SUMS OF PRODUCTS OF POSITIVE OPERATORS AND SPECTRA OF LÜDERS OPERATORS

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Abstract. Each bounded operator T on an infinite dimensional Hilbert space $\mathcal{H}$ is a sum of three operators that are similar to positive operators; two such operators are sufficient if T is not a compact perturbation of a scalar. The spectra of Lüders operators (elementary operators on $\mathrm{B}(\mathcal{H})$ with positive coefficients) of lengths at least three are not necessarily contained in $\mathbb{R}^+$. On the other hand, the spectra of such operators of lengths (at most) two are contained in $\mathbb{R}^+$ if the coefficients on one side commute.

1. Introduction

Completely positive maps on $\mathrm{B}(\mathcal{H})$ (the algebra of all bounded operators on a Hilbert space $\mathcal{H}$) of the form

$$\Psi(X) = \sum_{j=1}^{n} A_j^* X A_j, \tag{1.1}$$

have received a renewed interest recently especially in connection with quantum information theory (see [8], [9], [13], [18] and the references there). If all the coefficients A_j in (1.1) are positive operators such a map is called a Lüders operation. If n is finite then these are special cases of elementary operators, that is, maps of the form $X \mapsto \sum_{j=1}^{n} A_j X B_j$, whose spectra have been intensively studied in the past (see [5] and the references there), but only in the cases when both families of coefficients (A_j) and (B_j) are commutative. If $\mathcal{H}$ is finite dimensional, then $\mathrm{B}(\mathcal{H})$ is a Hilbert space for the inner product induced by the trace and it is easily verified that an elementary operator with positive coefficients A_j and B_j is a positive operator on this Hilbert space, so its spectrum is contained in $\mathbb{R}^+ := [0, \infty)$.

At the end of the paper [11] it was asked if the spectrum of a Lüders operator $X \mapsto \sum_{j=1}^{n} A_j X A_j$ with positive coefficients on $\mathrm{B}(\mathcal{H})$ is necessarily contained in $\mathbb{R}^+$ if $\mathcal{H}$ is infinite dimensional. We will show that, contrary to what one might expect, the answer to this question is negative. This will be a consequence of the fact that the operator $T = -1$ can be expressed as

$$T = \sum_{j=1}^{n} A_j B_j \quad \text{with positive } A_j, B_j \in \mathrm{B}(\mathcal{H}). \tag{1.2}$$

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At first the author did not know how do prove that every operator $T \in \mathrm{B}(\mathcal{H})$ is of the form (1.2), but then professor Heydar Radjavi told him that by [16] and [12] T is a sum of finitely many idempotents and, since every idempotent is similar to a projection, T is a sum of products of positive operators. To see this, note that an operator Q which is similar to a positive operator, say $Q = SPS^{-1}$, is a product of two positive operators: $Q = (SS^*)((S^{-1})^*PS^{-1})$. By Pearcy and Topping [12]) five idempotents are sufficient to express any T in this way and according to [19, Proposition 5.9] this is the minimal number since scalars are in general not sums of less than five idempotents. However, since idempotents are very special elements, we can not expect that 5 is the minimal n in (1.2).

One of the goals of this paper is to find the minimal n above. The result will imply that even the spectrum of a Lüders operator of small length is not necessarily contained in $\mathbb{R}^+$. More precisely, in the next section we will show that every $T \in \mathrm{B}(\mathcal{H})$ is a sum of three operators T_j each of which is similar to a positive operator. Moreover, if T is not a compact perturbation of a scalar, two operators T_j are sufficient. This result is optimal since compact perturbations of nonzero scalars can not be expressed in the form (1.2) with $n \leq 2$. We will also show that the trace class operators with trace not in $\mathbb{R}^+$ can not be expressed as $T_1 + T_2$ with both T_1 and T_2 similar to positive operators in $\mathrm{B}(\mathcal{H})$. As a preliminary step in the proof of the main result we will first show that T is a sum of four operators T_j similar to positive ones, with some additional properties needed.

In the last section we will first apply this result to answer the above mentioned question from [11]. Then we will prove that the spectra of operators of the form $X \mapsto \sum_{j=1}^{2} A_j X B_j$ with positive A_j and B_j are contained in $\mathbb{R}^+$ if $A_1A_2 = A_2A_1$ (or if $B_1B_2 = B_2B_1$).

Throughout the paper $\mathcal{H}$ denotes an infinite dimensional separable Hilbert space and $\mathrm{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. (The results hold also for non separable $\mathcal{H}$, but in their formulations the ideal of compact operators must be replaced by the unique proper maximal ideal of $\mathrm{B}(\mathcal{H})$.) An operator $T \in \mathrm{B}(\mathcal{H})$ is called positive if $\langle T\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$ (thus T is not necessarily definite) and the set of all positive operators is denoted by $\mathrm{B}(\mathcal{H})^+$.

2. Sums of operators similar to positive operators

We begin with a simple and well-known observation. Let $S \in \mathrm{B}(\mathcal{K} \oplus \mathcal{K})$ be a 2×2 operator matrix

$$S = \begin{bmatrix} u & x \\ y & z \end{bmatrix}, \tag{2.1}$$

where u is invertible. Then S is invertible if and only if $z - yu^{-1}x$ is invertible and in this case

$$S^{-1} = \begin{bmatrix} u^{-1}(1 + xdyu^{-1}) & -u^{-1}xd \\ -dyu^{-1} & d \end{bmatrix}, \quad \text{where } d = (z - yu^{-1}x)^{-1}. \tag{2.2}$$

To prove this, multiply S from the left by the invertible matrix

$$\begin{bmatrix} u^{-1} & 0 \\ -yu^{-1} & 1 \end{bmatrix}$$

to obtain an upper-triangular matrix with 1 and $z - yu^{-1}x$ along the diagonal.

The main assertion of the following lemma can be deduced from the proof of Theorem 1 in [12], but later we will need some additional information from its proof in the form presented below.

Lemma 2.1. *Every operator $T \in \mathrm{B}(\mathcal{H})$ is a sum of the form*

$$T = \sum_{j=1}^{4} S_j T_j S_j^{-1},$$

where $S_j \in \mathrm{B}(\mathcal{H})$ and the operators $T_j \in \mathrm{B}(\mathcal{H})$ are positive with disjoint spectra $\sigma(T_j)$, each $\sigma(T_j)$ consists of at most two points, $\sigma(T_1) \subset [0,1]$ and $\sigma(T_j) \subset (1,\infty)$ for $j \neq 1$. Moreover, the range of T_1 is closed and has infinite dimension and codimension.

In particular, T can be written as $T = \sum_{j=1}^{4} A_j B_j$, where $A_j, B_j \in \mathrm{B}(\mathcal{H})^+$.

Proof. Decompose $\mathcal{H}$ into an orthogonal sum of two isomorphic closed subspaces, $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$; then T is represented by an operator matrix of the form

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}. \tag{2.3}$$

First we will try to find diagonal positive operators $T_j = a_j \oplus b_j$ ($a_j, b_j \in \mathrm{B}(\mathcal{K})$) and invertible operators S_j ($j = 1,\ldots,4$) of the form (2.1) such that $T = \sum_{j=1}^{4} S_j T_j S_j^{-1}$. It turns out that we can even take S_j of the form

$$S_j = \begin{bmatrix} 1 & x_j \\ y_j & 1 + y_j x_j \end{bmatrix}.$$

Then

$$S_j T_j S_j^{-1} = \begin{bmatrix} a_j + s_j y_j & -s_j \\ y_j a_j - b_j y_j + y_j s_j y_j & b_j - y_j s_j \end{bmatrix}, \quad \text{where } s_j := a_j x_j - x_j b_j.$$

There are many appropriate choices for x_j, y_j, z_j, a_j, b_j in order to make the sum $\sum_{j=1}^{4} S_j T_j S_j^{-1}$ equal to T. For example, if we let $y_1 = 0 = x_2$, $y_3 = 1$, $b_1 = 0$ and for $j \geq 2$ choose all a_j and b_j to be positive scalars with $a_j - b_j = 1$, and denote $\beta = \sum_{j=2}^{4} b_j$ (so that $\sum_{j=2}^{4} a_j = \beta + 3$), then

$$\sum_{j=1}^{4} S_j T_j S_j^{-1} = \begin{bmatrix} a_1 + \beta + 3 + x_3 + x_4 y_4 & -a_1 x_1 - x_3 - x_4 \\ y_2 + x_3 + 1 + y_4 + y_4 x_4 y_4 & \beta - x_3 - y_4 x_4 \end{bmatrix}. \tag{2.4}$$

To achieve that the matrix in (2.4) will be equal to T, we only need to choose x_3, x_4, y_4 in $\mathrm{B}(\mathcal{K})$ and invertible $a_1 \in \mathrm{B}(\mathcal{K})^+$ so that

$$a_1 + \beta + 3 + x_3 + x_4 y_4 = A \quad \text{and} \quad \beta - x_3 - y_4 x_4 = D, \tag{2.5}$$

for then the off-diagonal terms of the matrix (2.4) can be made equal to B and C by a suitable choice of y_2 and x_1. Adding the two equations (2.5) we see, that we only need to choose x_4, y_4 and a_1 so that

$$x_4 y_4 - y_4 x_4 = A + D - a_1 - 2\beta - 3 =: T_0, \tag{2.6}$$

for then x_3 can be computed from either of the equations (2.5). So (for a fixed β), we first choose an invertible positive $a_1 \in \mathrm{B}(\mathcal{K})$ of the form $\lambda + \mu p$, where $\lambda, \mu \in \mathbb{R}^+$ and p is a projection of infinite rank and nullity, such that $\sigma(a_1) \subset (0,1]$ and T_0 is not a compact perturbation of a scalar. Then T_0 is a commutator by [2] (a simplified proof is in [1]), which means that there exist x_4 and y_4 satisfying

(2.6). By suitably choosing scalars a_j and b_j ($j \geq 2$) we can make the spectra of T_j disjoint for all j. $\square$

Remark 2.2. For a later use observe that in the above proof the spectra of a_j and b_j are disjoint for all j; in fact all a_j and b_j chosen above are scalars, except possibly a_1. Also note that the operator $S_1 T_1 S_1^{-1}$ has the form

$$\begin{bmatrix} a_1 & * \\ 0 & 0 \end{bmatrix},$$

where $a_1 \in \mathrm{B}(\mathcal{K})^+$.

Theorem 2.3. *Every $T \in \mathrm{B}(\mathcal{H})$ is of the form $T = \sum_{j=1}^{3} S_j T_j S_j^{-1}$, where $S_j \in \mathrm{B}(\mathcal{H})$ and the operators $T_j \in \mathrm{B}(\mathcal{H})$ are positive (and invertible for $j \leq 2$) with finite spectra $\sigma(T_j)$, each $\sigma(T_j)$ consists of at most four points. Moreover, 0 is an isolated point of $\sigma(T_3)$, the range of T_3 is closed and has infinite dimension and codimension.*

Proof. As in the proof of Lemma 2.1 we represent T by the operator matrix (2.3). Now we try to find positive block-diagonal operators $T_j = a_j \oplus b_j$ and invertible operators $S_j \in \mathrm{B}(\mathcal{H})$ of the form (2.1) (with $z - yu^{-1}x = 1$) such that $\sum_{j=1}^{3} S_j T_j S_j^{-1} = T$. Denoting

$$S_j = \begin{bmatrix} u_j & x_j \\ y_j & z_j \end{bmatrix}, \text{ where } u_j \text{ is invertible and } z_j - y_j u_j^{-1} x_j = 1,$$

we compute (using (2.2)) that

$$S_j T_j S_j^{-1} = \begin{bmatrix} c_j + s_j v_j & -s_j \\ v_j c_j - b_j v_j + v_j s_j v_j & b_j - v_j s_j \end{bmatrix},$$

where

$$c_j := u_j a_j u_j^{-1}, \quad v_j := y_j u_j^{-1}, \quad \text{and} \quad s_j := c_j x_j - x_j b_j. \tag{2.7}$$

Note that if the spectra of b_j and c_j are disjoint, then from (2.7) a_j, y_j, b_j and x_j can all be computed from c_j, u_j, v_j, b_j, and s_j. (That the equation $c_j x_j - x_j b_j = s_j$ can be solved for x_j is Rosenblum's theorem [14, p. 8].) Further, we assume that the matrix S_3 is diagonal (that is, $x_3 = 0 = y_3$, so we will only need that the spectra of c_j and b_j are disjoint for $j = 1, 2$). Then the condition $\sum S_j T_j S_j^{-1} = T$ is equivalent to the following four equations:

$$s_1 v_1 + s_2 v_2 = A - c_1 - c_2 - c_3, \quad s_1 + s_2 = -B, \tag{2.8}$$

$$v_1 c_1 - b_1 v_1 + v_2 c_2 - b_2 v_2 + v_1 s_1 v_1 + v_2 s_2 v_2 = C, \;\; v_1 s_1 + v_2 s_2 = -D + b_1 + b_2 + b_3. \tag{2.9}$$

Set $c := c_1 + c_2 + c_3$, $b := b_1 + b_2 + b_3$ and

$$s := s_1, \quad v := v_2, \quad w := v_2 - v_1.$$

Then from the second equation in (2.8) we get $s_2 = -(B + s)$; using this, the other three equations (2.8), (2.9) can be rewritten as

$$Bv + sw = c - A, \qquad vB + ws = D - b, \tag{2.10}$$

$$v(c_1 + c_2 - sw) - (b_1 + b_2 + ws)v - wc_1 + b_1 w + wsw - vBv = C. \tag{2.11}$$

From (2.10) we have that $c_1+c_2-sw = A-c_3+Bv$ and $b_1+b_2+ws = D-b_3-vB$, hence (2.11) can be rewritten as

$$wsw - wc_1 + b_1w = C - v(A-c_3) + (D-b_3)v - vBv. \tag{2.12}$$

We are going to show that the system of equations (2.10), (2.12) has a solution.

First suppose that T is not a compact perturbation of a scalar. Then we may assume that in the matrix representation of T we have that $D = 0$ and that B is an isometry with the range of B isomorphic to its orthogonal complement in $\mathcal{K}$ since by [2, Corollary 3.4] T is similar to such an operator. In this case we shall see that we can even afford to choose $s = 0$, so that the above system of equations simplifies to

$$Bv = c - A, \tag{2.13}$$

$$vB = -b, \tag{2.14}$$

$$b_1w - wc_1 = C - v(A-c_3) + (-b_3)v - vBv. \tag{2.15}$$

Since $B^*B = 1$, the equation (2.13) is equivalent to the following two:

$$v = B^*(c-A) \quad \text{and} \quad P^{\perp}(c-A) = 0, \text{ where } P := BB^* \text{ and } P^{\perp} := 1-P. \tag{2.16}$$

Using this expression for v, (2.14) can be rewritten as

$$b_1 + b_2 + b_3 = b = B^*(A-c)B. \tag{2.17}$$

If there exist v, c_j and b_j $(j = 1,2,3)$ such that the equations (2.16) and (2.17) are satisfied and the spectra of c_1 and b_1 are disjoint, then the equation (2.15) can be solved for w by Rosenblum's theorem.

To show that the system (2.16), (2.17) has a solution, represent A by a 2×2 operator matrix with respect to the decomposition $\mathcal{K} = P\mathcal{K}\oplus P^{\perp}\mathcal{K}$. By Lemma 2.1 $A = \sum_{j=1}^{4} A_j$ where each A_j is similar to a positive operator; moreover, by Remark 2.2 we may assume that (with respect to the decomposition $\mathcal{K} = P\mathcal{K}\oplus P^{\perp}\mathcal{K}$) A_4 is of the form

$$A_4 = \begin{bmatrix} a & r \\ 0 & 0 \end{bmatrix}, \text{ where } a \geq 0, \tag{2.18}$$

which means that $P^{\perp}A_4 = 0$. Thus, if we put $c_j = A_j$ for $j = 1,2,3$ (and $c = c_1+c_2+c_3$), then we have $P^{\perp}(A-c) = P^{\perp}A_4 = 0$, which is just the condition in (2.16). Further

$$B^*(A-c)B = B^*A_4B = B^*A_4PB = B^*GB, \tag{2.19}$$

where

$$G := A_4P = a\oplus 0.$$

Thus the operator $B^*(A-c)B$ is positive and hence it can be written (in many ways) as a sum of three positive operators b_j, which is just what the condition (2.17) requires. We may choose $b_3 = 0$. To see that it is possible to choose b_j and c_j $(j = 1,2)$ so that their spectra are disjoint, note that PB is a unitary operator from $\mathcal{K}$ onto $P\mathcal{K}$ which intertwines b and a by (2.19) and (2.17), hence b and a have the same spectrum. By Lemma 2.1 we may choose a and $c_j = A_j$ so that each of their spectra consists of at most two points, $\sigma(a)\subseteq(0,1]$ and $\sigma(A_j)\subset(1,\infty)$ $(j = 1,2,3)$. Since $b_j\geq 0$ and $b_1+b_2 = b$, the spectra of b_j are contained in $[0,1]$, hence $\sigma(b_j)\cap\sigma(c_j) = \emptyset$. Since $\sigma(b)$ consists of at most two points in $(0,1]$, we may choose b_1, b_2 to have the same property. (We may choose for b_1 a sufficiently small positive scalar, for example.)

Since T_j is similar to $a_j \oplus b_j$ and a_j is similar to $c_j = A_j$ $(j = 1, 2, 3)$, $\sigma(T_j) = \sigma(A_j) \cup \sigma(b_j)$ consists of at most four points. Other properties of operators T_j stated in the theorem also follows easily from that of c_j and a_j chosen above.

Now we consider the case when T is a compact perturbation of a scalar. In this case let $E = 1 \oplus 0$, the projection onto the first summand in the decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$. Then $\tilde{T} := T - E$ is not a compact perturbation of a scalar, so by the already proved case $\tilde{T}$ can be expressed as $\tilde{T} = \sum_{j=1}^{3} S_j(a_j \oplus b_j)S_j^{-1}$, where $a_j \geq 0$ and $b_j \geq 0$ and S_3 is block-diagonal. Since S_3 commutes with E, we have

$$T = \tilde{T} + E = \sum_{j=1}^{2} S_j(a_j \oplus b_j)S_j^{-1} + S_3((a_3 \oplus b_3) + E)S_3^{-1},$$

which is a sum of three operators similar to positive ones with (at most) four-point spectra. □

Remark 2.4. Observe that in the proof of Theorem 2.3 the operator T_3 is of the form $e \oplus 0$ (since $b_3 = 0$ and S_3 a diagonal 2×2 operator matrix), where e is similar to a positive invertible operator with at most two-point spectrum.

Corollary 2.5. *Each $T \in \mathrm{B}(\mathcal{H})$ can be expressed as $T = \sum_{j=1}^{3} A_j B_j$, where $A_j, B_j \in \mathrm{B}(\mathcal{H})^+$.*

Theorem 2.6. *If $T \in \mathrm{B}(\mathcal{H})$ is not a compact perturbation of a scalar, then T is a sum of two operators similar to positive operators.*

Proof. We have to show that in the proof of Theorem 2.3 a_3 and b_3 can be taken to be 0. That b_3 can be taken to be 0 has been already observed in that proof. Now note that in the matrix representation (2.3) of T we may assume, in addition to $D = 0$ and B is an isometry, that A is not a compact perturbation of a scalar. For this, we simply decompose the second copy of $\mathcal{K}$ into two orthogonal isomorphic closed subspaces, $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$, and decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{K}_1^{\perp} \oplus \mathcal{K}_1$. Since B maps $\mathcal{K}_1$ isometrically into $\mathcal{K}_1^{\perp}$ the matrix of T has 0 on the $(2, 2)$ position and an isometry with infinitely codimensional range on the $(1, 2)$ position. The new element on the position $(1, 1)$ is than not a compact perturbation of a scalar. So we will assume that already in the initial matrix representation of T the element A is not a compact perturbation of a scalar. Consider now the matrix of A relative to the decomposition of the Hilbert space of A into the range of B and its orthogonal complement. Since A is not a compact perturbation of a scalar, by Theorem 2.3 and Remark 2.4 A is of the form $A = \sum_{j=1}^{3} \tilde{A}_j$, where $\tilde{A}_1$ and $\tilde{A}_2$ are similar to positive invertible operators each with at most four points in its spectrum and $\tilde{A}_3$ is of the form $e \oplus 0$ with e similar to a positive invertible operator with a two-point spectrum. By the same reasoning as in the proof of Theorem 2.3 (see the paragraph containing (2.18); the role of A_4 is now played by $\tilde{A_3}$) we see that the system of equations (2.16), (2.17) has a solution such that $c_j = \tilde{A}_j$ for $j = 1, 2$ and $c_3 = 0 = b_3 = 0$. But we have to show also that we can achieve $\sigma(c_j) \cap \sigma(b_j) = \emptyset$ $(j = 1, 2)$ in order to assure that (2.15) has a solution for w and that x_j can be computed from the last equation in (2.7). For this we note now that the operator $B^*(A - c)B = B^*\tilde{A}_3 B$ is unitarily equivalent to e. Since $\sigma(c_j)$ $(j = 1, 2)$ is a finite subset of $(0, \infty)$ and $\sigma(B^*(A - c)B)$ consists of just two positive points, it follows that $B^*(A-c)B$ is similar to a sum b_1+b_2, where $b_j \geq 0$ and $\sigma(b_j)\cap\sigma(c_j)=\emptyset$ for both $j=1,2$. $\square$

An operator $T\in \mathrm{B}(\mathcal{H})$ of the form $\lambda+K$, where $\lambda\in\mathbb{C}\setminus\mathbb{R}^+$ and K is compact, is not of the form

$$PQ+RS \quad \text{for any } P,Q,R,S\in \mathrm{B}(\mathcal{H})^+. \tag{2.20}$$

To see this, just note that the spectrum of the coset $\dot{R}\dot{S}$ in the Calkin algebra is the same as the spectrum of $\dot{S}^{1/2}\dot{R}\dot{S}^{1/2}$, hence contained in $\mathbb{R}^+$, while the spectrum of $\lambda-\dot{P}\dot{Q}$ is contained in the ray $\lambda-\mathbb{R}^+$ which is disjoint with $\mathbb{R}^+$.

Each compact operator on a Hilbert space is an additive commutator of two bounded operators [1]. By an analogy one might conjecture that each compact operator is a sum of two operators similar to positive ones, but this is not true.

Proposition 2.7. *If $T\in \mathrm{C}^1(\mathcal{H})$ (the trace class) is nonzero and $Tr\,(T)$ is not positive, then T is not a sum of two operators in $\mathrm{B}(\mathcal{H})$ similar to positive ones.*

Proof. Assume the contrary, that $T=S_1AS_1^{-1}+S_2BS_2^{-1}$, where $A,B\in \mathrm{B}(\mathcal{H})^+$. Put $F:=-S_1^{-1}TS_1$ and $S=S_1^{-1}S_2$. Then

$$F+A=-SBS^{-1}. \tag{2.21}$$

Considering the essential spectra, it follows from (2.21) and the positivity of A and B that A and B must be compact. We claim that A and B must be in the Hilbert-Schmidt class $\mathrm{C}^2(\mathcal{H})$. For a proof we may first replace B by a unitarily equivalent operator (and modify S accordingly) to reduce to the situation when A and B can be diagonalized in the same orthonormal basis $\mathbb{B}$ of $\mathcal{H}$. Let (α_j) and (β_j) be the lists of eigenvalues of A and B in decreasing order (each eigenvalue repeated according to its multiplicity). From (2.21) we have $AS+SB=G$, where $G:=-FS$. Denoting by $\sigma_{i,j}$ and $\psi_{i,j}$ the entries of the matrices of S and G in the basis $\mathbb{B}$, this means that

$$(\alpha_i+\beta_j)\sigma_{i,j}=\psi_{i,j}. \tag{2.22}$$

Let $\gamma_j:=(\sum_i|\psi_{i,j}|^2)^{1/2}$ and note that $\sum_j\gamma_j^2<\infty$ since $G\in \mathrm{C}^2(\mathcal{H})$. Since S is invertible (in particular, bounded from below), there exists a scalar $\gamma>0$ such that $\sum_i|\sigma_{i,j}|^2\geq\gamma$ for all i, hence (2.22) implies that

$$\beta_j^{-2}\gamma_j^2=\beta_j^{-2}\sum_i|\psi_{i,j}|^2=\sum_i\frac{(\alpha_i+\beta_j)^2}{\beta_j^2}|\sigma_{i,j}|^2\geq\sum_i|\sigma_{i,j}|^2\geq\gamma,$$

whenever $\beta_j\neq 0$. Thus $\beta_j^2\leq\gamma_j^2\gamma^{-1}$ and consequently $\sum_j\beta_j^2<\infty$, which means that $B\in \mathrm{C}^2(\mathcal{H})$. Similarly (or from (2.21), since $F\in \mathrm{C}^2(\mathcal{H})$) we see that $A\in \mathrm{C}^2(\mathcal{H})$.

By considering the polar decomposition of S of the form $S=RU$, where R is positive and U is unitary, we may rewrite (2.21) in the form

$$F+A=-RCR^{-1}, \tag{2.23}$$

where $C:=UBU^*\geq 0$. Assume for a moment that in some orthonormal basis of $\mathcal{H}$ the operator R can be represented by a diagonal matrix and let $[\alpha_{i,j}]$, $[\phi_{i,j}]$ and

$[\gamma_{i,j}]$ be the matrices of A, F and C (respectively) in this basis. Then, considering the sums of diagonal terms of matrices, (2.23) implies that

$$-\sum_{j=1}^{n}\phi_{j,j}=\sum_{j=1}^{n}\alpha_{j,j}+\sum_{j=1}^{n}\gamma_{j,j}. \tag{2.24}$$

Letting $n\to\infty$, the first sum in (2.24) tends to $-Tr\,(F)=Tr\,(T)\in\mathbb{C}\setminus(0,\infty)$, while the second and the third sums converge to elements in $[0,\infty]$. This shows that the equality (2.24) can hold for all n only if $Tr\,(T)=0$ and $\phi_{j,j}=0=\alpha_{j,j}$ for all j. Since $A\in\mathrm{B}(\mathcal{H})^+$, the condition $\alpha_{j,j}=0$ for all j implies that $A=0$. But then B is similar to T, hence $Tr\,(B)=0$, which implies (since $B\geq 0$) that $B=0$. In this case $T=0$, which was excluded by the hypothesis of the proposition. Now we will show by an approximation argument that (2.23) leads to a contradiction even if R can not be diagonalized.

By the Weyl - von Neumann theorem [4, p. 214], given $\varepsilon>0$, there exist a diagonal hermitian operator D and an operator $H\in\mathrm{C}^2(\mathcal{H})$ with $\|H\|_2<\varepsilon$ (where $\|\cdot\|_2$ denotes the Hilbert - Schmidt norm) such that $R=D+H$. If ε is small enough then D is invertible (since $D=R-H=R(1-R^{-1}H)$) and

$$\|D^{-1}\|\leq\|R^{-1}\|\sum_{n=0}^{\infty}\|R^{-1}H\|^n\leq\frac{\|R^{-1}\|}{1-\varepsilon\|R^{-1}\|}.$$

Further, if ε is small enough then $1+HD^{-1}$ is invertible and

$$RCR^{-1}=(1+HD^{-1})DCD^{-1}(1+HD^{-1})^{-1}.$$

Since $(1+HD^{-1})^{-1}=1-HD^{-1}(1+HD^{-1})^{-1}$, we may write

$$RCR^{-1}=DCD^{-1}-DCD^{-1}HD^{-1}(1+HD^{-1})^{-1} \\ +HCD^{-1}\left[1-HD^{-1}(1+HD^{-1})^{-1}\right],$$

hence (since B and therefore also C is in $\mathrm{C}^2(\mathcal{H})$ by the first paragraph of this proof)

$$\|RCR^{-1}-DCD^{-1}\|_1\leq\|H\|_2\|C\|_2\|D^{-1}\|\cdot \\ \left[\|D\|\|D^{-1}\|\|(1+HD^{-1})^{-1}\|+\|1-HD^{-1}(1+HD^{-1})^{-1}\|\right].$$

It follows that $\|RCR^{-1}-DCD^{-1}\|_1\to 0$ as $\varepsilon\to 0$. This allows us to conclude in essentially the same way as in the previous paragraph (by considering the sums of diagonal entries of matrices) that (2.23) leads to a contradiction. □

For most of the above proof it would be sufficient if we assumed that $T\in\mathrm{C}^2(\mathcal{H})$ (instead of $T\in\mathrm{C}^1(\mathcal{H})$), but the problem is that for an operator T not in $\mathrm{C}^1(\mathcal{H})$ the sum of diagonal entries of its matrix relative to a general orthogonal basis can be quite arbitrary (it need not even be defined [7]).

Problem. Which compact operators on an infinite dimensional Hilbert space can be written as T_1+T_2, where T_1 and T_2 are similar to positive operators?

Theorem 2.6 implies that all operators can be approximated in norm by sums of two operators similar to positive ones; but concerning such approximation a much stronger result holds: it follows from [6, Theorem 3.10] that both summands can be taken to be similar to the same positive operator.

3. On spectra of Lüders operators

For two commutative m-tuples (A_j) and (B_j) of elements of $\mathrm{B}(\mathcal{H})$ the spectrum $\sigma(\Phi)$ of the map $\Phi(X) := \sum_{j=1}^{m} A_j X B_j$ on $\mathrm{B}(\mathcal{H})$ can be described in terms of spectra of (A_j) and (B_j) ([5], [11]); in particular $\sigma(\Phi) \subseteq \mathbb{R}^+$ if $A_j, B_j \in \mathrm{B}(\mathcal{H})^+$. For noncommutative (A_j) and (B_j) the situation may be completely different. One consequence of Theorem 2.3 is that for an infinite dimensional Hilbert space $\mathcal{H}$ the spectra of Lüders operators on $\mathrm{B}(\mathcal{H})$ are not necessarily contained in $\mathbb{R}^+$.

Proposition 3.1. *Let $\mathcal{H}$ be an infinite dimensional Hilbert space. Every complex number λ can be an eigenvalue of a Lüders operator on $\mathrm{B}(\mathcal{H})$ of length 3 (or more).*

Proof. Decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$. By Corollary 2.5 there exist $A_j, B_j \in \mathrm{B}(\mathcal{K})^+$ such that $\sum_{j=1}^{3} A_j B_j = \lambda$. By a simple calculation this implies that the operator

$$X_0 := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue λ of the Lüders operator Φ on $\mathrm{B}(\mathcal{H})$ defined by $\Phi(X) = \sum_{j=1}^{3} T_j X T_j$, where

$$T_j = \begin{bmatrix} A_j & 0 \\ 0 & B_j \end{bmatrix}.$$

□

Theorem 3.2. *Suppose that $A_j, B_j \in \mathrm{B}(\mathcal{H})^+$ ($j = 1, 2$) and let Φ be the map on $\mathrm{B}(\mathcal{H})$ defined by $\Phi(X) = \sum_{j=1}^{2} A_j X B_j$. If $A_1 A_2 = A_2 A_1$ (or if $B_1 B_2 = B_2 B_1$) then the spectrum of Φ is contained in $\mathbb{R}^+$.*

Proof. Since boundary points of the spectrum of any operator are approximate eigenvalues [3], it suffices to show that each approximate eigenvalue λ of Φ is in $\mathbb{R}^+$. By considering the space $\mathcal{B} := \ell_\infty(\mathrm{B}(\mathcal{H}))/c_0(\mathrm{B}(\mathcal{H}))$, where $\ell_\infty(\mathrm{B}(\mathcal{H}))$ is the space of all bounded sequences with the entries in $\mathrm{B}(\mathcal{H})$ and $c_0(\mathrm{B}(\mathcal{H}))$ is the subspace of all sequences converging (in norm) to 0, we may reduce the approximate eigenvalues of Φ to proper eigenvalues of the corresponding operator $\tilde{\Phi}$ on $\mathcal{B}$. Here of course $\tilde{\Phi}$ is defined by $\tilde{\Phi}([X_n]) = [\Phi(X_n)]$, where $[X_n]$ denotes the coset of a sequence $(X_n) \in \ell_\infty(\mathrm{B}(\mathcal{H}))$. Note that $\tilde{\Phi}$ is again an elementary operator, namely of the form

$$\tilde{\Phi}(Y) = \sum_{j=1}^{2} \tilde{A}_j Y \tilde{B}_j \quad (Y \in B), \tag{3.1}$$

where $\tilde{A}$ denotes the coset in $\mathcal{B}$ of the constant sequence $(A, A, \ldots) \in \ell_\infty(\mathrm{B}(\mathcal{H}))$ for each $A \in \mathrm{B}(\mathcal{H})$. Since $\mathcal{B}$ is a C^*-algebra, we can regard it as a subalgebra of $\mathrm{B}(\mathcal{K})$ for some (non-separable) Hilbert space $\mathcal{K}$ and by the formula (3.1) we may regard the map $\tilde{\Phi}$ to be defined on $\mathrm{B}(\mathcal{K})$. Any approximate eigenvalue λ of Φ is then an eigenvalue of $\tilde{\Phi}$. Choose a nonzero eigenvector Y corresponding to λ. $\mathcal{K}$ is not separable, but it can be expressed as an orthogonal sum of separable subspaces $\mathcal{K}_i$ that reduce all the operators A_j, B_j and Y. If i is such that $Y|\mathcal{K}_i \neq 0$, then λ is an eigenvalue of the operator Ψ on $\mathrm{B}(\mathcal{K}_i)$ defined by $\Psi(X) = \sum_{j=1}^{2} C_j X D_j$, where $C_j = A_j|\mathcal{K}_i$ and $D_j = B_j|\mathcal{K}_i$. So it suffices to show that all eigenvalues of such operators are in $\mathbb{R}^+$. Thus, (adapting the notation) we may assume that λ is an eigenvalue of Φ. Denote by X a corresponding eigenvector with $\|X\| = 1$, hence

$$\sum_{j=1}^{2} A_j X B_j = \lambda X. \tag{3.2}$$

Suppose that A_1 and A_2 commute. Then by Voiculescu's version [17] of the Weyl-von Neumann-Berg theorem, given $\varepsilon > 0$, there exist commuting diagonal hermitian operators $C_j \in \mathrm{B}(\mathcal{H})$ and Hilbert-Schmidt operators $H_j \in \mathrm{C}^2(\mathcal{H})$ such that $A_j = C_j + H_j$ and $\|H_j\|_2 < \varepsilon$ $(j = 1, 2)$. Let $C_j = C_j^+ - C_j^-$ be the decomposition of C into the positive and the negative part and denote by Q_j the range projection of C_j^-. Then $A_j + C_j^- = C_j^+ + H_j$, hence (since $Q_j C_j^+ = 0$ and $Q_j C_j^- = C_j^-$)

$$Q_j A_j Q_j + C_j^- = Q_j H_j Q_j \in \mathrm{C}^2(\mathcal{H}).$$

This implies that $C_j^- \in \mathrm{C}^2(\mathcal{H})$ and $\|C_j^-\|_2 \le \|H_j\|_2 < \varepsilon$. So, replacing C_j by C_j^+ and H_j by $H_j - C_j^-$ (and the initial ε by $\varepsilon/2$), we may assume that $C_j \ge 0$. Let P be any finite rank projection that commutes with C_1 and C_2. (Note that, since C_1, C_2 are commuting diagonal operators, there exist a net of such projections P converging strongly to the identity.) From (3.2) we have that $\sum P A_j X B_j X^* P = \lambda P X X^* P$, hence applying the trace Tr we obtain

$$\sum_{j=1}^{2} \left(Tr\,(P C_j X B_j X^* P) + Tr\,(P H_j X B_j X^* P) \right) = \lambda Tr\,(P X X^* P). \tag{3.3}$$

Since P commutes with C_j,

$$Tr\,(P C_j X B_j X^* P) = Tr\,(C_j P X B_j X^* P) = Tr\,(C_j^{1/2} P X B_j X^* P C_j^{1/2}) \ge 0. \tag{3.4}$$

Further (since $\|Z\|_2 = \|Z^*\|_2$ for all $Z \in \mathrm{B}(\mathcal{H})$) ,

$$|Tr\,(P H_j X B_j X^* P| \le \|H_j\|_2 \|X B_j X^* P\|_2 = \|H_j\|_2 \|P X B_j X^*\|_2 < \varepsilon \|PX\|_2, \tag{3.5}$$

where we have assumed (without lost of generality) that $\|B_j\| \le 1$. If P is sufficiently close to 1 so that $PX \ne 0$, then from (3.3) and (3.5) we have that

$$\begin{aligned} \left| \lambda - \textstyle\sum_{j=1}^{2} \frac{Tr\,(P C_j X B_j X^* P)}{Tr\,(P X X^* P)} \right| &\le \varepsilon \textstyle\sum_{j=1}^{2} \frac{\|PX\|_2}{Tr\,(P X X^* P)} \\ &= \frac{2\varepsilon}{\|PX\|_2}. \end{aligned}$$

Letting in this estimate $P \to 1$, $\varepsilon \to 0$ and using (3.4), we see that $\lambda \ge 0$. □

Remark 3.3. Theorem 3.2 can be extended to operators of the form

$$X \mapsto \sum_{j=1}^{n} A_j X B_j \tag{3.6}$$

if the coefficients on one side, say all the A_j, are smooth nonnegative functions $A_j = f_j(H_1, H_2)$ of a pair of commuting hermitian operators (H_1, H_2). Namely, in this case it can be shown (using the Fourier transform) that small Hilbert-Schmidt perturbations of (H_1, H_2) result in small Hilbert-Schmidt perturbations of $f_j(H_1, H_2)$. The author does not know if the theorem can be extended to the general situation, when all the A_j commute, but the B_j do not necessarily commute.

Problems. 1. Can Theorem 3.2 be generalized to operators of length greater than 2?

2. Suppose that all A_j, B_j are positive and for each j at least one of A_j, B_j is compact. Then it can be deduced from [15, Corollary 6.6] (see [10]) that all eigenvalues of the operator (3.6) are contained in $\mathbb{R}^+$. Is the same true for the entire spectrum?

3. Can in Theorem 3.2 the commutativity condition be replaced by commutativity modulo compact operators?

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