and let e be the spectral measure on $\Delta := \Delta(A)$ such that

$$c = \int_{\Delta} \hat{c}(\phi)\, de(\phi)$$

for all $c \in A$, where $\hat{c}$ is the Gelfand transform of c [10, p. 266]. It suffices to show that $xe(K) = e(K)x$ for each compact subset K of Δ or, equivalently, that $e(K)^{\perp} xe(K) = 0$, where $e(K)^{\perp} = 1 - e(K)$. Since $e(K)^{\perp} = e(\Delta \setminus K)$ is the join of all the projections $e(H)$ for compact subsets H of $K^c := \Delta \setminus K$, it suffices to show that $e(H)xe(K) = 0$ for all such H. Assume the contrary, that

$$e(H)xe(K) \neq 0$$

for some compact $H \subseteq K^c$. Consider the orthogonal decomposition

$$\mathcal{H} = e(H)\mathcal{H} \oplus e(K)\mathcal{H} \oplus e\bigl(H^c \cap K^c\bigr)\mathcal{H} \tag{4.6}$$

and let $x = [x_{k,l}]$ be the corresponding representation of x by a 3×3 operator matrix. With respect to the decomposition (4.6) each operator a_j is represented by a diagonal matrix $a_j = c_j \oplus d_j \oplus f_j$ (where, for example, $c_j = a_j e(H) \mid e(H)\mathcal{H}$). Then the (1, 2) entry of the matrix $\Theta(x) = \sum_{j=1}^{\infty} a_j x a_j^*$ is $\sum_{j=1}^{\infty} c_j x_{1,2} d_j^*$, where $x_{1,2} = e(H)xe(K) \neq 0$. From $\Theta(x) = x$ we have

$$\sum_{j=1}^{\infty} c_j x_{1,2} d_j^* = x_{1,2},$$

which means that 1 is an eigenvalue of the map $\Theta_{c,d^*} := \sum_{j=1}^{\infty} c_j \otimes d_j^*$. By Proposition 4.5

$$1 = \langle \lambda, \mu \rangle \quad \text{for some } \lambda \in \sigma_{Ae(H)}(c),\ \mu \in \sigma_{Ae(K)}(d). \tag{4.7}$$

Since $\sum_{j=1}^{\infty} c_j c_j^* = e(H)$ and $\sum_{j=1}^{\infty} d_j d_j^* = e(K)$, it follows that $\|\lambda\| \leqslant 1$ and $\|\mu\| \leqslant 1$, hence (4.7) implies that $\mu = \lambda$. Therefore

$$\sigma_{Ae(H)}(c) \cap \sigma_{Ae(K)}(d) \neq \emptyset. \tag{4.8}$$

On the other hand, the map $\hat{a} : \phi \mapsto (\phi(a_1), \phi(a_2), \ldots)$ from Δ into ℓ^2 is injective. Since the C*-algebra $Ae(H)$ is isomorphic to $C(H)$ (complex-valued continuous functions on H) by Tietze's theorem, $\Delta(Ae(H)) \cong H$. (That is, all characters of $Ae(H)$ are evaluations at points of H.) Hence $\sigma_{Ae(H)}(c) = \sigma_{Ae(H)}(ae(H)) = \hat{a}(H)$. Similarly $\sigma_{Ae(K)}(d) = \hat{a}(K)$. Since H and K are disjoint and $\hat{a}$ is injective, $\sigma_{Ae(H)}(c)$ and $\sigma_{Ae(K)}(d)$ must also be disjoint, but this is in contradiction with (4.8). □

Problem. Does the conclusion of Theorem 4.6 still hold if, instead of commutativity, we assume that each of the two sequences (a_j) and (b_j) is contained in a finite injective von Neumann algebra?

Note added in revision. After this paper had already been sent to publication, I received a letter from professor B. Prunaru in which he mentioned the paper [22]. In [22] an example of quantum operation with $\mathcal{F}_a \neq A'$ had already been found; but in the example the coefficients a_j are not positive. Further, the recent paper [31] contains a very short and elegant proof of the fact that $\mathcal{F}_a = A'$ if the C*-algebra $A = C^*(a_j)$ is abelian. I am grateful to professor Prunaru for this information.

Recently it has been shown in [27] that spectra of Lüders operators on $\mathrm{B}(\mathcal{H})$ are not necessarily contained in $\mathbb{R}^+$ if $\mathcal{H}$ is infinite dimensional.

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