STRONG OPERATOR MODULES AND THE HAAGERUP TENSOR PRODUCT

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1. *Introduction and basic definitions*

Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, cardinal numbers $\mathcal{I}$ and $\mathcal{J}$ and a subset $X \subseteq \mathrm{B}(\mathcal{H};\mathcal{K})$ (that is, the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$), we denote by $\mathcal{M}_{\mathcal{J},\mathcal{I}}(X)$ the set of all $\mathcal{J}\times\mathcal{I}$ matrices with entries in X that represent bounded operators from $\mathcal{H}^{(\mathcal{I})}$ (that is, the direct sum of $\mathcal{I}$ copies of $\mathcal{H}$) to $\mathcal{K}^{(\mathcal{J})}$. (The reader may assume that all Hilbert spaces considered here are separable; then the only relevant infinite cardinal number will be the countably infinite cardinal, denoted here by ∞. This assumption would not change the exposition presented here, except the proof of Theorem 7.1. Basic facts concerning computation with infinite matrices can be found in [**21**, § 2.6].) We use the abbreviations $\mathcal{M}_{\mathcal{I}}(X)=\mathcal{M}_{\mathcal{I},\mathcal{I}}(X)$, $\mathcal{R}_{\mathcal{I}}(X)=\mathcal{M}_{1,\mathcal{I}}(X)$ and $\mathcal{C}_{\mathcal{I}}(X)=\mathcal{M}_{\mathcal{I},1}(X)$. These sets should not be confused with the norm closures of finitely supported matrices (the latter will appear in Remark 5.2 and will be denoted differently). We denote by $a^{(\mathcal{I})}$ the direct sum of $\mathcal{I}$ copies of an operator a and, given a set of operators R, we use $R^{(\mathcal{I})}$ to mean $\{a^{(\mathcal{I})}\colon\ a\in R\}$. Given $x\in\mathcal{R}_{\mathcal{I}}(\mathrm{B}(\mathcal{H}))$ and $r\in\mathcal{C}_{\mathcal{I}}(\mathrm{B}(\mathcal{H}))$, the product xr means just the composition of operators; in components it can be expressed as $xr=\sum_{i\in\mathcal{I}}x_ir_i$, where the sum converges in the strong (and weak) operator topology.

DEFINITION 1.1. Given a von Neumann algebra $R\subseteq\mathrm{B}(\mathcal{H})$, a subspace $X\subseteq\mathrm{B}(\mathcal{H})$ is a *strong* right R-module if $xr\in X$ for all $x\in\mathcal{R}_{\mathcal{I}}(X)$ and $r\in\mathcal{C}_{\mathcal{I}}(R)$, where $\mathcal{I}$ is an arbitrary cardinal number. More generally, a *strong* right module over a W*-algebra R is an (abstract) operator space X (in the sense of [**29**] and [**19**]) which is a right R-module such that there exist a Hilbert space $\mathcal{H}$, a *normal* representation φ of R on $\mathcal{H}$ and a (linear) complete isometry $\Phi\colon\ X\to\mathrm{B}(\mathcal{H})$ such that $\Phi(X)$ is a strong right $\varphi(R)$-module and $\Phi(xr)=\Phi(x)\varphi(r)$ for all $x\in X$ and $r\in R$. The pair of maps (Φ,φ) is then called a *representation* of (X,R). Strong left operator modules are defined in the same way.

It can be shown (Proposition 2.1) that the notion of a strong module is independent of representations.

REMARK 1.2. If $\mathcal{H}$ is infinite dimensional and $x\in\mathrm{B}(\mathcal{H};\mathcal{H}^{(\mathcal{I})})=\mathcal{C}_{\mathcal{I}}(\mathrm{B}(\mathcal{H}))$, at most $\dim\mathcal{H}$ components of x are non-zero, where $\dim\mathcal{H}$ is the cardinality

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of an orthonormal basis of $\mathcal{H}$ (since for each $\xi \in \mathcal{H}$ at most countably many components of $x\xi$ can be non-zero). Therefore in Definition 1.1 we may take $\mathcal{I} = \dim \mathcal{H}$ (or $\mathcal{I} = \infty$, if $\mathcal{H}$ is finite dimensional). To simplify the notation, we shall write $\mathcal{M}(X)$, $\mathcal{R}(X)$ and $\mathcal{C}(X)$ instead of $\mathcal{M}_{\mathcal{I}}(X)$, $\mathcal{R}_{\mathcal{I}}(X)$ and $\mathcal{C}_{\mathcal{I}}(X)$, respectively, where $\mathcal{I} = \dim \mathcal{H}$. The letter $\mathcal{I}$ will always denote a sufficiently large cardinal (such as $\dim \mathcal{H}$). If $\mathcal{H}$ is separable, the notion of strong R-modules coincides with the notion of $\mathbf{M}_\infty(R)$-submodules of dual operator modules introduced by Effros and Ruan at the end of [**15**]. The term ‘strong module’ was suggested to the author by D. P. Blecher.

Clearly each weak* closed R-submodule of $\mathrm{B}(\mathcal{H})$ is strong, but the converse is not necessarily true. For example, every norm closed subspace X of $\mathrm{B}(\mathcal{H})$ is strong as a right (or left) $\mathbb{C}$-module since for each $x \in \mathcal{R}(X)$ and $c \in \mathcal{C}(\mathbb{C})$ the sum xc converges in norm (because the sum c^*c is norm convergent). Note also that for an arbitrary $R \subseteq \mathrm{B}(\mathcal{H})$ each strong R-module $X \subseteq \mathrm{B}(\mathcal{H})$ is necessarily norm closed. Namely, each y in the norm closure of X can be represented as

$$y = \sum_{i=1}^{\infty} \frac{1}{i} x_i \quad (x_i \in X),$$

where $\|x_i\| < i^{-1}$ for all $i > 1$. Hence $y = xr$, where x is the countably infinite row with components x_i in X and r is the countably infinite column with components i^{-1} in R. But, if $R \neq \mathbb{C}$, a norm closed R-submodule of $\mathrm{B}(\mathcal{H})$ is not necessarily strong; for example, it can be shown (but we shall not need this fact) that each strong R-submodule of R is necessarily weak* closed.

It turns out that strong modules are precisely the right kind of modules to which one can naturally extend the theory of the weak* Haagerup tensor product of Blecher and Smith [**7**]. In [**7**] only the tensor product of weak* closed subspaces over $\mathbb{C}$ is considered, but in our present approach the replacement of the scalar field $\mathbb{C}$ by an arbitrary von Neumann algebra does not make any essential difference in the proofs.

Given abstract operator spaces $X_1, \ldots, X_n$ and Z we denote by

$$\mathrm{CB}(X_1, \ldots, X_n; Z)$$

the space of all completely bounded multilinear maps from $X_1 \times \ldots \times X_n$ to Z (see [**10**] or [**11**]). If these are concrete operator spaces (hence they carry the relative weak* topologies inherited from $\mathrm{B}(\mathcal{H})$), we denote by $\mathrm{NCB}(X_1, \ldots, X_n; Z)$ the subspace of all normal (that is, weak* continuous in each variable separately) maps. (We shall restrict our attention almost exclusively to the cases $n = 1$ and $n = 2$. Basic theory of operator spaces and completely bounded maps is explained nicely in [**26**, **30**, **11**].)

Given von Neumann algebras $R_1, \ldots, R_n$ acting on a Hilbert space $\mathcal{H}$ with commutants R_j' and operators

$$x_0 \in \mathcal{R}(\mathrm{B}(\mathcal{H})), \quad x_1, \ldots, x_{n-1} \in \mathcal{M}(\mathrm{B}(\mathcal{H})) \quad \text{and} \quad x_n \in \mathcal{C}(\mathrm{B}(\mathcal{H})),$$

we denote by $x_0 \odot_{R_1} \ldots \odot_{R_n} x_n$ the operator in

$$\mathrm{NCB}(R_1', \ldots, R_n'; \mathrm{B}(\mathcal{H}))$$

defined by

$$x_0 \odot_{R_1} \dots \odot_{R_n} x_n(r'_1, \dots, r'_n) = x_0 r_1'^{(\mathcal{I})} x_1 r_2'^{(\mathcal{I})} x_2 \dots r_n'^{(\mathcal{I})} x_n \quad (r'_j \in R'_j).$$

The product on the right-hand side of this formula means just the composition of operators. (Occasionally we shall also use the above notation for operators between different Hilbert spaces, where the commutants should be replaced by spaces of operators intertwining different normal representations.) Given $x \in \mathcal{R}(\mathrm{B}(\mathcal{H}))$ and $y \in \mathcal{C}(\mathrm{B}(\mathcal{H}))$ with components x_i and y_i (respectively) in $\mathrm{B}(\mathcal{H})$, the operator $x \odot_R y$ can be expressed as

$$x \odot_R y = \sum_{i \in \mathcal{I}} x_i \otimes_R y_i,$$

where $x_i \otimes_R y_i$ denotes here the operator $r' \mapsto x_i r' y_i$ and the sum converges in the point strong operator topology.

Occasionally it will be necessary to consider also (infinite) matrices of completely bounded maps. If $x \in \mathcal{M}_{\mathcal{I},\mathcal{J}}(\mathrm{B}(\mathcal{H}))$ and $y \in \mathcal{M}_{\mathcal{J},\mathcal{K}}(\mathrm{B}(\mathcal{H}))$, where $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ are cardinal numbers, then, with $x_i \in \mathcal{R}_{\mathcal{J}}(\mathrm{B}(\mathcal{H}))$ the ith row of x and $y_k \in \mathcal{C}_{\mathcal{J}}(\mathrm{B}(\mathcal{H}))$ the kth column of y, we denote the $\mathcal{I} \times \mathcal{K}$ matrix of completely bounded maps $[x_i \odot_R y_j]$, obtained from x and y by formal matrix multiplication, simply by $x \odot_R y$. It is easy to see that $x \odot_R y$ considered as a map from R' to $\mathcal{M}_{\mathcal{I},\mathcal{K}}(\mathrm{B}(\mathcal{H})) = \mathrm{B}(\mathcal{H}^{\mathcal{K}}; \mathcal{H}^{\mathcal{I}})$ is given by $r' \mapsto x r'^{\mathcal{J}} y$, so that the new usage of the symbol $\odot_R$ agrees with the previous one.

Now we are going to define the extended Haagerup tensor product of concrete operator modules. A definition of this tensor product in the representation free manner (that is, without any reference to completely bounded operators and commutants) will be given for two spaces in Remark 3.7.

DEFINITION 1.3. Let $R_1, \dots, R_n \subseteq \mathrm{B}(\mathcal{H})$ be von Neumann algebras and let $X_0, \dots, X_n$ be subspaces of $\mathrm{B}(\mathcal{H})$ such that

$$R_i X_i R_{i+1} \subseteq X_i \quad (i = 0, \dots, n),$$

where $R_0 = \mathbb{C} = R_{n+1}$. Suppose that each X_i is strong as a left module over R_i and as a right module over R_{i+1}. Then the *extended Haagerup tensor product*

$$X_0 \bar{\otimes}^h_{R_1} \dots \bar{\otimes}^h_{R_n} X_n$$

is defined as the set of all

$$\vartheta \in \mathrm{NCB}(R'_1, \dots, R'_n; \mathrm{B}(\mathcal{H})) \text{ of the form } \vartheta = x_0 \odot_{R_1} \dots \odot_{R_n} x_n,$$

where $x_0 \in \mathcal{R}(X_0)$, $x_i \in \mathcal{M}(X_i)$ for $i = 1, \dots, n-1$ and $x_n \in \mathcal{C}(X_n)$.

Since $\mathcal{I}$ is infinite, it is easy to see that $X_0 \bar{\otimes}^h_{R_1} \dots \bar{\otimes}^h_{R_n} X_n$ is a vector subspace of $\mathrm{NCB}(R'_1, \dots, R'_n; \mathrm{B}(\mathcal{H}))$. Namely, we have

$$\begin{aligned} &x_0 \odot_{R_1} \dots \odot_{R_n} x_n + y_0 \odot_{R_1} \dots \odot_{R_n} y_n \\ &\quad = [x_0, y_0] \odot_{R_1} \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix} \odot_{R_2} \dots \odot_{R_{n-1}} \begin{bmatrix} x_{n-1} & 0 \\ 0 & y_{n-1} \end{bmatrix} \odot_{R_n} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \end{aligned}$$

where

$$[x_0, y_0] \in \mathcal{R}_{2\mathcal{I}}(X_0), \quad \begin{bmatrix} x_n \\ y_n \end{bmatrix} \in \mathcal{C}_{2\mathcal{I}}(X_n)$$

and

$$\begin{bmatrix} x_i & 0 \\ 0 & y_i \end{bmatrix} \in \mathcal{M}_{2\mathcal{I}}(X_i) \quad \text{for } i = 1, \ldots, n-1.$$

Choosing a bijection $\mathcal{I} \leftrightarrow 2\mathcal{I}$, we can identify $\mathcal{R}_{2\mathcal{I}}(X_0)$, $\mathcal{M}_{2\mathcal{I}}(X_i)$ and $\mathcal{C}_{2\mathcal{I}}(X_n)$ with $\mathcal{R}(X_0)$, $\mathcal{M}(X_i)$ and $\mathcal{C}(X_n)$, respectively; moreover, the sum is independent of the bijection chosen. Note also that the so-defined space deserves the name 'tensor product' since the symbol $x_0 \odot_{R_1} \ldots \odot_{R_n} x_n$ is linear (over $\mathbb{C}$) in each argument and satisfies

$$x_0 \odot_{R_1} \ldots \odot_{R_{i-1}} x_{i-1} r_i \odot_{R_i} x_i \odot_{R_{i+1}} \ldots \odot_{R_n} x_n \\ = x_0 \odot_{R_1} \ldots \odot_{R_{i-1}} x_{i-1} \odot_{R_i} r_i x_i \odot_{R_{i+1}} \ldots \odot_{R_n} x_n,$$

for all $r_i \in R_i$, $i = 1, \ldots, n$. Of course the extended Haagerup tensor product can also be defined if the modules are not required to be strong, but we shall need this requirement to establish most of the interesting properties of the product. It turns out that the norm of each element $\vartheta \in X_0 \otimes_{R_1} \ldots \otimes_{R_n} X_n$ is given by

$$\|\vartheta\| = \inf\{\|x_0\| \ldots \|x_n\| \colon \ \vartheta = x_0 \odot_{R_1} \ldots \odot_{R_n} x_n\},$$

where the infimum is over all $x_0 \in \mathcal{R}(X_0)$, $x_n \in \mathcal{C}(X_n)$ and $x_i \in \mathcal{M}(X_i)$ for $i = 1, \ldots, n-1$. This is similar to the expression for the usual Haagerup norm. We shall prove this formula for the case $n = 1$ in § 3. The case $n = 2$ will be studied in § 4, where the associativity of the product will also be proved. (The results of § 4 are not needed in the rest of the paper.) The case where there are more than three spaces can be dealt with by the same method as that used for three spaces and will be omitted here.

In § 2 some basic facts concerning strong modules are developed; these facts are needed in some later parts of the paper, but perhaps they are also of independent interest. Most of the rest of the paper is devoted to the study of the product $X \bar{\otimes}_R^h Y$ of two spaces, which contains the Haagerup tensor product $X \otimes_R^h Y$ (considered in [**5**, **24**]) completely isometrically by Lemma 2.3 of [**24**]. The relation between completely bounded operators and the Haagerup tensor product $X \otimes^h Y$ (over $\mathbb{C}$) has been studied first by Haagerup in an unpublished paper and later by several other authors (see [**13**, **27**, **8**, **9**, **2**]). The dual space of the Haagerup tensor product $X \otimes^h Y$ can be characterised in several ways (see [**17**, **7**]); one of these is the weak* Haagerup tensor product of [**7**]. In the case where $R = \mathbb{C}$ and the modules are weak* closed, the extended and the weak* Haagerup tensor products coincide. One might conjecture that for weak* closed submodules the tensor product $X \bar{\otimes}_R^h Y$ is just the quotient space of the weak* Haagerup tensor product $X \bar{\otimes}^h Y$ by the weak* closed subspace N generated by all elements of the form $xr \otimes y - x \otimes ry$ ($x \in X$, $y \in Y$, $r \in R$), but this is not true even in the case $X = Y = \mathrm{B}(\mathcal{H})$ if R' contains no non-zero compact operators (Example 3.8). Nevertheless, we shall see that many nice properties of the (weak*) Haagerup tensor product are preserved in our more general context.

In [**31**] and [**7**] slice maps are used to study subspaces of $X \otimes^h Y$ and $X \bar{\otimes}^h Y$. Since for a general von Neumann algebra $R \subseteq \mathrm{B}(\mathcal{H})$ there are not enough (normal) R-module homomorphisms from $\mathrm{B}(\mathcal{H})$ to R, slice maps are not so useful in our broader context. They are replaced in § 5 by a simple technique of projections. This technique is based on the observation that for each strong right R-module $X \subseteq \mathrm{B}(\mathcal{H})$ and each $a \in \mathcal{R}(\mathrm{B}(\mathcal{H}))$ the set

$$\{r \in \mathcal{M}(R) \colon\ ar \in \mathcal{R}(X)\}$$

is a weak* closed right ideal in the von Neumann algebra $\mathcal{M}(R)$, and is hence of the form $e\mathcal{M}(R)$ for some projection $e \in \mathcal{M}(R)$ (Proposition 2.3). Using this technique we will prove, for example, that if R is commutative and X, Y are von Neumann algebras containing R and contained in R' (so that $X \bar{\otimes}^h_R Y$ is a Banach algebra), then for arbitrary subalgebras X_0 of X and Y_0 of Y containing R which are strong as R-modules, the relative commutant of $X_0 \bar{\otimes}^h_R Y_0$ in $X \bar{\otimes}^h_R Y$ is simply $X_0^c \bar{\otimes}^h_R Y_0^c$, where X_0^c and Y_0^c are the relative commutants of X_0 in X and Y_0 in Y. In the case where $R = \mathbb{C}$ this was proved in [**7**].

In § 6 we study some canonical isomorphisms needed later. We prove, for example, that for an injective von Neumann algebra R and any operator spaces X and Y there is a natural completely isometric isomorphism

$$(X \check{\otimes} R) \otimes^h_R (R \check{\otimes} Y) \to R \check{\otimes} (X \otimes^h Y),$$

where $\check{\otimes}$ is the minimal (that is, spatial) tensor product of operator spaces. (For two operator spaces $X \subseteq \mathrm{B}(\mathcal{H})$ and $Y \subseteq \mathrm{B}(\mathcal{K})$ the product $X \check{\otimes} Y$ is the norm closed subspace of $\mathrm{B}(\mathcal{H} \otimes \mathcal{K})$ generated by the algebraic tensor product $X \otimes Y$. More about this product can be found in [**6**].)

Let C be the centre of R, denote by $R_\natural$ the space of all normal C-module homomorphisms from R to C equipped with the completely bounded norm and the structure of the C-bimodule defined by

$$c_1 \rho c_2(r) = \rho(c_2 r c_1) \quad (r \in R),$$

and let $(R_\natural \otimes^h_C R_\natural)^\natural$ be the space of all completely bounded C-module homomorphisms from $R_\natural \otimes^h_C R_\natural$ to C. Then the natural map from $R \bar{\otimes}^h_C R$ to $(R_\natural \otimes^h_C R_\natural)^\natural$ is shown in § 7 to be a completely isometric isomorphism. In the case $C = \mathbb{C}$ this was proved previously in [**7**]. (We assume the special case $R = \mathrm{B}(\mathcal{H})$ is given in our proof.) Since C is the operator space dual of its predual $C_\sharp$ (also called the standard dual by Blecher [**3**]), it follows that $R \bar{\otimes}^h_C R$ is the operator space dual of $(R_\natural \otimes^h_C R_\natural) \hat{\otimes}_C C_\sharp$, where for given operator C-modules X and Y the product $X \hat{\otimes}_C Y$ is the quotient of the maximal operator space tensor product $X \hat{\otimes} Y$ (defined in [**18**] and [**6**]) by the closed subspace generated by all elements of the form $xc \otimes y - x \otimes cy$ $(x \in X,\ y \in Y,\ c \in C)$.

In § 8 all weak* closed two-sided ideals of the Banach algebra $R \bar{\otimes}^h_C R$ are shown to be determined by central projections, a result which is well known for von Neumann algebras.

We end this introduction by mentioning the convention (also used by some other authors) that $[x_1, x_2, \ldots]$ denotes a row, while $(x_1, x_2, \ldots)$ means a column.

Each map between operator spaces $\Phi\colon X \to Y$ induces for all positive integers m and n the map $\Phi_{m,n}\colon \mathcal{M}_{m,n}(X) \to \mathcal{M}_{m,n}(Y)$ by applying Φ to the entries of matrices. By definition Φ is completely bounded if and only if $\|\Phi\|_{\mathrm{cb}} := \sup_{m,n} \|\Phi_{m,n}\| < \infty$. If Φ is completely bounded, then Φ induces a map on $\mathcal{M}_{m,n}$ even if m and n are infinite cardinals (see [**13**]). The induced map will be denoted simply by Φ again when m and n are clear from the context.

2. *Strong modules*

First we would like to show that the notion of a strong module (Definition 1.1) is independent of a representation. Note that for any W*-algebra R and any strong right R-module X, the set

$$\{r \in R\colon\ Xr = 0\}$$

is a weak* closed two-sided ideal in R, and hence of the form $p^{\perp}R$ for some central projection $p \in R$. We call p the *central carrier* of X.

PROPOSITION 2.1. *If X is a strong right module over a W*-algebra R, then $\Psi(X)$ is a strong right $\psi(R)$-module for every representation (Ψ, ψ) of (X, R).*

In the proof of Proposition 2.1 and also later we shall need the following fact explained in the proof of Theorem 4.2 in [**15**].

LEMMA 2.2. *If $Y \subseteq \mathrm{B}(\mathcal{H})$ and $Z \subseteq \mathrm{B}(\mathcal{K})$ are strong operator modules over von Neumann algebras $R \subseteq \mathrm{B}(\mathcal{H})$ and $S \subseteq \mathrm{B}(\mathcal{K})$, $\omega\colon R \to S$ is a $*$-isomorphism (or, more generally, a normal $*$-homomorphism) and $\Omega\colon Y \to Z$ is a completely bounded (not necessarily normal) map satisfying $\Omega(yr) = \Omega(y)\omega(r)$ for all $y \in Y$ and $r \in R$, then the same identity also holds for all $y \in \mathcal{R}_{\mathcal{J}}(Y)$ and $r \in \mathcal{C}_{\mathcal{J}}(R)$, where $\mathcal{J}$ is any cardinal number and Ω and ω have been extended to $\mathcal{R}_{\mathcal{J}}(Y)$ and $\mathcal{C}_{\mathcal{J}}(R)$ in the usual way.*

Proof of Proposition 2.1. By Definition 1.1 there exists a representation (Φ, φ) of (X, R) on a Hilbert space $\mathcal{H}$ such that $\Phi(X)$ is strong over $\varphi(R)$. Let p be the central carrier of X. Since

$$\Psi(X)\psi(p^{\perp}R) = \Psi(Xp^{\perp}R) = 0,$$

we may assume that $p = 1$ when proving that $\Psi(X)$ is strong over $\psi(R)$ (otherwise we just replace R by pR). Then it is easy to see that ψ (and similarly φ) is one-to-one since Ψ is one-to-one. It follows that $\omega := \psi\varphi^{-1}$ is a $*$-isomorphism from the von Neumann algebra $\varphi(R)$ onto $\psi(R)$, and clearly $\Omega := \Psi\Phi^{-1}$ is a complete isometry from $\Phi(X)$ onto $\Psi(X)$ such that

$$\Omega(ys) = \Omega(y)\omega(s) \tag{2.1}$$

for all $y \in \Phi(X)$ and $s \in \varphi(R)$. Applying Ω to components, we get a complete isometry from $\mathcal{R}(\Phi(X))$ onto $\mathcal{R}(\Psi(X))$, which we denote again by Ω. From

Lemma 2.2 we see that (2.1) holds also for all $y \in \mathcal{R}(\Phi(X))$ and $s \in \mathcal{C}(\varphi(R))$. Since $\Phi(X)$ is strong over $\varphi(R)$, it follows that $\Psi(X)$ is strong over $\psi(R)$.

PROPOSITION 2.3. *If $X \subseteq \mathrm{B}(\mathcal{H})$ is a strong right module over a von Neumann algebra $R \subseteq \mathrm{B}(\mathcal{H})$, then for each $a \in \mathcal{R}(\mathrm{B}(\mathcal{H}))$ the set*

$$N = \{r \in \mathcal{M}(R):\ ar \in \mathcal{R}(X)\}$$

is a weak closed right ideal in $\mathcal{M}(R)$, and hence of the form $e\mathcal{M}(R)$ for some projection $e \in \mathcal{M}(R)$.*

Proof. Obviously N is a right ideal in $\mathcal{M}(R)$ since X is strong; hence the weak* closure $\bar{N}$ of N is of the form $\bar{N} = e\mathcal{M}(R)$ for some projection $e \in \mathcal{M}(R)$. It suffices now to prove that $e \in N$. Let $\{e_j\}_{j\in\mathcal{J}}$ be a maximal orthogonal family of non-zero subprojections of e in $\mathcal{M}(R)$ such that $ae_j \in \mathcal{R}(X)$ and let $f = \sum e_j$ (if $\mathcal{J} = \emptyset$ put $f = 0$). Since the element

$$af = \sum_{j\in\mathcal{J}} (ae_j)e_j$$

is equal to the product bc, where $b \in \mathcal{R}_{\mathcal{J}}(\mathcal{R}(X)) \cong \mathcal{R}(X)$ has components $ae_j \in \mathcal{R}(X)$ and $r \in \mathcal{C}(\mathcal{M}(R)) \cong \mathcal{M}(R)$ has components $e_j \in \mathcal{M}(R)$, and X is strong, it follows that $af \in \mathcal{R}(X)$. Hence $f \in N$ and it suffices now to prove that $e = f$.

Note that for each $r \in N$ (with $\|r\| = 1$, say) the range projection $p(r)$ of r is in N. Indeed, for each positive integer n the spectral projection p_n of rr^* corresponding to the interval $((n+1)^{-1}, n^{-1}]$ is in N (since $p_n = rr^*s_n$ for some $s_n \in \mathcal{M}(R)$ and N is a right ideal) and, since $p(r) = \sum p_n$, an argument similar to the one given in the previous paragraph shows that $p(r) \in N$.

Since $f \vee p(r)$ is the range projection of $f + rr^* \in N$, it follows that $f \vee p(r) \in N$ for each $r \in N$. From the maximality of $\{e_j\}_{j\in\mathcal{J}}$ we now conclude that $(f \vee p(r)) - f = 0$, and hence $p(r) \leqslant f$ for each $r \in N$ and consequently $f = e$.

COROLLARY 2.4. *If a, R and X are as in Proposition 2.3, the set*

$$M = \{r \in \mathcal{C}(R):\ ar \in X\}$$

is a weak closed right R-submodule of $\mathcal{C}(R)$ and $M = e\mathcal{C}(R)$ for some projection $e \in \mathcal{M}(R)$.*

Proof. Just observe that the ideal N from Proposition 2.3 consists precisely of matrices that have columns in M.

The projection e that appears in Proposition 2.3 and Corollary 2.4 will be called the *X-projection of a*. For a strong left R-module $Y \subseteq \mathrm{B}(\mathcal{H})$ and a column $a \in \mathcal{C}(\mathrm{B}(\mathcal{H}))$ the *Y-projection* is defined similarly.

3. *Basic properties of the extended Haagerup tensor product*

Let $R \subseteq \mathrm{B}(\mathcal{H})$ be a von Neumann algebra, $X \subseteq \mathrm{B}(\mathcal{H})$ a strong right R-module and $Y \subseteq \mathrm{B}(\mathcal{H})$ a strong left R-module. In each of the spaces

$\mathcal{M}_n(X \bar{\otimes}^h_R Y)$ (n will always be a positive integer in this paper if not explicitly stated otherwise) we introduce a new norm by

$$\|\vartheta\| = \inf\{\|x\|\|y\|:\ \vartheta = x \odot_R y,\ \ x \in \mathcal{M}_{n,\mathcal{J}}(X),\ \ y \in \mathcal{M}_{\mathcal{J},n}(Y)\} \tag{3.1}$$

and we denote the space $X \bar{\otimes}^h_R Y$ equipped with this new matricial norm structure by $X \tilde{\otimes}^h_R Y$. (This is only temporary notation; we shall see that the new norm is the same as the old one.) The norm of an element $\vartheta \in \mathcal{M}_n(X \bar{\otimes}^h_R Y)$ regarded as a completely bounded map will be denoted by $\|\vartheta\|_{\mathrm{cb}}$. It is not obvious that $\|\vartheta\|$ is defined for all ϑ.

LEMMA 3.1. *Every element $\vartheta \in \mathcal{M}_n(X \tilde{\otimes}^h_R Y)$ can be expressed as*

$$\vartheta = x \odot_R y$$

for suitable $x \in \mathcal{M}_{n,\mathcal{J}}(X)$ and $y \in \mathcal{M}_{\mathcal{J},n}(Y)$. The function $\vartheta \to \|\vartheta\|$ is indeed a norm on $\mathcal{M}_n(X \tilde{\otimes}^h_R Y)$ for each n and $X \tilde{\otimes}^h_R Y$ is an L^∞-matricially normed space (and hence an operator space by Ruan's theorem [**29**, **19**]). *Moreover, $\mathcal{M}_n(X \tilde{\otimes}^h_R Y)$ is naturally completely isometrically isomorphic to the space $\mathcal{C}_n(X) \tilde{\otimes}^h_{R^{(n)}} \mathcal{R}_n(Y)$, where $\mathcal{C}_n(X)$ and $\mathcal{R}_n(Y)$ are regarded as subspaces of $\mathrm{B}(\mathcal{H}^n) = \mathcal{M}_n(\mathrm{B}(\mathcal{H}))$ by embedding onto the first column and row, respectively.*

Proof. Let $\vartheta = [x_{ij} \odot_R y_{ij}]$, where $x_{ij} \in \mathcal{R}(X)$ and $y_{ij} \in \mathcal{C}(Y)$ ($i, j = 1, \ldots, n$). For each i define $x_i \in \mathcal{R}_{n^2}(\mathcal{R}(X)) = \mathcal{R}_{n^2\mathcal{J}}(X)$ and $y_i \in \mathcal{C}_{n^2\mathcal{J}}(Y)$ by

$$x_i = [0, \ldots, x_{i1}, \ldots, 0; \ldots; 0, \ldots, x_{in}, \ldots, 0]$$

(where there are n blocks and x_{ij} is in the ith position in each block) and

$$y_i = (0, \ldots, 0; \ldots; y_{1i}, \ldots, y_{ni}; \ldots; 0, \ldots, 0).$$

Then $x_i \odot_R y_j = x_{ij} \odot_R y_{ij}$ for all i, j, and hence $\vartheta = x \odot_R y$, where $x \in \mathcal{M}_{n,n^2\mathcal{J}}(X)$ is the matrix with rows x_i and $y \in \mathcal{M}_{n^2\mathcal{J},n}(Y)$ is the matrix with columns y_j. Since $\mathcal{J}$ is infinite, we can identify $n^2\mathcal{J}$ with $\mathcal{J}$ and therefore regard x and y as elements of $\mathcal{M}_{n,\mathcal{J}}(X)$ and $\mathcal{M}_{\mathcal{J},n}(Y)$, respectively. This proves that $\|\ \|$ is everywhere defined.

The proof that $\|\ \|$ satisfies the triangle inequality is the same as for the usual Haagerup norm (see [**13**]). The definiteness of $\|\ \|$ follows from the inequality

$$\|\vartheta\|_{\mathrm{cb}} \leqslant \|\vartheta\|,$$

which is proved by the same computation as for the usual Haagerup norm [**11**]. The computation that $\|\ \|$ is an L^∞-matricial norm (that is, $\|\vartheta \oplus \omega\| = \max\{\|\vartheta\|, \|\omega\|\}$) is also the same as in the classical case [**29**].

Given $v \in \mathcal{C}_n(X)$ we denote by $\tilde{v} \in \mathcal{M}_n(X)$ the matrix with first column v and the remaining columns 0. Similarly, for $w \in \mathcal{R}_n(Y)$ let $\tilde{w} \in \mathcal{M}_n(Y)$ be the matrix with first row w and the remaining rows 0. Further, given a matrix $x \in \mathcal{M}_{n,\mathcal{J}}(X)$ with columns $x^i \in \mathcal{C}_n(X)$, let $\tilde{x} \in \mathcal{R}_{\mathcal{J}}(\mathcal{M}_n(X))$ have the components $\tilde{x}^i$. Similarly, we define $\tilde{y} \in \mathcal{C}_{\mathcal{J}}(\mathcal{M}_n(Y))$ for each $y \in \mathcal{M}_{\mathcal{J},n}(Y)$. Finally, to each $\vartheta \in \mathcal{M}_n(X \tilde{\otimes}^h_R Y)$ we associate an element $\tilde{\vartheta} \in \mathcal{C}_n(X) \tilde{\otimes}^h_{R^{(n)}} \mathcal{R}_n(Y)$ as follows. Choose $x \in \mathcal{M}_{n,\mathcal{J}}(X) = \mathcal{R}_{\mathcal{J}}(\mathcal{C}_n(X))$ and $y \in \mathcal{M}_{\mathcal{J},n}(Y) = \mathcal{C}_{\mathcal{J}}(\mathcal{R}_n(Y))$ such that

$\vartheta = x \odot_R y$ and put $\tilde{\vartheta} = \tilde{x} \odot_{R^{(n)}} \tilde{y}$. (Recall that $\mathcal{C}_n(X)$ and $\mathcal{R}_n(Y)$ are regarded as subspaces of $\mathcal{M}_n(X)$ and $\mathcal{M}_n(Y)$ by embedding onto the first column and row, respectively.)

We claim that the correspondence $\Omega\colon \vartheta \mapsto \tilde{\vartheta}$ is a well-defined linear bijection from $\mathcal{M}_n(X \tilde{\otimes}^h_R Y)$ to $\mathcal{C}_n(X) \tilde{\otimes}^h_{R^{(n)}} \mathcal{R}_n(Y)$. To see this, let $x_i \in \mathcal{R}_{\mathcal{J}}(X)$ and $y_i \in \mathcal{C}_{\mathcal{J}}(Y)$ be the rows and the columns of x and y, respectively, and put $\vartheta_{ij} = x_i \odot_R y_j$ $(i, j = 1, \dots, n)$, so that $\vartheta = [\vartheta_{ij}]$. (Thus ϑ is an $n \times n$ matrix of mappings $\vartheta_{ij} \in \mathrm{CB}(R'; \mathrm{B}(\mathcal{H}))$.) An easy computation (which we leave to the reader) shows that $\tilde{\vartheta}$, as a completely bounded map from $R^{(n)'} = \mathcal{M}_n(R')$ to $\mathrm{B}(\mathcal{H}^n)$, acts as follows:

$$\tilde{\vartheta}([r'_{ij}]) = [\vartheta_{ij}(r'_{11})] \quad ([r'_{ij}] \in \mathcal{M}_n(R')).$$

This shows that $\tilde{\vartheta}$ indeed depends only on ϑ (and not on the choice of x and y) and that ϑ is determined by $\tilde{\vartheta}$, and hence Ω is injective. Since by definition each element in $\mathcal{C}_n(X) \tilde{\otimes}^h_{R^{(n)}} \mathcal{R}_n(Y)$ is of the form $\tilde{x} \odot_{R^{(n)}} \tilde{y}$ for some $x \in \mathcal{M}_{n,\mathcal{J}}(X)$ and $y \in \mathcal{M}_{\mathcal{J},n}(Y)$, Ω is surjective.

It is easily seen from the definition of the norms that Ω is an isometry. That Ω is a complete isometry follows then by applying the result just proved to the spaces $\mathcal{C}_n(X)$ and $\mathcal{R}_n(Y)$ instead of X and Y. Namely, for each positive integer m we have (where the equalities really mean isometries)

$$\begin{aligned}\mathcal{M}_m(\mathcal{C}_n(X) \tilde{\otimes}^h_{R^{(n)}} \mathcal{R}_n(Y)) &= \mathcal{C}_m(\mathcal{C}_n(X)) \tilde{\otimes}^h_{R^{(n)(m)}} \mathcal{R}_m(\mathcal{R}_n(Y)) \\ &= \mathcal{C}_{mn}(X) \tilde{\odot}^h_{R^{(mn)}} \mathcal{R}_{mn}(Y) \\ &= \mathcal{M}_{mn}(X \tilde{\otimes}^h_R Y) \\ &= \mathcal{M}_m(\mathcal{M}_n(X \tilde{\otimes}^h_R Y)).\end{aligned}$$

The proof is complete.

Now we extend the definition of the tensor product $\tilde{\otimes}^h_R$ to abstract operator modules over W*-algebras. If R is a W*-algebra, X is a strong right R-module and Y is a strong left R-module, then there exist a representation (Φ, φ) of (X, R) on a Hilbert space $\mathcal{H}$ and a representation (Ψ, ψ) of (Y, R) on a Hilbert space $\mathcal{K}$ such that $\Phi(X)$ is a strong right module over the von Neumann algebra $\varphi(R)$, and $\Psi(Y)$ is a strong left module over $\psi(R)$ (Definition 1.1). Let $\mathcal{L} = \mathcal{H} \oplus \mathcal{K}$, $\pi = \varphi \oplus \psi$ and regard $\mathcal{H}$ and $\mathcal{K}$ as subspaces of $\mathcal{L}$ in the obvious way. Then we define

$$X \tilde{\otimes}^h_R Y := \Phi(X) \tilde{\otimes}^h_{\pi(R)} \Psi(Y).$$

We shall prove that this definition is independent of the choice of representations (up to a completely isometric isomorphism).

The following simple and well-known lemma will be used several times.

LEMMA 3.2. *Suppose that $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ are Hilbert spaces, R is a von Neumann algebra acting on $\mathcal{H}$, $x \in \mathrm{B}(\mathcal{H}; \mathcal{K})$ and $y \in \mathrm{B}(\mathcal{L}; \mathcal{H})$. Then $xR'y = 0$ if and only if there exists an element* (*in fact, a projection*) *$p \in R$ such that $xp = 0$ and $p^{\perp} y = 0$.*

The proof is trivial, the projection p with range $[R'y\mathcal{L}]$ satisfies the requirements.

Now we would like to prove that the product $X \tilde{\otimes}_R^h Y$ is independent of the choice of representations (Φ, φ) of X and (Ψ, ψ) of Y. Let $p, q \in R$ be the central carriers of X and Y. Since $\Phi(X)\varphi(p^\perp) = 0$ and $\psi(q^\perp)\Psi(Y) = 0$, it is easy to see that the replacement of R by pqR does not have any real effect on the product $\Phi(X) \tilde{\otimes}_{\pi(R)}^h \Psi(Y)$ (where $\pi = \varphi \oplus \psi$, as above). Thus we may assume without loss of generality that $p = 1 = q$; hence φ, ψ and π are one-to-one. That different representations give completely isometric products is now an immediate consequence of the following proposition.

PROPOSITION 3.3. *Suppose that $R_i \subseteq B(\mathcal{H}_i)$ $(i = 1, 2)$ are von Neumann algebras, $X_i \subseteq B(\mathcal{H}_i)$ are strong right R_i-modules and $Y_i \subseteq B(\mathcal{H}_i)$ are strong left R_i-modules. If $\Theta\colon X_1 \to X_2$ and $\Omega\colon Y_1 \to Y_2$ are completely isometric isomorphisms and $\nu\colon R_1 \to R_2$ is a *-isomorphism satisfying*

$$\Theta(xr) = \Theta(x)\nu(r) \quad \textit{and} \quad \Omega(ry) = \nu(r)\Omega(y) \quad (x \in X_1,\ r \in R_1,\ y \in Y_1),$$

then there is a completely isometric isomorphism

$$\Theta \tilde{\otimes}_\nu^h \Omega\colon\ X_1 \tilde{\otimes}_{R_1}^h Y_1 \to X_2 \tilde{\otimes}_{R_2}^h Y_2,$$

defined by

$$(\Theta \tilde{\otimes}_\nu^h \Omega)(x \odot_{R_1} y) = \Theta(x) \odot_{R_2} \Omega(y) \quad (x \in \mathcal{R}(X),\ y \in \mathcal{C}(Y)).$$

(Here $\mathcal{R}$ and $\mathcal{C}$ mean $\mathcal{R}_\mathcal{I}$ and $\mathcal{C}_\mathcal{I}$, where $\mathcal{I} = \max\{\dim \mathcal{H}_1, \dim \mathcal{H}_2\}$.)

Proof. The induced maps $\Theta\colon \mathcal{R}(X_1) \to \mathcal{R}(X_2)$ and $\Omega\colon \mathcal{C}(Y_1) \to \mathcal{C}(Y_2)$ are complete isometries and by Lemma 2.2 we have

$$\Theta(xr) = \Theta(x)\nu(r) \quad \text{and} \quad \Omega(ry) = \nu(r)\Omega(y) \tag{3.2}$$

for all $x \in \mathcal{R}(X_1)$, $y \in \mathcal{C}(Y_1)$ and $r \in \mathcal{M}(R_1)$. This implies that $\Theta \tilde{\otimes}_\nu^h \Omega$ is well defined. Indeed, suppose that $x \odot_{R_1} y = z \odot_{R_1} w$ in $X_1 \tilde{\otimes}_{R_1}^h Y_1$. This means that $xr'^{(\mathcal{I})}y = zr'^{(\mathcal{I})}w$ for all $r' \in R_1'$. In other words

$$[x, z]{R_1'}^{(2\mathcal{I})} \begin{bmatrix} y \\ -w \end{bmatrix} = 0.$$

Hence by Lemma 3.2 there exists a projection $p \in \mathcal{M}_{2\mathcal{I}}(R_1)$ such that

$$[x, z]p = 0 \quad \text{and} \quad p^\perp \begin{bmatrix} y \\ -w \end{bmatrix} = 0.$$

Using (3.2) we see that

$$[\Theta(x), \Theta(z)]\nu(p) = 0 \quad \text{and} \quad \nu(p)^\perp \begin{bmatrix} \Omega(y) \\ -\Omega(w) \end{bmatrix} = 0,$$

which implies that

$$[\Theta(x), \Theta(z)]{R_2'}^{(2\mathcal{I})} \begin{bmatrix} \Omega(y) \\ -\Omega(w) \end{bmatrix} = 0,$$

and hence $\Theta(x) \odot_{R_2} \Omega(y) = \Theta(z) \odot_{R_2} \Omega(w)$.

It is now trivial to verify that $\Theta \tilde{\otimes}^h_\nu \Omega$ is a contraction. Hence by symmetry (that is, by applying the above arguments to Θ^{-1}, Ω^{-1} and ν^{-1}) it follows that $\Theta \tilde{\otimes}^h_\nu \Omega$ must be an isometric isomorphism. To see that $\Theta \tilde{\otimes}^h_\nu \Omega$ is in fact a complete isometry, we use the last sentence of Lemma 3.1 and the isometry (just proved) for the spaces $\mathcal{C}_n(X_i)$ and $\mathcal{R}_n(Y_i)$ instead of X_i and Y_i. We have

$$\begin{aligned}\mathcal{M}_n(X_1 \tilde{\otimes}^h_{R_1} Y_1) &= \mathcal{C}_n(X_1) \tilde{\otimes}^h_{R_1^{(n)}} \mathcal{R}_n(Y_1)\\ &\cong \mathcal{C}_n(X_2) \tilde{\otimes}^h_{R_2^{(n)}} \mathcal{R}_n(Y_2)\\ &= \mathcal{M}_n(X_2 \tilde{\otimes}^h_{R_2} Y_2),\end{aligned}$$

where the equalities denote isometric isomorphisms.

The following lemma is suggested by an observation in [**24**].

LEMMA 3.4. *The map*

$$\Phi\colon\ X \tilde{\otimes}^h_R Y \to \mathcal{R}(X) \tilde{\otimes}^h_{\mathcal{M}(R)} \mathcal{C}(Y), \quad \Phi(x \odot_R y) = x \otimes_{\mathcal{M}(R)} y$$

is a completely isometric isomorphism.

Proof. An application of Lemma 3.2 shows that Φ is well defined. Obviously Φ is a contraction. To prove that Φ is surjective and an isometry, we define a map in the opposite direction as follows. Choose any bijection $\sigma\colon\ \mathcal{I} \to \mathcal{I} \times \mathcal{I}$ and let $\sigma_j\colon\ \mathcal{I} \to \mathcal{I}$ $(j = 1, 2)$ be the components of σ. Then σ induces a map $\psi_1\colon\ \mathcal{R}(\mathcal{R}(X)) \to \mathcal{R}(X)$ by $\psi_1(x)_i = [x_{\sigma_1(i)}]_{\sigma_2(i)}$ and a map $\psi_2\colon\ \mathcal{C}(\mathcal{C}(Y)) \to \mathcal{C}(Y)$ by $\psi_2(y)_i = (y_{\sigma_1(i)})_{\sigma_2(i)}$. Now define

$$\Psi\colon\ \mathcal{R}(X) \tilde{\otimes}^h_{\mathcal{M}(R)} \mathcal{C}(Y) \to X \tilde{\otimes}^h_R Y$$

by

$$\Psi(x \odot_{\mathcal{M}(R)} y) = \psi_1(x) \odot_R \psi_2(y) \quad (x \in \mathcal{R}(\mathcal{R}(X)),\ y \in \mathcal{C}(\mathcal{C}(Y))).$$

To see that Ψ is well defined, suppose that $x \odot_{\mathcal{M}(R)} y = z \odot_{\mathcal{M}(R)} w$, which means that

$$\sum_{i\in\mathcal{I}} x_i r'^{(\mathcal{I})} y_i = \sum_{i\in\mathcal{I}} z_i r'^{(\mathcal{I})} w_i$$

for all $r' \in R'$, where $x_i, z_i \in \mathcal{R}(X)$ and $y_i, w_i \in \mathcal{C}(Y)$ are the components of $x, z \in \mathcal{R}(\mathcal{R}(X))$ and $y, w \in \mathcal{C}(\mathcal{C}(Y))$. Thus, denoting by x_{ij}, $z_{ij} \in X$ and y_{ij}, $w_{ij} \in Y$ the components of x_i, z_i, y_i and w_i, respectively, we have

$$\sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}} x_{ij} r' y_{ij} = \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}} z_{ij} r' w_{ij} \quad (r' \in R'). \tag{3.3}$$

Since x, y, z and w are bounded operators, it is easy to verify that the sums in (3.3) converge absolutely in the weak operator topology. Hence the order of summation is not important and it follows that

$$\sum_{i\in\mathcal{I}} x_{\sigma(i)} r' y_{\sigma(i)} = \sum_{i\in\mathcal{I}} z_{\sigma(i)} r' w_{\sigma(i)} \quad (r' \in R').$$

By the definition of Ψ this means that $\Psi(x \odot_{\mathcal{M}(R)} y) = \Psi(z \odot_{\mathcal{M}(R)} y)$; hence Ψ is well defined. It is not hard to verify that Ψ is a contraction and inverse to Φ; hence Φ (and Ψ) must be isometric.

That Φ is a complete isometry, now follows by an application of what we have just proved (with $\mathcal{C}_n(X)$ and $\mathcal{R}_n(Y)$ instead of X and Y respectively) and the last part of Lemma 3.1. Namely, for each n we have the following chain of isometries (denoted by =):

$$\begin{aligned} \mathcal{M}_n(X \tilde{\otimes}^h_R Y) &= \mathcal{C}_n(X) \tilde{\otimes}^h_R \mathcal{R}_n(Y) \\ &= \mathcal{R}(\mathcal{C}_n(X)) \tilde{\otimes}^h_{\mathcal{M}(R)} \mathcal{C}(\mathcal{R}_n(Y)) \\ &= \mathcal{C}_n(\mathcal{R}(X)) \tilde{\otimes}^h_{\mathcal{M}(R)} \mathcal{R}_n(\mathcal{C}(Y)) \\ &= \mathcal{M}_n(\mathcal{R}(X) \tilde{\otimes}^h_{\mathcal{M}(R)} \mathcal{C}(Y)). \end{aligned}$$

The proof is complete.

We shall need an isomorphism for the product $\bar{\otimes}^h_R$ similar to that in Lemma 3.1 for the product $\tilde{\otimes}^h_R$. The second part of the following lemma is needed only for studying the products of more than two spaces.

LEMMA 3.5. *Let $R \subseteq \mathrm{B}(\mathcal{H})$ be a von Neumann algebra, $X \subseteq \mathrm{B}(\mathcal{H})$ a strong right R-module and $Y \subseteq \mathrm{B}(\mathcal{H})$ a strong left R-module. Suppose that $m \leqslant n \leqslant \mathcal{I}$ are cardinal numbers (where $\mathcal{I} = \dim \mathcal{H}$) and regard $\mathcal{C}_m(\mathrm{B}(\mathcal{H}))$ and $\mathcal{R}_n(\mathrm{B}(\mathcal{H}))$ as subspaces in $\mathcal{M}_n(\mathrm{B}(\mathcal{H})) = \mathrm{B}(\mathcal{H}^n)$ by embedding into the first row and column, respectively (where we first embed $\mathcal{C}_m(\mathrm{B}(\mathcal{H}))$ into $\mathcal{C}_n(\mathrm{B}(\mathcal{H}))$ on the first m positions). Then*

(i) $\mathcal{M}_{m,n}(X \bar{\otimes}^h_R Y) = \mathcal{C}_m(X) \bar{\otimes}^h_{R^{(n)}} \mathcal{R}_n(Y)$ *completely isometrically.*

If, in addition, $S \subseteq \mathrm{B}(\mathcal{H})$ is a von Neumann algebra, Y is a strong right S-module and $Z \subseteq \mathrm{B}(\mathcal{H})$ a strong left S-module, then

(ii) $\mathcal{M}_{m,n}(X \bar{\otimes}^h_R Y \bar{\otimes}^h_S Z) = \mathcal{C}_m(X) \bar{\otimes}^h_{R^{(n)}} Y^{(n)} \bar{\otimes}^h_{S^{(n)}} \mathcal{R}_n(Z)$ *completely isometrically.*

Proof. Both parts have similar proofs, so we sketch only the proof of (ii) which is harder (the proof of (i) can be found in the expository article [**25**]). For notational simplicity we assume that $m = n$ (the case $m < n$ can be treated then by embedding $\mathcal{C}_m(X)$ into $\mathcal{C}_n(X)$ on the first m positions). Elements of $\mathcal{M}_n(X \bar{\otimes}^h_R Y \bar{\otimes}^h_S Z)$ are matrices $\vartheta = [\vartheta_{ij}]$, where each $\vartheta_{ij} \in X \bar{\otimes}^h_R Y \bar{\otimes}^h_S Z$ is of the form $\vartheta_{ij} = x_{ij} \odot_R y_{ij} \odot_S z_{ij}$ with $x_{ij} \in \mathcal{R}(X)$, $y_{ij} \in \mathcal{M}(Y)$ and $z_{ij} \in \mathcal{C}(Z)$. Since $\mathcal{I}$ is infinite and $n \leqslant \mathcal{I}$, one can show that all ϑ_{ij} can be expressed in the form

$$\vartheta_{ij} = x_i \odot_R y \odot_S z_j \quad (i, j < n),$$

where $x_i \in \mathcal{R}(X)$, $y \in \mathcal{M}(Y)$ and $z_j \in \mathcal{C}(Z)$. (We have assumed that n is infinite; if n is finite, the inequality sign $<$ should be replaced by $\leqslant$. We identify $\mathcal{R}(X)$ with $\mathcal{R}_n(\mathcal{R}_n(\mathcal{R}(X)))$, $\mathcal{C}(Y)$ with $\mathcal{C}_n(\mathcal{C}_n(\mathcal{C}(Y)))$, and $\mathcal{M}(Y)$ with $\mathcal{M}_n(\mathcal{M}_n(\mathcal{M}(Y)))$ (since $\mathcal{I}$ is infinite) and we construct elements

$$x_i \in \mathcal{R}_n(\mathcal{R}_n(\mathcal{R}(X))) \quad \text{and} \quad z_j \in \mathcal{C}_n(\mathcal{C}_n(\mathcal{C}(Y)))$$

as in the proof of Lemma 3.1, namely, $x_i(k)(l) = \delta_{il}x_{ik}$ $(i, k, l < n)$ and $z_j(k)(l) = \delta_{jk}z_{lj}$ $(j, k, l < n)$, where δ_{ij} denotes the Kronecker symbol. Finally, let $y \in \mathcal{M}_n(\mathcal{M}_n(\mathcal{M}(Y)))$ be the diagonal matrix $y = \bigoplus_{k,l<n} y_{lk}$.) It follows that

$$\vartheta = x \odot_R y \odot_S z,$$

where $x \in \mathcal{C}_n(\mathcal{R}(X)) = \mathcal{M}_{n,\mathcal{J}}(X)$ has rows x_i, $z \in \mathcal{R}_n(\mathcal{C}(Z)) = \mathcal{M}_{\mathcal{J},n}(Z)$ has columns z_j, and $y \in \mathcal{M}_{\mathcal{J}}(Y) = \mathcal{M}(Y)$. We associate with x and z the elements $\tilde{x} \in \mathcal{R}(\mathcal{M}_n(X))$ and $\tilde{z} \in \mathcal{C}(\mathcal{M}_n(Z))$ in the same way as in the proof of Lemma 3.1. Further, let $\tilde{y} \in \mathcal{M}(\mathcal{M}_n(Y))$ be the matrix that corresponds to $y^{(n)}$ under the canonical isomorphism $\mathcal{M}_n(\mathcal{M}(Y)) \cong \mathcal{M}(\mathcal{M}_n(Y))$. (The entries of $\tilde{y}$ are the diagonal matrices $y_{ij}^{(n)}$ in $\mathcal{M}_n(Y)$.) Finally, define

$$\Omega\colon\ \mathcal{M}_n(X \mathbin{\bar{\otimes}}^h_R Y \mathbin{\bar{\otimes}}^h_S Z) \to \mathcal{C}_n(X) \mathbin{\bar{\otimes}}^h_{R^{(n)}} Y^{(n)} \mathbin{\bar{\otimes}}^h_{S^{(n)}} \mathcal{R}_n(Z)$$

by $\Omega(\vartheta) = \tilde{\vartheta}$, where $\tilde{\vartheta} = \tilde{x} \odot_{R^{(n)}} \tilde{y} \odot_{S^{(n)}} \tilde{z}$.

We claim that Ω is a well-defined isometry. To see this, denote by $\rho\colon \mathcal{M}_n(R') \to \mathcal{R}_n(R')$ the projection onto the first row and by $\kappa\colon \mathcal{M}_n(S') \to \mathcal{C}_n(S')$ the projection onto the first column. Further, let $\iota\colon R' \to \mathcal{M}_n(R')$ and $\mu\colon S' \to \mathcal{M}_n(S')$ be the embeddings onto the $(1, 1)$ positions. Note that in the computation of norms we regard ϑ as an element of

$$\mathcal{M}_n(\mathrm{CB}(R', S'; \mathrm{B}(\mathcal{H})) = \mathrm{CB}(R', S'; \mathrm{B}(\mathcal{H}^n))$$

and $\tilde{\vartheta}$ as an element of

$$\mathrm{CB}(\mathcal{M}_n(R'), \mathcal{M}_n(S'); \mathrm{B}(\mathcal{H}^n))$$

(since $(R^{(n)})' = \mathcal{M}_n(R')$ and similarly for S). A computation (left to the reader) shows that

$$\tilde{\vartheta}(r', s') = [\vartheta_{ij}(\rho(r'), \kappa(s'))] = \vartheta(\rho(r'), \kappa(s'))$$

for arbitrary $r' \in \mathcal{M}_n(R')$ and $s' \in \mathcal{M}_n(S')$. (Here the canonical bilinear extension of ϑ to matrices is denoted by ϑ again.) This shows that $\tilde{\vartheta}$ is determined by ϑ, and hence Ω is well defined. Since ρ and κ are complete contractions, it follows that $\|\tilde{\vartheta}\|_{\mathrm{cb}} \leqslant \|\vartheta\|_{\mathrm{cb}}$. The reverse inequality follows from the relation

$$\vartheta(r', s') = \tilde{\vartheta}(\iota(r'), \mu(s')) \quad (r' \in R',\ s' \in S'),$$

since ι and μ are also complete contractions. This proves that Ω is an isometry. That it is in fact a complete isometry follows then as in the last part of the proof of Lemma 3.1 and, by definition, Ω is surjective.

THEOREM 3.6. *The operator spaces $X \mathbin{\tilde{\otimes}}^h_R Y$ and $X \mathbin{\bar{\otimes}}^h_R Y$ are the same for every von Neumann algebra $R \subseteq \mathrm{B}(\mathcal{H})$, each strong right R-module $X \subseteq \mathrm{B}(\mathcal{H})$ and each strong left R-module $Y \subseteq \mathrm{B}(\mathcal{H})$.*

Proof. By definition $X \mathbin{\tilde{\otimes}}^h_R Y$ and $X \mathbin{\bar{\otimes}}^h_R Y$ are the same set. It suffices to prove that the two norms $\|\ \|$ and $\|\ \|_{\mathrm{cb}}$ agree on $X \mathbin{\bar{\otimes}}^h_R Y$ for arbitrary X, Y and R, for then we shall have for each n, by Lemma 3.1 and Lemma 3.5(i),

$$\begin{aligned}\mathcal{M}_n(X \mathbin{\tilde{\otimes}}^h_R Y) &= \mathcal{C}_n(X) \mathbin{\tilde{\otimes}}^h_{R^{(n)}} \mathcal{R}_n(Y)\\ &= \mathcal{C}_n(X) \mathbin{\bar{\otimes}}^h_{R^{(n)}} \mathcal{R}_n(Y)\\ &= \mathcal{M}_n(X \mathbin{\bar{\otimes}}^h_R Y).\end{aligned}$$

We have already mentioned that the proof of the inequality $\|\vartheta\|_{\mathrm{cb}} \leqslant \|\vartheta\|$ ($\vartheta \in X \bar{\otimes}^h_R Y$) is easy (as in the classical case). The proof of the reverse inequality is essentially the same as the proof of Lemma 2.3 in [**24**], but since the argument is short we repeat it here for the convenience of the reader. Suppose that $\|\vartheta\|_{\mathrm{cb}} < 1$; we shall prove that then $\|\vartheta\| < 1$. Since $\vartheta \in X \bar{\otimes}^h_R Y$, $\vartheta = x \odot_R y$ for some $x \in \mathcal{R}(X)$ and $y \in \mathcal{C}(Y)$. On the other hand, by the representation theorem for normal completely bounded linear mappings [**12**] there exist $z \in \mathcal{R}(\mathrm{B}(\mathcal{H}))$ and $w \in \mathcal{C}(\mathrm{B}(\mathcal{H}))$ such that $\|z\|, \|w\| < 1$ and $z \odot_R w = x \odot_R y$. By Lemma 2.1 from [**24**] (see also formulas (2.4) and (2.4′) in [**24**]) there exist four sequences (u_k), (v_k), (e_k) and (e'_k) in $\mathcal{M}(R)$ such that $e_k + e'_k + u_k v_k = 1$, $\|xe_k\| \to 0$, $\|e'_k y\| \to 0$, $\|xu_k\| \leqslant \|z\|$ and $\|v_k y\| \leqslant \|w\|$. Note that

$$\begin{aligned} \|x \otimes_{\mathcal{M}(R)} y - xu_k \otimes_{\mathcal{M}(R)} v_k y\| &= \|x \otimes_{\mathcal{M}(R)} (1 - u_k v_k) y\| \\ &= \|x \otimes_{\mathcal{M}(R)} (e_k + e'_k) y\| \\ &= \|xe_k \otimes_{\mathcal{M}(R)} y + x \otimes_{\mathcal{M}(R)} e'_k y\| \\ &\leqslant \|xe_k\|\|y\| + \|x\|\|e'_k y\| \\ &\to 0. \end{aligned}$$

By Lemma 3.4, it follows that

$$\lim_{k\to\infty} \|xu_k \odot_R v_k y - x \odot_R y\| = 0.$$

Since $\|xu_k\| \leqslant \|z\| < 1$ and $\|v_k y\| \leqslant \|w\| < 1$ (and $xu_k \in \mathcal{R}(X)$, $v_k y \in \mathcal{C}(Y)$, since X and Y are strong), we see that $\|\vartheta\| = \|x \odot_R y\| < 1$.

REMARK 3.7. Now the product $X \bar{\otimes}^h_R Y$ can be defined without any reference to completely bounded operators or to R'. Namely, in view of Lemma 3.2 we may regard elements of $X \bar{\otimes}^h_R Y$ as equivalence classes of pairs $(x, y) \in \mathcal{R}(X) \times \mathcal{C}(Y)$, where two pairs (x, y) and (z, w) are equivalent if and only if there exists an element $p \in \mathcal{M}_{2\mathcal{J}}(R)$ (in fact, a projection) such that $[x, z]p = 0$ and $p^{\perp}(y, -w) = 0$. Denoting the equivalence class of (x, y) by $x \odot_R y$, we see that the sum is defined by $x \odot_R y + z \odot_R w = [x, z] \odot_R (y, w)$, the multiplication by complex numbers is defined in the obvious way and the norm is defined by (3.1).

From now on we also denote by $X \bar{\otimes}^h_R Y$ (instead of $X \tilde{\otimes}^h_R Y$) the extended Haagerup tensor product of abstract operator modules.

The Haagerup tensor product $X \otimes^h_R Y$ is defined in [**5**] and [**24**] as the quotient of $X \otimes^h Y$ by the closed subspace generated by all elements of the form

$$xr \otimes y - x \otimes ry \quad (x \in X,\ y \in Y,\ r \in R). \tag{3.4}$$

By analogy one might guess that for weak* closed R-submodules $X, Y \subseteq \mathrm{B}(\mathcal{H})$ the product $X \bar{\otimes}^h_R Y$ coincides with the quotient of $X \bar{\otimes}^h Y$ (the weak* Haagerup tensor product of [**7**]) by the weak* closed subspace generated by all elements of the form (3.4). However, the following example shows that this is not true, since the quotient space is often 0.

EXAMPLE 3.8. Let $X = Y = B := \mathrm{B}(\mathcal{H})$ and let $R \subseteq \mathrm{B}(\mathcal{H})$ be a von Neumann algebra such that R' contains no non-zero compact operators. Denote by N the weak* closed subspace of $B \bar{\otimes}^h B$ generated by all elements of the form (3.4). We shall prove that $N = B \bar{\otimes}^h B$ by showing that the only element ρ of the predual $K \hat{\otimes} T$ of $B \otimes^h B$ (see [**7**]) that annihilates N is $\rho = 0$; here K and T denote the space of all compact and all trace class operators on $\mathcal{H}$. Each $\rho \in K \hat{\otimes} T$ can be expressed as

$$\rho = \sum_{n=1}^{\infty} \rho_n,$$

where the ρ_n are from the algebraic tensor product $K \otimes T$ and $\sum \|\rho_n\|_\wedge < \infty$ (here $\| \ \|_\wedge$ denotes the maximal operator tensor norm). Thus, there is an increasing sequence (m_n) of positive integers such that

$$\rho_n = \sum_{i=m_n}^{m_{n+1}} a_i \otimes t_i \quad (a_i \in K,\ t_i \in T)$$

for each n and then

$$\rho = \sum_{i=1}^{\infty} a_i \otimes t_i.$$

If ρ annihilates N, then

$$\sum_{i=1}^{\infty} \langle x(ra_i - a_i r)y, t_i \rangle = 0 \quad (\text{for all } x, y \in B,\ r \in R), \tag{3.5}$$

where $\langle\ ,\ \rangle$ denotes the duality between B and T. We shall show that the series

$$\sum_{i=1}^{\infty} (ra_i - a_i r) y t_i$$

converges in the usual operator norm of K. Hence (3.5) implies (by varying x over T) that

$$\sum_{i=1}^{\infty} (ra_i - a_i r) y t_i = 0. \tag{3.6}$$

Indeed, since the maximal operator tensor norm $\| \ \|_\wedge$ dominates the Haagerup norm $\| \ \|_h$ (see [**18**]), we have $\sum \|\rho_n\|_h < \infty$. Hence we may suppose that

$$\sum_{n=1}^{\infty} \|\tilde{a}_n\|^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \|\tilde{t}_n\|_\wedge^2 < \infty,$$

where

$$\tilde{a}_n = [a_{m_n+1}, \ldots, a_{m_{n+1}}] \quad \text{and} \quad \tilde{t}_n = (t_{m_n+1}, \ldots, t_{m_{n+1}})$$

and $\| \ \|_\wedge$ denotes the operator predual matrix norms on columns with the entries in T. Since the matrix norms $\| \ \|_\wedge$ dominate the usual operator norm (namely, by [**17**] or [**4**], $T = \mathcal{H}_r \hat{\otimes} \mathcal{H}_c$ is the maximal and $K = \mathcal{H}_r \check{\otimes} \mathcal{H}_c$ is the minimal operator tensor product of the row Hilbert space $\mathcal{H}_r$ and the column Hilbert space $\mathcal{H}_c$ of $\mathcal{H}$), it follows that the two series

$$\sum_{i=1}^{\infty} a_i a_i^* \quad \text{and} \quad \sum_{i=1}^{\infty} t_i^* t_i$$

are both convergent in the usual operator norm. This easily implies the convergence of the series (3.6).

Put $\tilde{t} = (t_1, t_2, \dots) \in \mathrm{B}(\mathcal{H}, \mathcal{H}^\infty)$ and let $p \in \mathrm{B}(\mathcal{H}^\infty)$ be the projection with range $[B^{(\infty)}\tilde{t}\mathcal{H}]$. Then $p \in \mathcal{M}_\infty(\mathbb{C}1_{\mathcal{H}})$ and (3.6) can be written as

$$[ra_1 - a_1 r, ra_2 - a_2 r, \dots]p = 0 \quad (r \in R). \tag{3.7}$$

If $\tilde{t} = 0$, then $\rho = 0$. If $\tilde{t} \neq 0$, then $p \neq 0$ and, denoting by $(\lambda_1, \lambda_2, \dots)$ any non-zero column of p, we have from (3.7),

$$\sum_{i=1}^{\infty} \lambda_i (ra_i - a_i r) = 0 \quad (r \in R).$$

This means that the compact operator $\sum \lambda_i a_i$ commutes with R; hence it must be 0 by assumption. Since this holds for each column of p, we have now $\tilde{a}p = 0$, where $\tilde{a} = [a_1, a_2, \dots]$. This implies that

$$\sum_{i=1}^{\infty} a_i \otimes t_i = \tilde{a} \odot \tilde{t} = \tilde{a} \odot p\tilde{t} = \tilde{a}p \odot \tilde{t} = 0 \quad \text{in } K \otimes^h K. \tag{3.8}$$

To see that (3.8) implies that $\rho = 0$, it suffices now to show that the natural complete contraction $\Omega\colon\ K \hat{\otimes} T \to K \otimes^h K$ is one-to-one.

So, suppose that $\rho \in \ker \Omega$ and $\|\rho\| < 1$. Then we can write $\rho = \sum_{n=1}^{\infty} \omega_n$, where $\omega_n \in K \otimes T$ and $\|\omega_n\| < 2^{-n}$ for $n \geqslant 2$. By the definition of the maximal operator tensor norm [**18**] there exist matrices $a_n \in \mathcal{M}_{k_n}(K)$, $b_n \in \mathcal{M}_{m_n}(T)$, $\alpha_n \in \mathcal{R}_{k_n m_n}(\mathbb{C})$ and $\beta_n \in \mathcal{C}_{k_n m_n}(\mathbb{C})$ such that $\omega_n = \alpha_n(a_n \otimes t_n)\beta_n$ for all n, $\|a_n\| = \|t_n\|_\wedge = 1$ and $\|\alpha_n\|\|\beta_n\| < 2^{-n}$ for $n \geqslant 2$. There is a net of finite rank complete contractions $\kappa_j\colon\ K \to K$ converging to the identity in the point norm topology (namely, the compressions onto the finite dimensional subspaces of $\mathcal{H}$). Let $\kappa_j \hat{\otimes} 1$ and $\kappa_j \otimes^h 1$ be the contractions induced by κ_j on $K \hat{\otimes} T$ and $K \otimes^h K$, respectively. Then

$$\Omega((\kappa_j \hat{\otimes} 1)(\rho)) = (\kappa_j \otimes^h 1)\Omega(\rho) = 0. \tag{3.9}$$

Since κ_j is of finite rank, $(\kappa_j \hat{\otimes} 1)(\rho)$ is contained in the algebraic tensor product $K \otimes T$. Hence (3.9) implies that $(\kappa_j \hat{\otimes} 1)(\rho) = 0$. But it is easy to see that the net $(\kappa_j \hat{\otimes} 1)(\rho)$ converges to ρ (approximate ρ by finite sums of elementary tensors); hence $\rho = 0$.

We would like to end this section with a result concerning the extension of bilinear maps to the extended Haagerup tensor product. To this end we shall need a variant of the Christensen–Sinclair representation theorem (Theorem 3.9 below).

If X is a right operator module and Y is a left operator module over a C*-algebra A, then a completely bounded bilinear map Ω from $X \times Y$ to $\mathrm{B}(\mathcal{H})$ (or to any operator space) is called *A-balanced* if $\Omega(xa, y) = \Omega(x, ay)$

for all $x \in X$, $y \in Y$ and $a \in A$. Further, if A is a von Neumann algebra and the map $a \mapsto \Omega(xa, y)$ is normal for all $x \in X$ and $y \in Y$, then we say that Ω is *A-normal*.

THEOREM 3.9. *If X is a right operator module and Y is a left operator module over a C^*-algebra A and $\Omega\colon X \times Y \to B(\mathcal{H})$ is an A-balanced complete contraction, then there exist a Hilbert space $\mathcal{K}$, a representation $\pi\colon A \to B(\mathcal{K})$, and complete contractions $\Phi\colon X \to B(\mathcal{K}; \mathcal{H})$ and $\Psi\colon Y \to B(\mathcal{H}; \mathcal{K})$ such that*

$$\Phi(xa) = \Phi(x)\pi(a), \quad \Psi(ay) = \pi(a)\Psi(y) \quad \textit{and} \quad \Omega(x, y) = \Phi(x)\Psi(y)$$

for all $x \in X$, $y \in Y$ and $a \in A$. Moreover, if A is a von Neumann algebra and Ω is A-normal, then π can be taken to be normal.

Proof. By the representation theorem of Christensen, Effros and Sinclair, we may assume that Y and A are contained in a C^*-algebra Y_1 such that the module multiplication is the internal multiplication in Y_1. By the well-known Christensen–Sinclair representation theorem (see [**10**] or [**30**]) there exist a Hilbert space $\mathcal{K}_0$ and complete contractions $\Phi\colon X \to B(\mathcal{K}_0; \mathcal{H})$ and $\Psi\colon Y \to B(\mathcal{H}; \mathcal{K}_0)$ such that $\Omega(x, y) = \Phi(x)\Psi(y)$ for all $x \in X$ and $y \in Y$. Moreover, Ψ is of the form $\Psi(y) = \sigma(y)b$, where σ is a representation on $\mathcal{K}_0$ of Y_1 and $b \in B(\mathcal{H}; \mathcal{K}_0)$ is a contraction. (By the Christensen–Sinclair theorem Φ has a similar description which, however, is not needed here.) Put $\mathcal{K} = [\Psi(Y)\mathcal{H}]$. For each $a \in A$ define $\pi(a)$ on the dense subspace $\Psi(Y)\mathcal{H}$ of $\mathcal{K}$ by

$$\pi(a) \sum \Psi(y_i)\xi_i = \sum \Psi(ay_i)\xi_i \quad (\xi_i \in \mathcal{H}).$$

To see that $\pi(a)$ is well defined and bounded, note that

$$\Big\| \sum \Psi(ay_i)\xi_i \Big\|^2 = \sum_{i,j} \langle \sigma(y_i^* a^* a y_j) b\xi_j, b\xi_i \rangle \leqslant \|a\|^2 \Big\| \sum \Psi(y_i)\xi_i \Big\|^2.$$

Thus, $\pi(a)$ can be uniquely extended to $\pi(a) \in B(\mathcal{K})$ and it is easy to verify that π is a representation of A on $\mathcal{K}$ satisfying $\pi(a)\Psi(y) = \Psi(ay)$ for all $a \in A$ and $y \in Y$. (To see that π preserves the involution $*$, we use the fact that π is a contraction.) We may replace Φ with the map $x \mapsto \Phi(x)|\mathcal{K}$ without violating the identity $\Omega(x, y) = \Phi(x)\Psi(y)$ (for all $x \in X$ and $y \in Y$). Since

$$(\Phi(x)\pi(a))\Psi(y)\xi = \Phi(x)\Psi(ay)\xi = \Omega(x, ay)\xi = \Omega(xa, y)\xi = \Phi(xa)\Psi(y)\xi$$

for all $a \in A$, $x \in X$, $y \in Y$ and $\xi \in \mathcal{H}$ and since $[\Psi(Y)\mathcal{H}] = \mathcal{K}$, it follows that $\Phi(x)\pi(a) = \Phi(xa)$.

To prove the last statement of the theorem, let $\pi = \pi_n + \pi_s$ be the decomposition of π into the normal and the singular part (see [**21**, Chapter 10]). Since Ω is A-normal, the map

$$a \mapsto \langle \Phi(xa)\Psi(y)\xi, \eta \rangle = \langle \Omega(xa, y)\xi, \eta \rangle$$

is normal for all $\xi, \eta \in \mathcal{H}$. From $[\Psi(Y)\mathcal{H}] = \mathcal{K}$, it follows then by standard arguments that the map $a \mapsto \Phi(xa)$ is normal and similarly the map $a \mapsto \Psi(ay)$ is normal. (To prove the normality of the last map, we need that

$[\Phi(X)^*\mathcal{H}] = \mathcal{K}$, which can be achieved by replacing $\mathcal{K}$ with $(1-P)\mathcal{K}$, where P is the projection onto the space of all vectors in $\mathcal{K}$ annihilated by $\Phi(X)$, and modifying Φ and Ψ accordingly.) Finally, from

$$\Phi(x)\pi_s(a) = \Phi(xa) - \Phi(x)\pi_n(a),$$

we conclude that $\Phi(x)\pi_s(a) = 0$ since the left-hand side of the above identity is singular while the right-hand side is normal in a. Similarly $\pi_s(a)\Psi(y) = 0$ for all $a \in A$ and $y \in Y$ and therefore replacing π by π_n we complete the proof.

PROPOSITION 3.10. *Let R be a von Neumann algebra, X and Y strong modules over R, and $\Omega\colon\ X \times Y \to \mathrm{B}(\mathcal{H})$ an R-balanced completely bounded bilinear map. Then Ω can be extended to a completely bounded map $\bar{\Omega}\colon\ X \bar{\otimes}^h_R Y \to \mathrm{B}(\mathcal{H})$ satisfying*

$$\bar{\Omega}\Big(\sum_{i\in\mathcal{I}} x_i \otimes_R y_i\Big) = \sum_{i\in\mathcal{I}} \Omega(x_i, y_i)$$

(for all index sets $\mathcal{I}$) if and only if Ω is R-normal. In this case $\|\bar{\Omega}\|_{\mathrm{cb}} = \|\Omega\|_{\mathrm{cb}}$.

Proof. Assume that the extension $\bar{\Omega}$ exists. Choose $x \in X$, $y \in Y$ and let $\{e_i\colon\ i \in \mathcal{I}\}$ be any orthogonal set of projections in R. Then

$$\Omega\Big(\sum xe_i, y\Big) = \bar{\Omega}\Big(\sum xe_i \otimes_R y\Big) = \bar{\Omega}\Big(\sum xe_i \otimes_R e_i y\Big) = \sum \Omega(xe_i, y).$$

Hence the map $a \mapsto \Omega(xa, y)$ is completely additive on projections of R and therefore normal by [**21**]. This means that Ω is R-normal.

To prove the converse, let $\mathcal{K}$, Φ, Ψ and π be as in Theorem 3.9 with π normal and put $\bar{\Omega} = \mu \circ (\Phi \bar{\otimes}^h_\pi \Psi)$, where

$$\Phi \bar{\otimes}^h_\pi \Psi\colon\ X \bar{\otimes}^h_R Y \to \mathrm{B}(\mathcal{K};\mathcal{H}) \bar{\otimes}^h_{\pi(R)} \mathrm{B}(\mathcal{H};\mathcal{K})$$

is defined as in the proof of Proposition 3.3 and

$$\mu\colon\ \mathrm{B}(\mathcal{K};\mathcal{H}) \bar{\otimes}^h_{\pi(R)} \mathrm{B}(\mathcal{H};\mathcal{K}) \to \mathrm{B}(\mathcal{H})$$

is the multiplication map.

4. *Products of three spaces*

By a strong R, S-bimodule we shall mean an operator R, S-bimodule which is strong as a right R-module and as a left S-module.

PROPOSITION 4.1. *If R and S are W^*-algebras, X is a strong right R-module and Y is a strong R, S-bimodule, then $X \bar{\otimes}^h_R Y$ is a strong right S-module for the operation*

$$(x \odot_R y)s := x \odot_R ys \quad (x \in \mathcal{R}(X),\ \ y \in \mathcal{C}(Y),\ \ s \in S).$$

Proof. We may assume that R, S, X and Y are contained in $\mathrm{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Since $\mathrm{B}(\mathcal{H})$ is the operator dual of $\mathrm{T}(\mathcal{H})$ (the trace class of $\mathcal{H}$, see [**3**] or [**17**]), the space

$$D := \mathrm{CB}(R'; \mathrm{B}(\mathcal{H}))$$

is the operator dual of the maximal operator tensor product $D_\sharp = R' \hat{\otimes} \mathrm{T}(\mathcal{H})$ [**6**]. Here D is an operator right S-module (that is, a $\mathbb{C}, S$-operator bimodule in the sense of [**15**]) by

$$(\vartheta s)(r') := \vartheta(r')s \quad (\vartheta \in D,\ r' \in R',\ s \in S).$$

Since $D_\sharp$ can be regarded also as an L^1-predual of D (see the argument in [**16**, p. 186]), it follows easily that D is a normal dual operator right S-module in the sense of [**15**]. Hence by [**15**, Theorem 4.4] there exist a Hilbert space $\mathcal{K}$, a completely isometric weak* homeomorphism Φ of D onto a weak* closed subspace of $\mathrm{B}(\mathcal{K})$ and a (faithful) normal *-representation φ of S on $\mathcal{K}$ such that $\Phi(\vartheta s) = \Phi(\vartheta)\varphi(s)$ for all $\vartheta \in D$ and $s \in S$. It suffices now to prove that $\Phi(X \bar{\otimes}^h_R Y)$ is strong over $\varphi(S)$.

So, let $s \in \mathcal{C}(S)$ and $\vartheta \in \mathcal{R}(X \bar{\otimes}^h_R Y)$ and denote the components of s and ϑ by s_i and ϑ_i (respectively). We have to show that

$$\sum_{i \in \mathcal{I}} \Phi(\vartheta_i)\varphi(s_i) \in \Phi(X \bar{\otimes}^h_R Y). \tag{4.1}$$

Since the sum in (4.1) converges in the weak* topology of $\mathrm{B}(\mathcal{K})$ and Φ is a weak* homeomorphism, the sum $\sum_{i \in \mathcal{I}} \vartheta_i s_i$ converges in the weak* topology to an element $\psi \in D$, and it suffices to show that $\psi \in X \bar{\otimes}^h_R Y$. Since $\vartheta \in \mathcal{R}(X \bar{\otimes}^h_R Y)$, by Lemma 3.5(i) (applied to the case $m = 1$ and $n = \mathcal{I}$) there exist $x \in \mathcal{R}(X)$ and $y \in \mathcal{C}(\mathcal{R}(Y)) = \mathcal{M}(Y)$ such that

$$\vartheta = x \odot_R y = \left[\sum_{i \in \mathcal{I}} x_i \otimes_R y_{ij} \right]_{j \in \mathcal{I}}.$$

Now we have

$$\psi = \sum_{j \in \mathcal{I}} \vartheta_j s_j = \sum_{j \in \mathcal{I}} \sum_{i \in \mathcal{I}} x_i \otimes_R y_{ij} s_j = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} x_i \otimes_R y_{ij} s_j = x \odot_R (ys). \tag{4.2}$$

(Here the change of the order of summation can be justified by the fact that for each $r' \in R'$ we have $(xr'^{(\mathcal{I})}y)s = (xr'^{(\mathcal{I})})(ys)$.) Since Y is strong over S, $ys \in \mathcal{C}(Y)$, and hence (4.2) shows that $\psi \in X \bar{\otimes}^h_R Y$.

Let R, S, X, Y be as in Proposition 4.1 and let Z be a strong left S-module. Then we can form

$$(X \bar{\otimes}^h_R Y) \bar{\otimes}^h_S Z \quad \text{and} \quad X \bar{\otimes}^h_R (Y \bar{\otimes}^h_S Z)$$

and we would like to prove associativity. (The associativity of the usual Haagerup tensor product of operator modules is proved in [**5**], but we shall not use this result here.) We assume that X, Y, Z, R and S are all contained in $\mathrm{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Recall that the triple product $X \bar{\otimes}^h_R Y \bar{\otimes}^h_S Z$

is the space of all mappings from $R' \times S'$ to $\mathrm{B}(\mathcal{H})$ of the form $x \odot_R y \odot_S z$, where $x \in \mathcal{R}(X)$, $y \in \mathcal{M}(Y)$, $z \in \mathcal{C}(Z)$ and by definition

$$(x \odot_R y \odot_S z)(r', s') = x r'^{(\mathcal{I})} y s'^{(\mathcal{I})} z \quad (r' \in R', s' \in S').$$

Motivated by Theorem 3.6, first we would like to prove that the norm $\|\vartheta\|_{\mathrm{cb}}$ in $X \bar{\otimes}_R^h Y \bar{\otimes}_S^h Z$ is equal to the quantity

$$\|\vartheta\| := \inf\{\|x\|\|y\|\|z\| \colon \vartheta = x \odot_R y \odot_S z, x \in \mathcal{R}(X), \\ y \in \mathcal{M}(Y), \ z \in \mathcal{C}(Z)\}. \tag{4.3}$$

By Lemma 3.5(i) each element of $(X \bar{\otimes}_R^h Y) \bar{\otimes}_S^h Z$ is of the form

$$(x \odot_R y) \odot_S z,$$

where $x \in \mathcal{R}(X)$, $y \in \mathcal{M}(Y)$ and $z \in \mathcal{C}(Z)$. Thus, to see that

$$V := (X \bar{\otimes}_R^h Y) \bar{\otimes}_S^h Z \quad \text{and} \quad W := X \bar{\otimes}_R^h Y \bar{\otimes}_S^h Z$$

are essentially the same as vector spaces, it suffices to prove the following lemma.

LEMMA 4.2. *For arbitrary $x, \tilde{x} \in \mathcal{R}(X)$, $y, \tilde{y} \in \mathcal{M}(Y)$ and $z, \tilde{z} \in \mathcal{C}(Z)$ the identities*

$$(x \odot_R y) \odot_S z = (\tilde{x} \odot_R \tilde{y}) \odot_S \tilde{z} \quad \textit{and} \quad x \odot_R y \odot_S z = \tilde{x} \odot_R \tilde{y} \odot_S \tilde{z} \tag{4.4}$$

are equivalent.

Proof. We can write the first identity of (4.4) in the form

$$[x \odot_R y, \tilde{x} \odot_R \tilde{y}] \odot_S \begin{bmatrix} z \\ -\tilde{z} \end{bmatrix} = 0$$

and, using Lemma 3.2 or Remark 3.7, it is not difficult to show that this is equivalent to the existence of a projection $q \in \mathcal{M}_{2\mathcal{I}}(S)$ satisfying

$$[x \odot_R y, \tilde{x} \odot_R \tilde{y}] q = 0 \quad \text{and} \quad q^{\perp} \begin{bmatrix} z \\ -\tilde{z} \end{bmatrix} = 0. \tag{4.5}$$

The first identity in (4.5) can be written as

$$[x, \tilde{x}] \odot_R \begin{bmatrix} y & 0 \\ 0 & \tilde{y} \end{bmatrix} q = 0,$$

and by Lemma 3.2 again we see that this is equivalent to the existence of a projection $p \in \mathcal{M}_{2\mathcal{I}}(R)$ satisfying

$$[x, \tilde{x}] p = 0 \quad \text{and} \quad p^{\perp} \begin{bmatrix} y & 0 \\ 0 & \tilde{y} \end{bmatrix} q = 0. \tag{4.6}$$

Thus, the first equality in (4.4) holds if and only if there exist projections $p \in \mathcal{M}_{2\mathcal{I}}(R)$ and $q \in \mathcal{M}_{2\mathcal{I}}(S)$ such that (4.6) and the second identity in (4.5) hold. It is easy to verify that the same two conditions are together also equivalent to the second identity in (4.4).

We shall now verify that V and W agree as normed spaces, where in W we take the norm defined by (4.3). Indeed, choose any $\tilde{x} \in \mathcal{R}(X)$, $\tilde{y} \in \mathcal{M}(Y)$ and $\tilde{z} \in \mathcal{C}(Z)$ and put $\tilde{\omega} = (\tilde{x} \odot_R \tilde{y}) \odot_S \tilde{z}$ and $\omega = \tilde{x} \odot_R \tilde{y} \odot_S \tilde{z}$. Then by Theorem 3.6, Lemma 3.5(i) and Lemma 4.2 we have

$$\begin{aligned}\|\tilde{\omega}\| &= \inf\{\|\vartheta\|\|z\|: \ \vartheta \odot_S z = \tilde{\omega}, \ \vartheta \in \mathcal{R}(X \mathbin{\bar{\otimes}}^h_R Y), \ z \in \mathcal{C}(Z)\} \\ &= \inf\{\|x\|\|y\|\|z\|: \ (x \odot_R y) \odot_S z = \tilde{\omega}, \ x \in \mathcal{R}(X), \ y \in \mathcal{M}(Y), \ z \in \mathcal{C}(Z)\} \\ &= \inf\{\|x\|\|y\|\|z\|: \ x \odot_R y \odot_S z = \omega, \ x \in \mathcal{R}(X), \ y \in \mathcal{M}(Y), \ z \in \mathcal{C}(Z)\} \\ &= \|\omega\|.\end{aligned}$$

Now we shall show that the norm in V agrees also with the norm $\|\cdot\|_{\mathrm{cb}}$ in W. Since by definition the norm in the extended Haagerup tensor product does not change on enlarging the spaces, we may assume that $X = Y = Z = B := \mathrm{B}(\mathcal{H})$. We have just seen above that the norm on $(B \mathbin{\bar{\otimes}}^h_R B) \mathbin{\bar{\otimes}}^h_S B$ agrees with the norm $\|\cdot\|$ on $B \mathbin{\bar{\otimes}}^h_R B \mathbin{\bar{\otimes}}^h_S B$ defined by (4.3). Hence it suffices to show that the norms $\|\cdot\|$ and $\|\cdot\|_{\mathrm{cb}}$ on $B \mathbin{\bar{\otimes}}^h_R B \mathbin{\bar{\otimes}}^h_S B$ agree. But the inequality $\|\cdot\|_{\mathrm{cb}} \leqslant \|\cdot\|$ is easy and the reverse inequality follows from the representation theorem for bilinear normal completely bounded maps [**10**].

THEOREM 4.3. *If R and S are von Neumann algebras, X is a strong right R-module, Y is a strong R,S-bimodule and Z is a strong left S-module, then the operator spaces $(X \mathbin{\bar{\otimes}}^h_R Y) \mathbin{\bar{\otimes}}^h_S Z$ and $X \mathbin{\bar{\otimes}}^h_R Y \mathbin{\bar{\otimes}}^h_S Z$ are naturally completely isometrically isomorphic. Hence (by symmetry) the product $\bar{\otimes}^h_{(\cdot)}$ is associative.*

Proof. We have seen above that the two spaces are isometric. That they are in fact completely isometric now follows by an application of Lemma 3.5 ((i) and (ii)), which is left to the reader (we shall not use this result later in this paper).

5. *Projections and slice maps*

In this section we shall use the notion of X-projection (introduced in § 2) to study the extended Haagerup tensor product.

LEMMA 5.1. *Let $X, Y \subseteq \mathrm{B}(\mathcal{H})$ be a strong right and a strong left module, respectively, over a von Neumann algebra $R \subseteq \mathrm{B}(\mathcal{H})$, let $a \in \mathcal{R}(\mathrm{B}(\mathcal{H}))$ and $b \in \mathcal{C}(\mathrm{B}(\mathcal{H}))$. Then $a \odot_R b \in X \mathbin{\bar{\otimes}}^h_R Y$ if and only if*

$$a \odot_R b = ae \odot_R fb,$$

where $e \in \mathcal{M}(R)$ is the X-projection of a and $f \in \mathcal{M}(R)$ is the Y-projection of b.

Proof. If $a \odot_R b = ae \odot_R fb$, then $a \odot_R b \in X \mathbin{\bar{\otimes}}^h_R Y$ since $ae \in \mathcal{R}(X)$ and $fb \in \mathcal{C}(Y)$ by Proposition 2.3. Suppose now conversely, that

$$a \odot_R b = x \odot_R y$$

for some $x \in \mathcal{R}(X)$ and $y \in \mathcal{C}(Y)$. Then $[a, x] \odot_R (b, -y) = 0$, and hence by Lemma 3.2 there exists a projection $p \in \mathbb{M}_2(\mathcal{M}(R))$ satisfying

$$(5.1) \qquad [a, x]p = 0 \quad \text{and} \quad p^\perp \begin{bmatrix} b \\ -y \end{bmatrix} = 0.$$

Let

$$p = \begin{bmatrix} r & s \\ s^* & t \end{bmatrix} \quad (r, s, t \in \mathcal{M}(R)).$$

Then from the first identity in (5.1) we have $ar = -xs^* \in \mathcal{R}(X)$ and $as = -xt \in \mathcal{R}(X)$; hence $r = er$ and $s = es$ by the definition of e. This implies that $p \leqslant e \oplus 1$. Therefore we have from the second identity in (5.1) that

$$\begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b \\ -y \end{bmatrix} = \begin{bmatrix} b \\ -y \end{bmatrix}.$$

In particular, $eb = b$, and hence

$$ae^\perp \odot_R b = a \odot_R e^\perp b = 0.$$

This proves that $a \odot_R b = ae \odot_R b$. A similar argument shows now that $ae \odot_R b = ae \odot_R fb$, and hence $a \odot_R b = ae \odot_R fb$.

Denote by $\tilde{\mathcal{R}}(X)$ the space of all rows $[x_1, x_2, \dots]$ with entries in X such that the series $\sum x_i x_i^*$ converges in norm, and by $\tilde{\mathcal{C}}(X)$ the space of columns $(x_1, x_2, \dots)$ such that the series $\sum x_i^* x_i$ converges in norm.

REMARK 5.2. The conclusion of Lemma 5.1 also holds for the usual Haagerup tensor product $X \otimes^h Y$ of norm closed subspaces of $\mathrm{B}(\mathcal{H})$. More precisely, an element $a \odot b$ (with $a \in \tilde{\mathcal{R}}(\mathrm{B}(\mathcal{H}))$ and $b \in \tilde{\mathcal{C}}(\mathrm{B}(\mathcal{H}))$) is in $X \otimes^h Y$ if and only if $a \odot b = ae \odot fb$, and $ae \in \tilde{\mathcal{R}}(X)$, $fb \in \tilde{\mathcal{C}}(Y)$. The reason for this is that here $e, f \in \mathcal{M}(\mathbb{C})$ and the spaces $\tilde{\mathcal{R}}(X)$ and $\tilde{\mathcal{C}}(Y)$ are stable under the multiplication by $\mathcal{M}(\mathbb{C})$ (to see this, consider first the columns and rows with only finitely many non-zero entries).

If $R \subseteq \mathrm{B}(\mathcal{H})$ is an Abelian von Neumann algebra and $X, Y \subseteq \mathrm{B}(\mathcal{H})$ are algebras containing R and contained in R' and are strong as R-modules, then $X \bar{\otimes}_R^h Y$ is a Banach algebra for the product defined as the composition of maps. (Note that $(x \odot_R y)(R') \subseteq R'$ for all $x \in \mathcal{R}(X)$ and $y \in \mathcal{C}(Y)$ since $X, Y \subseteq R'$, so the composition is defined. Proposition 4.1 tells us that $X \bar{\otimes}_R^h Y$ is a strong $\mathbb{C}$-module; hence it is indeed complete as a normed space.) In this Banach algebra a strong analogue of the Tomita commutation theorem holds, which in the case $R = \mathbb{C}$ is proved in [**31**] for the usual Haaagerup tensor product and in [**7**] for the weak* Haagerup tensor product. In general, the result is as follows.

THEOREM 5.3. *Let $R \subseteq \mathrm{B}(\mathcal{H})$ be a commutative von Neumann algebra, $X_0 \subseteq X \subseteq R'$ and $Y_0 \subseteq Y \subseteq R'$ algebras containing R which are strong as R-modules. Denote by X_0^c and Y_0^c the* (*relative*) *commutants of X_0 and Y_0 in*

X and Y, respectively, and let $(X_0 \bar{\otimes}_R^h Y_0)^c$ be the commutant of $X_0 \bar{\otimes}_R^h Y_0$ in $X \bar{\otimes}_R^h Y$. Then

$$(X_0 \bar{\otimes}_R^h Y_0)^c = X_0^c \bar{\otimes}_R^h Y_0^c.$$

Proof. By definition each element $x \odot_R y \in (X_0 \bar{\otimes}_R^h Y_0)^c$ is an X_0, Y_0-bimodule homomorphism, and hence necessarily of the form $z \odot_R w$ for some $z \in \mathcal{R}(X_0')$ and $w \in \mathcal{C}(Y_0')$ by [**24**, Theorem 1.2]. In other words, $x \odot_R y \in X_0' \bar{\otimes}_R^h Y_0'$. Hence by Lemma 5.1, $x \odot_R y = xe \odot_R fy$, where $e \in \mathcal{M}(R)$ is the X_0'-projection of x and $f \in \mathcal{M}(R)$ is the Y_0'-projection of y. Since X and Y are strong over R, we now have

$$xe \in \mathcal{R}(X_0') \cap \mathcal{R}(X) = \mathcal{R}(X_0^c) \quad \text{and} \quad fy \in \mathcal{C}(Y_0') \cap \mathcal{C}(Y_0) = \mathcal{C}(Y_0^c)$$

and it follows that $x \odot_R y \in X_0^c \bar{\otimes}_R^h Y_0^c$. This proves that $(X_0 \bar{\otimes}_R^h Y_0)^c \subseteq X_0^c \bar{\otimes}_R^h Y_0^c$, while the reverse inclusion is obvious.

REMARK 5.4. Lemma 5.1 can also be used to prove the following generalization of Theorem 2.4 from [**1**] (see also the remark in [**7**, p. 137]). Let $\Phi\colon X_1 \to X_2$ and $\Psi\colon Y_1 \to Y_2$ be completely bounded homomorphisms of strong right and left R-modules, respectively, where R is a W*-algebra. Then for arbitrary strong R-submodules $Z \subseteq X_2$ and $V \subseteq Y_2$ we have

$$(\Phi \bar{\otimes}_R^h \Psi)^{-1}(Z \bar{\otimes}_R^h V) = \Phi^{-1}(0) \bar{\otimes}_R^h Y_1 + \Phi^{-1}(Z) \bar{\otimes}_R^h \Psi^{-1}(V) + X_1 \bar{\otimes}_R^h \Psi^{-1}(0),$$

where by definition $(\Phi \bar{\otimes}_R^h \Psi)(x \odot_R y) = \Phi(x) \odot_R \Psi(y)$ for $x \in \mathcal{R}(X)$ and $y \in \mathcal{C}(Y)$. (That $\Phi \bar{\otimes}_R^h \Psi$ is a well-defined completely bounded map follows as in Proposition 3.3.) Since we shall not need this result here, we omit the proof.

Let us now briefly consider slice maps. Given a W*-algebra R and a right or left R-module X, we denote by $X^\natural$ the space of all completely bounded R-module homomorphisms from X to R. Let X be a strong right and Y a strong left R-module. For each $\rho \in Y^\natural$ the *left slice map* is defined by

$$\Phi_\rho\colon X \bar{\otimes}_R^h Y \to X, \quad \Phi_\rho(x \odot_R y) = x\rho(y) \quad (x \in \mathcal{R}(X),\ y \in \mathcal{C}(Y)).$$

Note that Φ_ρ is the composition of the map

$$1 \bar{\otimes}_R^h \rho\colon X \bar{\otimes}_R^h Y \to X \bar{\otimes}_R^h R$$

with the natural completely isometric isomorphism $X \bar{\otimes}_R^h R \to X$ $(x \odot_R r \mapsto xr)$; this shows that Φ_ρ is a well-defined completely bounded map. The *right slice maps* $X \bar{\otimes}_R^h Y \to Y$ are defined similarly.

We say that $X^\natural$ *has enough elements* if and only if

$$\bigcap_{\rho \in X^\natural} \ker \rho = 0.$$

Now we can generalize the essential part of Theorem 4.1 from [**7**].

THEOREM 5.5. *Let X and Y be a strong right and a strong left module* (*respectively*) *over a von Neumann algebra R and let $X_0 \subseteq X$ and $Y_0 \subseteq Y$ be strong R-submodules. If $X^\natural$ and $Y^\natural$ have enough elements, then the following two statements are equivalent for an element $\vartheta = x \odot_R y \in X \bar{\otimes}_R^h Y$:*

(i) $\vartheta \in X_0 \bar{\otimes}_R^h Y_0$;

(ii) $\Phi_\rho(\vartheta) \in X_0$ *for each* $\rho \in Y^\natural$ *and* $\Psi_\omega(\vartheta) \in Y_0$ *for each* $\omega \in X^\natural$, *where* Φ_ρ *and* Ψ_ω *are the left and the right slice maps corresponding to* ρ *and* ω, *respectively.*

Proof. Obviously (i) implies (ii). Assume (ii). Then $x\rho(y) \in X_0$ for each $\rho \in Y^\natural$. Hence $\rho(y) = e\rho(y)$, where $e \in \mathcal{M}(R)$ is the X_0-projection of $x \in \mathcal{R}(X)$. It follows that $\rho(y - ey) = 0$ for all $\rho \in Y^\natural$ by Lemma 2.2, which implies that $y = ey$ since $Y^\natural$ has enough elements. Thus, $\vartheta = x \odot_R ey = z \odot_R y$, where $z := xe \in \mathcal{R}(X_0)$. A similar argument shows now that $\vartheta = z \odot_R y = z \odot_R w$ for some $w \in \mathcal{C}(Y_0)$; hence $\vartheta \in X_0 \bar{\otimes}_R^h Y_0$.

REMARK 5.6. There are many situations where, for a right R-module X, $X^\natural$ has enough elements. For example, let $X \subseteq S$, where S is a von Neumann algebra containing R such that there exists a faithful conditional expectation $\varepsilon\colon S \to R$. In this case for each $s \in S$ the map $\rho_s\colon X \to R$ defined by $\rho_s(x) = \varepsilon(sx)$ is in $X^\natural$, and $\rho_{x^*}(x) \neq 0$ for each non-zero $x \in X$.

If R is commutative and $X \subseteq R'$, then by Theorem 3 from [**20**], X has enough elements.

On the other hand, if $X = \mathrm{B}(\mathcal{H})$ and $R \subseteq \mathrm{B}(\mathcal{H})$ is a continuous von Neumann algebra, then it is not hard to prove that

$$\bigcap_{\rho \in X^\natural} \ker \rho \supseteq \mathrm{K}(\mathcal{H}),$$

so $X^\natural$ does not have enough elements in this case. (We shall not need this result later, so we omit the details.)

6. *Some canonical isomorphisms*

Throughout the rest of this paper C will be the centre of a von Neumann algebra R. In [**9**] the product $R \otimes_C^h R$ was studied by Chatterjee and Smith; here we shall study the product $R \bar{\otimes}_C^h R$. In particular we shall show that $R \bar{\otimes}_C^h R$ is a dual Banach space, but first we need some preliminary results of independent interest. Recall that a von Neumann algebra $R \subseteq \mathrm{B}(\mathcal{H})$ is called injective if there is a projection of norm 1 from $\mathrm{B}(\mathcal{H})$ to R (see [**14**] for more). For operator spaces $X \subseteq \mathrm{B}(\mathcal{H})$ and $Y \subseteq \mathrm{B}(\mathcal{K})$ we denote by $X \check{\otimes} Y$ and $X \bar{\otimes} Y$ the norm closure and the weak* closure, respectively, of $X \otimes Y$ in $\mathrm{B}(\mathcal{H} \otimes \mathcal{K})$.

THEOREM 6.1. *If R is an injective von Neumann algebra and X, Y are arbitrary operator spaces* (*not necessarily R-modules*), *then the correspondence*

$$(x \otimes r) \otimes_R (s \otimes y) \mapsto rs \otimes (x \otimes y) \quad (r, s \in R,\ \ x \in X,\ \ y \in Y)$$

induces a completely isometric isomorphism

$$\Phi\colon\ (X \check{\otimes} R) \otimes^h_R (R \check{\otimes} Y) \to R \check{\otimes} (X \otimes^h Y).$$

Proof. It suffices to prove that Φ is isometric for *all* X and Y. Indeed, then for each positive integer n we shall have (using the analogy of Lemma 3.5(i) for the product $\otimes^h_R$ instead of $\bar{\otimes}^h_R$, which can be proved in the same way or deduced from Lemma 3.5(i)) the following chain of natural isometries (denoted as equalities):

$$\begin{aligned}\mathcal{M}_n((X \check{\otimes} R) \otimes^h_R (R \check{\otimes} Y)) &= \mathcal{C}_n(X \check{\otimes} R) \otimes^h_R \mathcal{R}_n(R \check{\otimes} Y)\\ &= (\mathcal{C}_n(X) \check{\otimes} R) \otimes^h_R (R \check{\otimes} \mathcal{R}_n(Y))\\ &= R \check{\otimes} (\mathcal{C}_n(X) \otimes^h \mathcal{R}_n(Y))\\ &= R \check{\otimes} \mathcal{M}_n(X \otimes^h Y)\\ &= \mathcal{M}_n(R \check{\otimes} (X \otimes^h Y)).\end{aligned}$$

This shows that Φ will be a complete isometry (once it has been checked that the map between the beginning and the end space in the above calculation agrees with the map induced by Φ). Since the range of Φ is obviously dense in $R \check{\otimes} (X \otimes^h Y)$, Φ will be surjective. So, it remains to prove that Φ is isometric. We may assume that X, Y and R are all contained in $B := \mathrm{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Identify B with $\mathcal{M}(\mathbb{C})$ (recall that $\mathcal{M}(\mathbb{C})$ means $\mathcal{M}_{\mathcal{J}}(\mathbb{C})$, where $\mathcal{J} = \dim \mathcal{H}$). Then we can regard $X \check{\otimes} R$ as a subspace of $X \bar{\otimes} B = \mathcal{M}(X)$, $R \check{\otimes} Y$ as a subspace of $B \bar{\otimes} Y = \mathcal{M}(Y)$, and $R \check{\otimes} (X \otimes^h Y)$ as a subspace of $B \bar{\otimes} (X \bar{\otimes}^h Y) = \mathcal{M}(X \bar{\otimes}^h Y)$. Let

$$\Psi\colon\ \mathcal{M}(X) \bar{\otimes}^h_{\mathcal{M}(\mathbb{C})} \mathcal{M}(Y) \to \mathcal{M}(X \bar{\otimes}^h Y)$$

be the composition of the following complete isometries (denoted as =):

$$\begin{aligned}\mathcal{M}(X) \bar{\otimes}^h_{\mathcal{M}(\mathbb{C})} \mathcal{M}(Y) &= \mathcal{R}(\mathcal{C}(X)) \bar{\otimes}^h_{\mathcal{M}(\mathbb{C})} \mathcal{C}(\mathcal{R}(Y))\\ &= \mathcal{C}(X) \bar{\otimes}^h \mathcal{R}(Y)\\ &= \mathcal{M}(X \bar{\otimes}^h Y),\end{aligned}$$

where we have used Lemma 3.4 (together with Theorem 3.6) and Lemma 3.5(i). It is easy to verify that

$$\Psi([x_{ij}] \otimes_{\mathcal{M}(\mathbb{C})} [y_{kl}]) = \left[\sum_{k \in \mathcal{J}} x_{ik} \otimes y_{kj}\right] \quad (x_{ij} \in X,\ \ y_{kl} \in Y),$$

and that in the tensor notation

$$\Psi\colon\ (X \bar{\otimes} B) \bar{\otimes}^h_B (B \bar{\otimes} Y) \to B \bar{\otimes} (X \bar{\otimes}^h Y)$$

we have

$$\Psi((x \otimes r) \otimes_B (s \otimes y)) = rs \otimes (x \otimes y) \quad (x \in X,\ \ y \in Y,\ \ r, s \in B).$$

Consider the diagram

$$\begin{array}{ccc} (X \check{\otimes} R) \otimes_R^h (R \check{\otimes} Y) & \xrightarrow{\Phi} & R \check{\otimes} (X \otimes^h Y) \\ \downarrow \sigma & & \downarrow \iota \\ (X \bar{\otimes} B) \bar{\otimes}_B^h (B \bar{\otimes} Y) & \xrightarrow{\Psi} & B \bar{\otimes} (X \bar{\otimes}^h Y) \end{array}$$

where ι is the inclusion and σ is the natural contraction given by

$$\sigma((x \otimes r) \otimes_R (s \otimes y)) = (x \otimes r) \otimes_B (s \otimes y).$$

It is easy to verify that the diagram commutes. Since Ψ and ι are isometries, it suffices now to prove that σ is isometric, for then Φ will be isometric, as required. To do this, we embed everything into $\mathrm{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$; thus, denoting by 1 the identity operator on $\mathcal{H}$, we identify $X \bar{\otimes} B$ with $X \bar{\otimes} B \bar{\otimes} 1$, $B \bar{\otimes} Y$ with $1 \bar{\otimes} B \bar{\otimes} Y$ and B with $1 \bar{\otimes} B \bar{\otimes} 1$. (These identifications respect the module actions of B and R.) Now consider the diagram

$$\begin{array}{ccc} (X \bar{\otimes} R \bar{\otimes} 1) \otimes_R^h (1 \bar{\otimes} R \bar{\otimes} Y) & \xrightarrow{\mu_R} & \mathrm{CB}(B \bar{\otimes} R' \bar{\otimes} B; \mathrm{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})) \\ \downarrow \tilde{\sigma} & & \downarrow \tau \\ (X \bar{\otimes} B \bar{\otimes} 1) \bar{\otimes}_B^h (1 \bar{\otimes} B \bar{\otimes} Y) & \xrightarrow{\mu_B} & \mathrm{CB}(B \bar{\otimes} 1 \bar{\otimes} B; \mathrm{B}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})) \end{array}$$

where μ_R is the restriction of the natural complete isometry from the tensor product $(B \bar{\otimes} B \bar{\otimes} B) \otimes_R^h (B \bar{\otimes} B \bar{\otimes} B)$ into the completely bounded operators (here we have used the fact that $(1 \bar{\otimes} R \bar{\otimes} 1)' = B \bar{\otimes} R' \bar{\otimes} B$) and μ_B is defined in the same way as μ_R, $\tilde{\sigma}$ is the extension of σ defined by the same rule as σ, and τ is induced by restricting the domain (that is, $\tau(\alpha) = \alpha|(B \bar{\otimes} 1 \bar{\otimes} B)$). We would like to prove that $\tilde{\sigma}$ is isometric (for then the restriction σ of $\tilde{\sigma}$ will also be isometric) and for this it suffices to show that the restriction of τ to the range of μ_R is isometric. Note that the von Neumann algebra $M := B \bar{\otimes} 1 \bar{\otimes} B$ is contained in the injective von Neumann algebra $N := B \bar{\otimes} R \bar{\otimes} B$, and the von Neumann algebra generated by $M \cup N'$ is $\overline{MN'} = B \bar{\otimes} R' \bar{\otimes} B$. Now it follows directly from Lemma 2.6 of [**24**] that the restriction of τ to a dense subspace of the range of μ_R is isometric (namely, the restriction to a certain space of elementary operators). Hence the restriction of τ to the range of μ_R is also isometric.

It can be proved that the assumption of injectivity in Theorem 6.1 is not redundant, but we shall not need this fact here, so we omit the proof.

Throughout the rest of this section Λ will be a measurable space (thus Λ is a set together with a σ-algebra of subsets) equipped with a finite positive measure μ for which Λ is complete (a subset of a set with measure 0 is measurable) and A will be the commutative von Neumann algebra $\mathrm{L}^\infty(\mu)$ acting on $\mathrm{L}^2(\mu)$ in the usual way. Given a separable Banach space X, we shall denote by $\mathrm{L}^\infty(\mu, X^\sharp)$ the space of (equivalence classes) of essentially bounded weak* measurable mappings from Λ to the dual $X^\sharp$ of X, equipped with the norm

$$\|f\| = \operatorname{essup}_{\lambda\in\Lambda}\|f(\lambda)\|.$$

(A mapping $f\colon \Lambda \to X^\sharp$ is weak* measurable if and only if for each $x \in X$ the numerical function $\lambda \mapsto f(\lambda)(x)$ is measurable.) If X is in addition an operator space and $X^\sharp$ is the operator dual of X, then $\mathrm{L}^\infty(\mu, X^\sharp)$ has the structure of a dual operator space by the identifications

$$\mathcal{M}_n(\mathrm{L}^\infty(\mu, X^\sharp)) = \mathrm{L}^\infty(\mu; \mathcal{M}_n(X^\sharp)) \quad (n \text{ a positive integer}),$$

where $\mathcal{M}_n(X^\sharp) = \mathrm{CB}(X; \mathcal{M}_n(\mathbb{C}))$.

Given a separable Hilbert space $\mathcal{K}$, the Hilbert space $\mathrm{L}^2(\mu, \mathcal{K}) = \mathcal{K} \otimes \mathrm{L}^2(\mu)$ consists of all (weakly) measurable mappings $\xi\colon \Lambda \to \mathcal{K}$ such that $\|\xi\|^2 = \int \|\xi(\lambda)\|^2\, d\mu(\lambda) < \infty$ (see [**21**, p. 958]). As $A = \mathrm{L}^\infty(\mu)$ acts on $\mathrm{L}^2(\mu)$, the von Neumann algebra $\mathrm{B}(\mathcal{K}) \bar{\otimes} A$ acts on $\mathrm{L}^2(\mu, \mathcal{K})$. Choosing an orthonormal basis of $\mathcal{K}$ with cardinality $\mathcal{J} \leq \infty$, we see that $\mathrm{B}(\mathcal{K})$ is identified with $\mathcal{M}_{\mathcal{J}}(\mathbb{C})$ and $\mathrm{B}(\mathcal{K}) \bar{\otimes} A$ is identified with the space of all mappings $a\colon \Lambda \to \mathcal{M}_{\mathcal{J}}(\mathbb{C})$ such that the entries $a_{ij}(\ )$ are measurable functions on Λ and $\|a\| = \operatorname{essup}\|a(\lambda)\|_{\lambda\in\Lambda} < \infty$. It is easy to see (and well known) that $\mathrm{B}(\mathcal{K}) \bar{\otimes} A = \mathrm{L}^\infty(\mu, \mathrm{B}(\mathcal{K}))$.

REMARK 6.2. If $\mathcal{K}$ is separable, then $X \bar{\otimes} A = \mathrm{L}^\infty(\mu, X)$ for each weak* closed subspace X of $\mathrm{B}(\mathcal{K})$. Indeed, it is well known that each $f \in \mathrm{L}^\infty(\mu, X)$ is a pointwise limit almost everywhere of a bounded sequence of step maps f_n (see [**22**, p. 257]), and an application of the Lebesgue dominated convergence theorem shows that $\rho(f_n) \to \rho(f)$ for each vector functional ρ on $\mathrm{L}^\infty(\mu, \mathrm{B}(\mathcal{K}))$, which proves the inclusion $\mathrm{L}^\infty(\mu, X) \subseteq X \bar{\otimes} A$. For the reverse inclusion, it suffices to show that $\mathrm{L}^\infty(\mu, X)$ is weak* closed in $\mathrm{L}^\infty(\mu, \mathrm{B}(\mathcal{K}))$. So, let (f_j) be a net in $\mathrm{L}^\infty(\mu, X)$ converging to an element $f \in \mathrm{L}^\infty(\mu, \mathrm{B}(\mathcal{K}))$ in the weak* topology. Then, in particular,

$$\int f_j(\lambda)(t) g(\lambda)\, d\mu \to \int f(\lambda)(t) g(\lambda)\, d\mu$$

for each $t \in \mathrm{T}(\mathcal{K}) = \mathrm{B}(\mathcal{K})_\sharp$ and each $g \in \mathrm{L}^1(\mu)$ (since $\mathrm{T}(\mathcal{K}) \otimes \mathrm{L}^1(\mu)$ is contained in the predual of $\mathrm{L}^\infty(\mu, \mathrm{B}(\mathcal{K}))$). If $t \in X_\perp$ (the preannihilator of X in $\mathrm{T}(\mathcal{K})$), then $f_j(\cdot)(t) = 0$. Hence $\int f_j(\lambda)(t) g(\lambda)\, d\mu = 0$ for each j and, consequently, $\int f(\lambda)(t) g(\lambda)\, d\mu = 0$ for each $g \in \mathrm{L}^1(\mu)$. By choosing a dense sequence (t_n) in $X_\perp$, we see that $f(\lambda)(t_n) = 0$ almost everywhere, and hence $f(\lambda) \in X$ almost everywhere.

The author was not able to prove an appropriate variant of Theorem 6.1 for the extended Haagerup tensor product in full generality, so we shall present here only a special case needed later. Given weak* closed subspaces $X \subseteq \mathrm{B}(\mathcal{H})$ and $Y \subseteq \mathrm{B}(\mathcal{K})$, the weak* Haagerup tensor product $X \bar{\otimes}^h Y$ can be represented as a weak* closed subspace of $\mathrm{B}(\mathcal{L})$ for some Hilbert space $\mathcal{L}$ (see [**7**]); moreover, if $\mathcal{H}$ and $\mathcal{K}$ are separable, then $X \otimes^h Y$ has a separable predual. Hence (by an easy modification of the proof of Proposition 2.1 from [**3**]) $\mathcal{L}$ can be chosen to be separable.

LEMMA 6.3. *Let $\mathcal{H}_0$, $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces, A a maximal Abelian von Neumann algebra on $\mathcal{H}_0$, and $X \subseteq \mathrm{B}(\mathcal{H})$, $Y \subseteq \mathrm{B}(\mathcal{K})$ weak* closed subspaces. Then*

$$(X \bar{\otimes} A) \bar{\otimes}^h_A (A \bar{\otimes} Y) = A \bar{\otimes} (X \bar{\otimes}^h Y)$$

completely isometrically. (*Here, for the right-hand side of this identity to be defined, we assume that $X \bar{\otimes}^h Y$ is represented completely isometrically and weak* homeomorphically as a weak* closed subspace of* $\mathrm{B}(\mathcal{L})$ *for some separable Hilbert space* $\mathcal{L}$.)

Proof. Since A is maximal Abelian and $\mathcal{H}_0$ separable, A is (unitarily equivalent to) $\mathrm{L}^\infty(\mu)$ for some finite positive Borel measure μ on a (compact) Polish space Λ and $\mathcal{H}_0 = \mathrm{L}^2(\mu)$. Recalling Remark 6.2, we realise that it suffices to prove that the map

$$\Phi\colon\ \mathrm{L}^\infty(\mu, X) \bar{\otimes}^h_A \mathrm{L}^\infty(\mu, Y) \to \mathrm{L}^\infty(\mu, X \bar{\otimes}^h Y),$$

$$\Phi(f \odot_A g)(\lambda) = f(\lambda) \odot g(\lambda) \quad (f \in \mathrm{L}^\infty(\mu, \mathcal{R}_\infty(X)),\ g \in \mathrm{L}^\infty(\mu, \mathcal{C}_\infty(Y)))$$

is an isometric isomorphism, for then it follows easily that Φ is in fact completely isometric (see the beginning of the proof of Theorem 6.1). Obviously Φ is a contraction. To prove that Φ is surjective and isometric, choose any $\epsilon > 0$ and let $h \in \mathrm{L}^\infty(\mu, X \bar{\otimes}^h Y)$ be a Borel map with $\|h\| = 1$. For $\lambda \in \Lambda$ we can choose $x_\lambda \in \mathcal{R}_\infty(X)$ and $y_\lambda \in \mathcal{C}_\infty(Y)$ such that

$$h(\lambda) = x_\lambda \odot y_\lambda$$

and

$$\|x_\lambda\| = \|y_\lambda\| < 1 + \epsilon.$$

Consider the set

$$\Gamma = \{(\lambda, x, y) \in \Lambda \times \mathcal{R}_\infty(X) \times \mathcal{C}_\infty(Y)\colon\ x \odot y = h(\lambda),\ \|x\| = \|y\| < 1 + \epsilon\}.$$

The closed balls B_1 and B_2 of radius $1 + \epsilon$ and centre 0 in $\mathcal{R}_\infty(X)$ and $\mathcal{C}_\infty(Y)$ are Polish spaces for the strong operator topology and Γ is easily seen to be an analytic (in fact Borel) subset of $\Lambda \times B_1 \times B_2$. Hence by the principle of measurable selection (see [**21**, p. 1041]) there exists a measurable mapping $(f, g)\colon\ \Lambda \to B_1 \times B_2$ such that $(\lambda,\ f(\lambda), g(\lambda)) \in \Gamma$ for each $\lambda \in \Lambda$. Now we have $f \in \mathcal{R}_\infty(\mathrm{L}^\infty(\mu, X))$, $g \in \mathcal{C}_\infty(\mathrm{L}^\infty(\mu, Y))$, $\|f \odot_A g\| \leqslant (1 + \epsilon)^2$ and $\Phi(f \odot_A g) = h$. This proves that Φ is surjective and (by letting $\epsilon \to 0$) isometric.

For a separable Banach space X a function $f\colon\ \Lambda \to X$ is called (strongly) measurable if and only if $f^{-1}(U)$ is measurable for each open ball U in X. We denote by $\mathrm{L}^\infty(\mu, X)$ the space of all essentially bounded measurable mappings from Λ to X. If X is a dual of a (necessarily separable) Banach space $X_\sharp$, then this notation agrees with the previous one since a bounded mapping $f\colon\ \Lambda \to X$ is measurable if and only if it is weak* measurable. (To see this, first note that the closed unit ball B_X of X is a countable intersection of polars of finite subsets of $X_\sharp$; hence $f^{-1}(B_X)$ is measurable if f is weak* measurable.)

Recall that $(X \check{\otimes} A)^\natural$ is the space of all (completely) bounded A-module homomorphisms from $X \check{\otimes} A$ to A equipped with the completely bounded norm.

LEMMA 6.4. *For every separable operator space X we have*

$$(X \check{\otimes} A)^\natural = \mathrm{L}^\infty(\mu, X^\sharp) = \mathrm{L}^\infty(\mu, X)^\natural$$

completely isometrically.

Proof. We regard $X \check{\otimes} A$ as a subspace of $\mathrm{L}^\infty(\mu, X)$ in the usual way. Define a map

$$\Phi\colon\ \mathrm{L}^\infty(\mu, X^\sharp) \to (X \check{\otimes} A)^\natural$$

by

$$(\Phi(\rho)(f))(\lambda) = \rho(\lambda)(f(\lambda)) \quad (f \in X \check{\otimes} A).$$

By approximating f with finite sums f_n of elementary tensors, we see that the function $\lambda \to \rho(\lambda)(f(\lambda))$ is measurable. Moreover, since each f_n can be expressed as a finite sum $\sum_i x_{in} \otimes g_{in}$, where $x_{in} \in X$, $\|g_{in}\| = 1$ and the functions g_{in} have disjoint supports (for a fixed n), it follows that

$$|\rho(\lambda)(f_n(\lambda))| = \left| \sum_i g_{in}(\lambda)\rho(\lambda)(x_{in}) \right| = \max_i |\rho(\lambda)(x_{in})| \leqslant \|\rho\| \|f_n\|.$$

Since $\|f_n - f\| \to 0$, we have $f_n(\lambda) \to f(\lambda)$ almost everywhere. Hence $\rho(\lambda)(f_n(\lambda)) \to \rho(\lambda)(f(\lambda))$ almost everywhere and it follows that $|\rho(\lambda)(f(\lambda))| \leqslant \|\rho\| \|f\|$ almost everywhere for each $f \in X \check{\otimes} A$. This proves that Φ is a contraction and a similar argument shows that Φ is in fact a complete contraction.

To prove that Φ is surjective and completely isometric, we define a map

$$\Psi\colon\ (X \check{\otimes} A)^\natural \to \mathrm{L}^\infty(\mu, X^\sharp)$$

by

$$(\Psi(\omega)(\lambda))(x) = (\omega(x \otimes 1))(\lambda),$$

where 1 denotes the constant function with value 1 on Λ. It is routine to verify that Ψ is a complete contraction and that $\Psi\Phi$ is the identity. The proof that $\Phi\Psi$ is the identity reduces to the equality

$$(\omega(f(\lambda) \otimes 1))(\lambda) = \omega(f)(\lambda), \tag{6.1}$$

which holds, for almost all $\lambda \in \Lambda$, for each fixed $f \in X \check{\otimes} A$ and $\omega \in (X \check{\otimes} A)^\natural$. To prove (6.1), assume first that $f = x \otimes g$ for some $x \in X$ and $g \in A$. Then, since ω is a homomorphism of A-modules, we have

$$\omega(f(\lambda) \otimes 1 - f) = \omega(g(\lambda)x \otimes 1 - x \otimes g) = \omega(x \otimes 1)(g(\lambda) - g)$$

for each λ; hence we indeed have $(\omega(f(\lambda) \otimes 1 - f))(\lambda) = 0$. For a general $f \in X \check{\otimes} A$ the identity (6.1) then follows by approximating f with finite sums of elementary tensors. This proves the first equality of the lemma.

The proof of the second equality is similar, but with one additional argument. As above we can define the two maps

$$\Phi\colon\ \mathrm{L}^\infty(\mu, X^\sharp) \to \mathrm{L}^\infty(\mu, X)^\natural \quad \text{and} \quad \Psi\colon\ \mathrm{L}^\infty(\mu, X)^\natural \to \mathrm{L}^\infty(\mu, X^\sharp).$$

The only difference is that the measurability of the function $\lambda \mapsto \rho(\lambda)(f(\lambda))$ ($f \in \mathrm{L}^\infty(\mu, X)$, $\rho \in \mathrm{L}^\infty(\mu, X^\sharp)$) is now less obvious. Here we have to use the fact that there exists a sequence of maps f_n of the form $f_n(\lambda) = \sum_i g_{in}(\lambda) x_{in}$ (finite sum, $x_{in} \in X$, $g_{in} \in A$) converging to f almost everywhere and the convergence is uniform outside subsets of arbitrarily small measure (see [**22**, pp. 256, 257]). The rest of the proof is the same as above.

We shall also need a bilinear variant of Lemma 6.4 which follows, however, immediately from Lemma 6.4.

COROLLARY 6.5. *For separable operator spaces X and Y we have*

$$(\mathrm{L}^\infty(\mu, X) \otimes_A^h \mathrm{L}^\infty(\mu, Y))^\natural = ((X \check{\otimes} A) \otimes_A^h (A \check{\otimes} Y))^\natural$$

completely isometrically.

Proof. By the extension theorem for completely bounded module mappings, the map

$$\Theta\colon\ (\mathrm{L}^\infty(\mu, X) \otimes_A^h \mathrm{L}^\infty(\mu, Y))^\natural \to ((X \check{\otimes} A) \otimes_A^h (A \check{\otimes} Y))^\natural,$$

$$\Theta(\vartheta) = \vartheta | (X \check{\otimes} A) \otimes_A^h (A \check{\otimes} Y)$$

is a completely quotient map; hence it suffices to show that Θ is one-to-one, for then Θ is a completely isometric isomorphism.

Suppose that $\vartheta \in \ker \Theta$. Then for each $g \in A \check{\otimes} Y$, the map

$$\vartheta_g \in \mathrm{L}^\infty(\mu, X)^\natural, \quad \vartheta_g(f) := \vartheta(f \otimes_A g) \quad (f \in \mathrm{L}^\infty(\mu, X))$$

annihilates $X \check{\otimes} A$; hence $\vartheta_g = 0$ by Lemma 6.4. Thus, for each $f \in \mathrm{L}^\infty(\mu, X)$, the map

$$\vartheta^f \in \mathrm{L}^\infty(\mu, Y)^\natural, \quad \vartheta^f(g) := \vartheta(f \otimes_A g) \quad (g \in \mathrm{L}^\infty(\mu, Y))$$

annihilates $A \check{\otimes} Y$. Hence by Lemma 6.4, $\vartheta^f = 0$, which means that $\vartheta = 0$.

Given a von Neumann algebra R and an R-module $X \subseteq \mathrm{B}(\mathcal{H})$ (left or right) we denote by $X_\natural$ the space of all (completely bounded) normal (that is, weak* continuous) R-module homomorphisms from X to R equipped with the completely bounded norm. In particular, $\mathrm{L}^\infty(\mu, \mathrm{B}(\mathcal{K}))_\natural$ is the space of all normal A-module homomorphisms from $\mathrm{L}^\infty(\mu, \mathrm{B}(\mathcal{K}))$ to $A = \mathrm{L}^\infty(\mu)$.

We shall need a generalization of the well-known fact that each bounded linear functional on the space $\mathrm{K}(\mathcal{H})$ of compact operators is normal.

LEMMA 6.6. *Suppose that $R \subseteq \mathrm{B}(\mathcal{H})$ is an algebra, $\mathcal{K}$ and $\mathcal{L}$ are Hilbert spaces and $\nu\colon R \to \mathrm{B}(\mathcal{L})$ is a weak* continuous representation* (*in particular,* $\mathrm{B}(\mathcal{L})$ *can be regarded as a right R-module*). *Then each bounded R-module homomorphism ρ from $\mathrm{K}(\mathcal{K}) \check{\otimes} R$ to $\mathrm{B}(\mathcal{L})$ is normal.*

Proof. Choose an orthonormal basis $\mathcal{J}$ of $\mathcal{K}$ and for each finite subset $\mathcal{F} \subseteq \mathcal{J}$ denote by $e_{\mathcal{F}}$ the projection onto the linear span of $\mathcal{F}$. Then $\mathrm{K}(\mathcal{K}) \check{\otimes} R$ is identified with the set of all matrices $a \in \mathcal{M}_{\mathcal{J}}(R)$ satisfying

$$\lim_{\mathcal{F} \to \mathcal{J}} \|a - e_{\mathcal{F}} a e_{\mathcal{F}}\| = 0.$$

It suffices to prove that for each normal linear functional ω on $\mathrm{B}(\mathcal{L})$ with $\|\omega\| = 1$, the composite $\sigma := \omega\rho$ is normal. Since ρ is a homomorphism of R-modules, the functional $e_{\mathcal{F}}\sigma e_{\mathcal{F}}$ is normal for each finite $\mathcal{F} \subseteq \mathcal{J}$. Indeed, denoting by $\{e_{ij}\}_{i,j\in\mathcal{J}}$ the matrix unit in $\mathrm{K}(\mathcal{K})$ corresponding to the orthonormal basis $\mathcal{J}$, we have for each $x = [r_{ij}] \in \mathrm{K}(\mathcal{K}) \check{\otimes} R$ that

$$(e_{\mathcal{F}}\sigma e_{\mathcal{F}})(x) := \sigma(e_{\mathcal{F}} x e_{\mathcal{F}}) = \omega\Bigg(\sum_{i,j\in\mathcal{F}} \rho(e_{ij} \otimes 1)\nu(r_{ij})\Bigg)$$

and the normality of $e_{\mathcal{F}}\sigma e_{\mathcal{F}}$ now follows from the normality of ω and ν. Since the set of normal functionals on $\mathrm{K}(\mathcal{K}) \check{\otimes} R$ is norm closed, it suffices now to show that

$$\lim_{\mathcal{F}\to\mathcal{J}} \|\sigma - e_{\mathcal{F}}\sigma e_{\mathcal{F}}\| = 0.$$

Given $\epsilon > 0$, choose $b \in \mathrm{K}(\mathcal{K}) \check{\otimes} R$ such that $\|b\| = 1$, $\sigma(b) > 1 - \frac{1}{2}\epsilon^2$ and $b = e_{\mathcal{F}} b e_{\mathcal{F}}$ for some finite set $\mathcal{F} \subseteq \mathcal{J}$. Then essentially the same computation as in [**21**, p. 749] shows that $|\sigma(a e_{\mathcal{F}}{}^{\perp})| < \epsilon\|a\|$ for all $a \in \mathrm{K}(\mathcal{K}) \check{\otimes} R$. Hence $\|e_{\mathcal{F}}{}^{\perp}\sigma\| \leqslant \epsilon$ for all sufficiently large finite subsets $\mathcal{F} \subseteq \mathcal{J}$. Similarly, $\|\sigma e_{\mathcal{F}}{}^{\perp}\| < \epsilon$; hence $\|\sigma - e_{\mathcal{F}}\sigma e_{\mathcal{F}}\| \to 0$.

COROLLARY 6.7. *If $\mathcal{K}$ is a separable Hilbert space, then* $\mathrm{L}^\infty(\mu, \mathrm{B}(\mathcal{K}))_\natural = \mathrm{L}^\infty(\mu, \mathrm{T}(\mathcal{K}))$ *completely isometrically.*

Proof. By Lemma 6.6 each (completely) bounded A-module homomorphism from $\mathrm{K}(\mathcal{K}) \check{\otimes} A$ to $\mathcal{M}_n(A)$ is normal (for arbitrary n). Therefore it can be extended without increasing the completely bounded norm to a normal A-module homomorphism from the weak* closure $\mathrm{B}(\mathcal{K}) \bar{\otimes} A$ of $\mathrm{K}(\mathcal{K}) \check{\otimes} A$ to $\mathcal{M}_n(A)$ (this is an easy consequence of the Kaplansky density theorem and the weak* compactness of unit balls). This shows that $(\mathrm{B}(\mathcal{K}) \bar{\otimes} A)_\natural = (\mathrm{K}(\mathcal{K}) \check{\otimes} A)^\natural$. If $\mathcal{K}$ is separable, Lemma 6.4 now implies that $\mathrm{L}^\infty(\mu, \mathrm{B}(\mathcal{K}))_\natural = \mathrm{L}^\infty(\mu, \mathrm{T}(\mathcal{K}))$ since $\mathrm{T}(\mathcal{K}) = \mathrm{K}(\mathcal{K})^\sharp$.

7. *The extended central Haagerup tensor product*

Recall that for an operator module $X \subseteq \mathrm{B}(\mathcal{H})$ over an (Abelian) von Neumann algebra C we denote by $X^\natural$ the space of all completely bounded C-module homomorphisms from X to C, and by $X_\natural$ the subspace of all normal C-module homomorphisms.

It was proved in [**7**] that $\mathrm{B}(\mathcal{H}) \bar{\otimes}^h \mathrm{B}(\mathcal{H}) = (\mathrm{T}(\mathcal{H}) \otimes^h \mathrm{T}(\mathcal{H}))^\sharp$. This result can be extended to general von Neumann algebras as follows.

THEOREM 7.1. *For each von Neumann algebra $R \subseteq \mathrm{B}(\mathcal{H})$ with centre C there is a natural completely isometric isomorphism*

$$\Phi\colon\ R \bar{\otimes}^h_C R \to (R_\natural \otimes^h_C R_\natural)^\natural,$$

defined by

$$\Phi(x \odot_C y)(\omega \odot_C \rho) = \omega(x)\rho(y) \quad (x \in \mathcal{R}(R),\ y \in \mathcal{C}(R),\ \omega, \rho \in R_\natural).$$

We shall first prove Theorem 7.1 in the case where $\mathcal{H}$ is separable and then the proof in general will be reduced to this case. In this reduction we shall need the following simple and perhaps well-known fact, which we state as a lemma for easier referencing.

LEMMA 7.2. *If* $\omega\colon R \to \mathrm{B}(\mathcal{K})$ *is a normal completely bounded map and* (p_m) *is a net of projections in* R *increasing to the identity* 1, *then* $\lim_m \|\omega - p_m\omega p_m\|_{\mathrm{cb}} = 0$.

Proof. Note that ω is a linear combination of normal completely positive maps, and for a completely positive ω with $\|\omega\| = 1$ we have by the Schwarz inequality [**26**, p. 39] that $\omega(rp_m^{\perp})^*\omega(rp_m^{\perp}) \leqslant \omega(p_m^{\perp}r^*rp_m^{\perp}) \leqslant \|r\|^2\omega(p_m^{\perp})$ for each $r \in \mathcal{M}_n(R)$ and each positive integer n; hence $\|p_m^{\perp}\omega\|_{\mathrm{cb}} \leqslant \|\omega(p_m^{\perp})\|^{1/2}$. Since ω is normal, we have $\omega(p_m^{\perp}) \to 0$; hence $\|p_m^{\perp}\omega\|_{\mathrm{cb}} \to 0$. Similarly $\|\omega p_m^{\perp}\|_{\mathrm{cb}} \to 0$.

Proof of Theorem 7.1. Suppose first that $R = C'$. Then R is isomorphic to a direct sum of algebras of the form $\mathrm{B}(\mathcal{K}) \bar{\otimes} A$, where A is a maximal commutative von Neumann algebra acting on some Hilbert space $\mathcal{L}$ with a cyclic vector. We may restrict our attention to one such summand. Then $\mathcal{L} = \mathrm{L}^2(\mu)$ and $A = \mathrm{L}^{\infty}(\mu)$ for some positive finite measure μ on a measurable space Λ, and $R = \mathrm{L}^{\infty}(\mu, \mathrm{B}(\mathcal{K}))$. If $\mathcal{K}$ is separable, then we have

$$
\begin{aligned}
(R_\natural \otimes_C^h R_\natural)^\natural &= (\mathrm{L}^{\infty}(\mu, \mathrm{T}(\mathcal{K})) \otimes_A^h \mathrm{L}^{\infty}(\mu, \mathrm{T}(\mathcal{K})))^\natural && \text{(Corollary 6.7)}\\
&= ((\mathrm{T}(\mathcal{K}) \check{\otimes} A) \otimes_A^h (A \check{\otimes} \mathrm{T}(\mathcal{K})))^\natural && \text{(Corollary 6.5)}\\
&= (A \check{\otimes} (\mathrm{T}(\mathcal{K}) \otimes^h \mathrm{T}(\mathcal{K})))^\natural && \text{(Theorem 6.1)}\\
&= \mathrm{L}^{\infty}(\mu, (\mathrm{T}(\mathcal{K}) \otimes^h \mathrm{T}(\mathcal{K}))^\sharp) && \text{(Lemma 6.4)}\\
&= \mathrm{L}^{\infty}(\mu, \mathrm{B}(\mathcal{K}) \bar{\otimes}^h \mathrm{B}(\mathcal{K})) && \text{([7])}\\
&= A \bar{\otimes} (\mathrm{B}(\mathcal{K}) \bar{\otimes}^h \mathrm{B}(\mathcal{K})) && \text{(Remark 6.2)}\\
&= (\mathrm{B}(\mathcal{K}) \bar{\otimes} A) \bar{\otimes}_A^h (A \bar{\otimes} \mathrm{B}(\mathcal{K})) && \text{(Lemma 6.3)}\\
&= R \bar{\otimes}_C^h R.
\end{aligned}
$$

It is easy to check that the map from $R \bar{\otimes}_C^h R$ to $(R_\natural \otimes_C^h R_\natural)^\natural$ which results from the above identifications is the same natural map Φ as in the statement of the theorem.

Now we remove the restriction that $\mathcal{K}$ is separable. (If the reader is interested only in separably acting von Neumann algebras, he or she may now skip one or two pages and read the last two paragraphs of the proof only.) Choose a net of projections $\{q_m\colon m \in \mathcal{M}\}$ in $\mathrm{B}(\mathcal{K})$ with separable ranges $\mathcal{K}_m := q_m\mathcal{K}$, converging to 1 and set $p_m = q_m \otimes 1 \in \mathrm{B}(\mathcal{K}) \bar{\otimes} A = R$. Let

$R_m = \mathrm{B}(\mathcal{K}_m) \bar{\otimes} A$ for each m. For simplicity of notation we shall identify the centre $p_m C$ of R_m with isomorphic algebra C. Consider the diagram

$$\begin{array}{ccc} R_m \bar{\otimes}^h_C R_m & \xrightarrow{\ \Phi_m\ } & (R_{m\natural} \otimes^h_C R_{m\natural})^\natural \\ \big\downarrow \iota_m & & \big\downarrow \kappa_m \\ R \bar{\otimes}^h_C R & \xrightarrow{\ \Phi\ } & (R_\natural \otimes^h_C R_\natural)^\natural \end{array}$$

where Φ and Φ_m are the natural maps, ι_m is the inclusion and κ_m is a complete isometry defined as follows. The restriction of the domain induces a natural complete contraction $\Omega_m\colon\ R_\natural \to R_{m\natural}$, which has a completely contractive right inverse

$$\Gamma_m\colon\ R_{m\natural} \to R_\natural, \quad \Gamma_m(\rho)(r) = \rho(p_m r p_m) \quad (\rho \in R_{m\natural},\ r \in R)$$

(we have identified $p_m R p_m$ and R_m). It follows that Ω_m is a completely quotient map, which induces a completely quotient map

$$\Psi_m\colon\ R_\natural \otimes^h_C R_\natural \to R_{m\natural} \otimes^h_C R_{m\natural}.$$

This map Ψ_m then induces the complete isometry κ_m in the above diagram, which is given explicitly by

$$\kappa_m(\tau)(\omega \otimes_C \rho) = \tau((\omega|R_m) \otimes_C (\rho|R_m)) \quad (\omega, \rho \in R_\natural).$$

It is easy to verify that the above diagram is commutative. Let $\pi_m\colon\ R \to R_m$ be defined by $\pi_m(r) = p_m r p_m$ and let $\sigma_m\colon\ R \bar{\otimes}^h_C R \to R_m \bar{\otimes}^h_C R_m$ and $\gamma_m\colon$ $(R_\natural \otimes^h_C R_\natural)^\natural \to (R_{m\natural} \otimes^h_C R_{m\natural})^\natural$ be the complete contractions induced by π_m. (Explicitly, $\gamma_m(\tau)(\omega \otimes_C \rho) = \tau(\tilde{\omega} \otimes_C \tilde{\rho})$, where, for a given $\omega \in R_{m\sharp}$, $\tilde{\omega}$ is the extension of ω to R defined by $\tilde{\omega}(r) = \omega(p_m r p_m)$.) Then it is easy to verify that

$$\gamma_m(\Phi(\vartheta)) = \Phi_m(\sigma_m(\vartheta)) \quad (\vartheta \in R \bar{\otimes}^h_C R). \tag{7.1}$$

Since $\mathcal{K}_m$ is separable, Φ_m is a complete isometry for each $m \in \mathcal{M}$. It is not hard to verify that $\|\vartheta\|_{\mathrm{cb}} = \lim_m \|\sigma_m(\vartheta)\|_{\mathrm{cb}}$ for each $\vartheta \in R \bar{\otimes}^h_C R$ and that

$$\|\tau\|_{\mathrm{cb}} = \lim_m \|\gamma_m(\tau)\|_{\mathrm{cb}}$$

for each $\tau \in (R_\natural \otimes^h_C R_\natural)^\natural$, where for the last equality Lemma 7.2 is needed. It follows then from (7.1) that Φ is an isometry and a similar argument shows that Φ is in fact a complete isometry.

To prove that Φ is surjective, first recall that $R \bar{\otimes}^h_C R$ is just the space of all normal completely bounded R'-bimodule homomorphisms from C' to $\mathrm{B}(\mathcal{H})$ (see Theorem 1.2 in [**24**]), which in our case is the space of all normal completely bounded A-module homomorphisms from $\mathrm{B}(\mathcal{K}) \bar{\otimes} A$ to $\mathrm{B}(\mathcal{H})$, and it follows from Lemma 6.6 that this space is completely isometrically isomorphic to $\mathrm{CB}_A(\mathrm{K}(\mathcal{K}) \check{\otimes} A; \mathrm{B}(\mathcal{H}))$. In particular, $R \bar{\otimes}^h_C R$ can be regarded as the dual of $(\mathrm{K}(\mathcal{K}) \check{\otimes} A) \hat{\otimes}_C \mathrm{T}(\mathcal{H})$. A bounded net $\{\vartheta_s\}$ in $R \bar{\otimes}^h_C R$ converges in the weak* topology to an element ϑ if and only if for each $b \in \mathrm{K}(\mathcal{K}) \check{\otimes} A$ the net $\{\vartheta_s(b)\}$ converges to $\vartheta(b)$ in the weak* topology of $\mathrm{B}(\mathcal{H})$. In fact, for bounded nets it suffices to check this condition only for elements of the form $b = f_{kl} \otimes a$, where $a \in A$ and $\{f_{kl}\colon\ k, l \in \mathcal{J}\}$ is a matrix unit in $\mathcal{K}$. Identify $R = \mathrm{B}(\mathcal{K}) \bar{\otimes} A$ with $\mathcal{M}_{\mathcal{J}}(A)$. Let $\vartheta = \sum_{i\in\mathcal{I}} x_i \otimes_C y_i$, where $x_i = [x_i(j,n)]$ and $y_i = [y_i(j,n)]$ are in $\mathcal{M}_{\mathcal{J}}(A)$. A straightforward computation shows that

$$\vartheta(f_{kl} \otimes a) = \left[a \sum_{i\in\mathcal{I}} x_i(j,k) y_i(l,n)\right]_{j,n}. \tag{7.2}$$

On the other hand, $(R_\natural \otimes_C^h R_\natural)^\natural$ is the (operator) dual of $(R_\natural \otimes_C^h R_\natural) \hat{\otimes}_C C_\sharp$ and a bounded net $\{\tau_s\}$ in $(R_\natural \otimes_C^h R_\natural)^\natural$ converges to an element τ in the weak* topology if and only if for every $\omega, \rho \in R_\natural$ the net $\{\tau_s(\omega \otimes_C \rho)\}$ converges to $\tau(\omega \otimes_C \rho)$ in the weak* topology of C. For each finite set $\mathcal{F} \subseteq \mathcal{J}$ denote by $e_{\mathcal{F}}$ the projection $e_{\mathcal{F}} = \sum_{j\in\mathcal{F}} e_j$, where $e_j = f_{jj} \otimes 1 \in \mathrm{K}(\mathcal{K}) \check{\otimes} A$. Since ω and ρ are normal, we have by Lemma 7.2 that $\lim_{\mathcal{F}\to\mathcal{J}} \|\omega - e_{\mathcal{F}}\omega e_{\mathcal{F}}\| = 0$ and a similar relation for ρ. Hence

$$\lim_{\mathcal{F}\to\mathcal{J}} \|\omega \otimes_C \rho - e_{\mathcal{F}}\omega e_{\mathcal{F}} \otimes_C e_{\mathcal{F}}\rho e_{\mathcal{F}}\| = 0.$$

(Here $R_\natural$ has the usual structure of an R-bimodule, that is, $(a\rho b)(r) = \rho(bra)$, where $a, b, r \in R$.) Thus, to check the weak* convergence of a bounded net in $(R_\natural \otimes_C^h R_\natural)^\natural$, we may assume that ω and ρ are of the form $\omega = e_k \omega e_j$ and $\rho = e_n \rho e_l$ for some $j, k, l, n \in \mathcal{J}$. Note that such an element ω acts on R in a simple way; namely, there is an $a_\omega \in A$ such that

$$\omega([a_{pq}]) = a_\omega a_{jk}^{(\mathcal{J})}$$

for each matrix $[a_{pq}] \in R = \mathcal{M}_{\mathcal{J}}(A)$. Another straightforward computation shows that, with

$$\vartheta = \sum_{i\in\mathcal{I}} [x_i(j,n)] \otimes_C [y_i(j,n)],$$

we have

$$\Phi(\vartheta)(\omega \otimes_C \rho) = \left(a_\omega a_\rho \sum_{i\in\mathcal{I}} x_i(j,k) y_i(l,n)\right)^{(\mathcal{J})}. \tag{7.3}$$

Using the above descriptions of the weak* convergence of bounded nets (in particular (7.2) and (7.3), where a_ω, a_ρ and a can be arbitrary elements of A), it is easy to see that Φ is weak* continuous on bounded sets (and hence weak* continuous by the Krein–Smulian theorem [**28**]). To see that Φ is surjective, let $\tau \in (R_\natural \otimes_C^h R_\natural)^\natural$, put $\tau_m = \gamma_m(\tau)$ for each $m \in \mathcal{M}$, let $\vartheta_m \in R_m \otimes_C^h R_m$ satisfy $\Phi_m(\vartheta_m) = \tau_m$, and let $\vartheta \in R \bar{\otimes}_C^h R$ be a weak* limit point of the bounded net $\{\iota_m(\vartheta_m)\colon\ m \in \mathcal{M}\}$. Then, by choosing a subnet, we may assume that $\lim \iota_m(\vartheta_m) = \vartheta$ in the weak* topology and we have

$$\Phi(\vartheta) = \lim \Phi(\iota_m(\vartheta_m)) = \lim \kappa_m(\Phi_m(\vartheta_m)) = \lim \kappa_m(\gamma_m(\tau)) = \tau.$$

To justify the last equality, observe that

$$(\kappa_m \gamma_m)(\tau)(\omega \otimes_C \rho) = \tau(p_m \omega p_m \otimes_C p_m \rho p_m)$$

for all $\omega, \rho \in R_\natural$ and recall that $\|\omega - p_m \omega p_m\| \to 0$ (and similarly for ρ) by Lemma 7.2.

Now let R be an arbitrary von Neumann algebra. Consider the diagram

$$\begin{array}{ccc} R \bar{\otimes}^h_C R & \xrightarrow{\Phi_R} & (R_\natural \otimes^h_C R_\natural)^\natural \\ \big\downarrow \iota & & \big\downarrow \mu \\ C' \bar{\otimes}^h_C C' & \xrightarrow{\Phi_{C'}} & (C_\natural \otimes^h_C C_\natural)^\natural \end{array}$$

where ι is the inclusion and μ is the complete isometry induced by the completely quotient map $C'_\natural \to R_\natural$ given by restricting the domain from C' to R. Since the diagram commutes and ι, μ and $\Phi_{C'}$ are complete isometries, Φ_R must be a complete isometry.

It remains to prove that Φ_R is surjective. Given $\tau \in (R_\natural \otimes^h_C R_\natural)^\natural$ there exists $\vartheta \in C' \bar{\otimes}^h_C C'$ such that $\mu(\tau) = \Phi_{C'}(\vartheta)$ and we have to show that $\vartheta \in R \bar{\otimes}^h_C R$. Let

$$\vartheta = x \odot_C y \quad (x \in \mathcal{R}(C'),\ \ y \in \mathcal{C}(C')).$$

Note that for arbitrary $\omega, \rho \in C'_\natural$ we have $\Phi_{C'}(\vartheta)(\omega \otimes_C \rho) = 0$ whenever $\omega|R = 0$ or $\rho|R = 0$ (since $\mu(\tau)(\omega \otimes_C \rho) = \tau((\omega|R) \otimes_C (\rho|R))$). Now for arbitrary $\omega, \rho \in C'_\natural$ and $r' \in R'$ the element $r'\omega - \omega r' \in C'_\natural$ satisfies $(r'\omega - \omega r')|R = 0$. Hence

$$\Phi_{C'}(\vartheta)((r'\omega - \omega r') \otimes_C \rho) = 0$$

or

$$\omega(x r'^{(\mathcal{I})} - r'x)\rho(y) = 0.$$

(Recall here that for a completely bounded map ω, we also denote its extension to various spaces of matrices by ω.) Since ρ is C-linear (and hence also linear over infinite matrices with entries in C by Lemma 2.2), this identity can be written as

$$\rho(\omega(x r'^{(\mathcal{I})} - r'x)y) = 0.$$

Since this holds for all $\rho \in R_\natural$, it follows now from Theorem 3 of [**20**] that

$$\omega(x r'^{(\mathcal{I})} - r'x)y = 0.$$

Denoting by $f \in \mathcal{M}(C)$ the projection with range $[C'^{(\mathcal{I})} y \mathcal{H}]^\perp$, we now have

$$\omega(x r'^{(\mathcal{I})} - r'x) f^\perp = 0$$

for each $\omega \in R_\natural$, which implies that

$$(x r'^{(\mathcal{I})} - r'x) f^\perp = 0.$$

It follows that the components of $x f^\perp$ commute with each $r' \in R'$; hence $x f^\perp \in \mathcal{R}(R)$. Moreover, from $fy = 0$ we have $\vartheta = x \odot_C y = x \odot_C f^\perp y = x f^\perp \odot_C y$. Thus, $\vartheta = z \odot_C y$, where $z := x f^\perp \in \mathcal{R}(R)$. Repeating the same arguments for y instead of x, we see that ϑ can be expressed as $\vartheta = z \odot_C w$, where $z \in \mathcal{R}(R)$ and $w \in \mathcal{C}(R)$; hence $\vartheta \in R \bar{\otimes}^h_C R$.

COROLLARY 7.3. *For each von Neumann algebra R the space $R \bar{\otimes}_C^h R$ is (completely isometrically isomorphic to) the operator dual of $(R_\natural \otimes_C^h R_\natural) \hat{\otimes}_C C_\sharp$.*

8. *The weak* closed ideals in $R \bar{\otimes}_C^h R$*

We shall determine all weak* closed two-sided ideals in $R \bar{\otimes}_C^h R$, where the weak* topology is introduced to $R \bar{\otimes}_C^h R$ through the isomorphism $R \bar{\otimes}_C^h R \to (R_\natural \otimes_C^h R_\natural)^\natural$ (Theorem 7.1 and Corollary 7.3). The product in $R \bar{\otimes}_C^h R$ is defined by the composition of mappings; in particular,

$$(8.1)\quad (x \odot_C y)(a \otimes_C b) = xa^{(\mathcal{I})} \odot_C b^{(\mathcal{I})} y \quad \text{and} \quad (a \otimes_C b)(x \odot_C y) = ax \odot_C yb$$

for all $x \in \mathcal{R}(R)$, $y \in \mathcal{C}(R)$ and $a, b \in R$ (where $\mathcal{I}$ is as in Remark 1.2).

The norm closed two-sided ideals in the usual Haagerup tensor product of C*-algebras were studied in [**1**] by Allen, Sinclair and Smith. Using a modification of the methods explained below, one can obtain some information about norm closed ideals, but since the author was not able to prove in this way the main result of [**1**] (that each non-zero norm closed two-sided ideal contains non-zero elementary tensors), we shall restrict our discussion to the weak* closed ideals of $R \bar{\otimes}_C^h R$. We need some general preliminary results.

PROPOSITION 8.1. *The slice maps from $R \bar{\otimes}_C^h R$ to R are weak* continuous.*

Proof. Let $\{x_j \odot_C y_j\}$ be a net converging to $x \odot y$ in the weak* topology (where $x, x_j \in \mathcal{R}(R)$ and $y, y_j \in \mathcal{C}(R)$). Then for every $\omega, \rho \in R_\natural$ the net $\{\omega(x_j)\rho(y_j)\}$ converges to $\omega(x)\rho(y)$ in the weak* topology of C. Hence (using Lemma 2.2) we have

$$\lim_j \nu\omega(x_j \rho(y_j)) = \nu\omega(x\rho(y))$$

for each $\nu \in C_\sharp$. Since by [**20**] each normal functional on R is of the form $\nu\omega$ for some $\omega \in R_\natural$ and $\nu \in C_\sharp$, we see that the net $\{x_j \rho(y_j)\}$ converges to $x\rho(y)$ in the weak* topology of R. This shows that the left slice map corresponding to ρ is weak* continuous. The continuity of right slice maps is proved similarly.

COROLLARY 8.2. *If $X, Y \subseteq R$ are weak* closed C-submodules, then $X \bar{\otimes}_C^h Y$ is a weak* closed subspace of $R \bar{\otimes}_C^h R$.*

Proof. This follows immediately from Theorem 5.5 and Proposition 8.1.

The following proposition shows in particular that the weak* topology on $X \bar{\otimes}_C^h Y$ is independent of the particular embedding of X and Y as weak* closed subspaces of a von Neumann algebra.

PROPOSITION 8.3. *If S is a von Neumann algebra with the same centre C as R, X and Y are weak* closed C-submodules of R, and $\Phi\colon X \to S$ and Ψ:*

$Y \to S$ are weak continuous completely bounded C-module homomorphisms, then the completely bounded map*

$$\Phi \,\bar{\otimes}_C^h\, \Psi\colon\ X \,\bar{\otimes}_C^h\, Y \to S \,\bar{\otimes}_C^h\, S$$

is weak continuous. Moreover, if Φ and Ψ are completely isometric, then $\Phi \,\bar{\otimes}_C^h\, \Psi$ is a weak* homeomorphism onto the weak* closed subspace of $S \,\bar{\otimes}_C^h\, S$.*

Proof. As a consequence of the Krein–Smulian theorem, it suffices to prove the weak* continuity on the unit ball, but this follows easily from the definition of the weak* topology in $R \,\bar{\otimes}_C^h\, R$ and $S \,\bar{\otimes}_C^h\, S$. It is also a well-known consequence of the Krein–Smulian theorem (and the weak* compactness of the unit ball) that a weak* continuous isometry (such as $\Phi \,\bar{\otimes}_C^h\, \Psi$) maps weak* closed subspaces homeomorphically onto weak* closed subspaces. Since $X \,\bar{\otimes}_C^h\, Y$ is weak* closed in $R \,\bar{\otimes}_C^h\, R$ by Corollary 8.2, the proposition is proved.

Recall that the 0-projection of a row $x \in \mathcal{R}(X)$ is the projection in $\mathcal{M}(R)$ which generates the weak* closed right ideal $\{r \in \mathcal{M}(R)\colon\ xr = 0\}$ and the 0-projection of a column is defined analogously.

LEMMA 8.4. *Given a von Neumann algebra R and strong right and left R-modules X and Y, each element $\vartheta \in X \,\bar{\otimes}_R^h\, Y$ can be expressed as $\vartheta = x \odot_R y \quad (x \in \mathcal{R}(X),\ \ y \in \mathcal{C}(Y))$, where the 0-projections of x and y in $\mathcal{M}(R)$ are diagonal.*

Proof. Let $\vartheta = z \odot_R w$ be any representation of ϑ with $z \in \mathcal{R}(X)$ and $w \in \mathcal{C}(Y)$. Denote by $e^\perp$ the 0-projection of z in $\mathcal{M}(R)$. Let $p \in \mathcal{M}(R)$ be the diagonal projection equivalent to e (see Lemma 4.1 in [**24**]) and let $u \in \mathcal{M}(R)$ be a partial isometry satisfying $u^*u = e$ and $uu^* = p$. Then $\vartheta = ze \odot_R w = zu^* \odot_R uw = s \odot_R t$, where $s := zu^* \in \mathcal{R}(X)$ and $t := uw \in \mathcal{C}(Y)$. It is easy to verify that the 0-projection of s is $p^\perp$, and is hence diagonal. Since $\vartheta = sp \odot_R t = s \odot_R pt$, we may replace t by pt; then the 0-projection of t, denoted by $f^\perp$, satisfies $f \leqslant p$. Choose $v \in \mathcal{M}(R)$ such that $v^*v = f$ and vv^* is a diagonal projection $q \in \mathcal{M}(R)$. Then

$$\vartheta = s \odot_R ft = sv^* \odot_R vt = x \odot_R y,$$

where the 0-projection of $y := vt$ is $q^\perp$, and hence diagonal. Moreover, the 0-projection of $x := sv^*$ is also $q^\perp$. Indeed, for each $r \in \mathcal{M}(R)$ we have $xr = sv^*r = 0$ if and only if $pv^*r = 0$ (since $p^\perp$ is the 0-projection of s), which can be written as $v^*r = 0$ (since $pv^* = p(fv^*) = v^*$), and this is equivalent to $qr = 0$ or $r = q^\perp r$.

LEMMA 8.5. *If $\{x^j\} \subseteq \mathcal{R}(R)$ and $\{y^j\} \subseteq \mathcal{C}(R)$ are bounded nets converging to x and y, respectively, in the strong operator topology, then the net $\{x^j \odot_C y^j\}$ converges to $x \odot_C y$ in the weak* topology of $R \,\bar{\otimes}_C^h\, R$.*

Proof. First note that $\omega(x^j) \to \omega(x)$ in the strong operator topology for each $\omega \in R_\natural$. (Recall again here that the extension of a completely bounded map ω to various spaces of matrices is denoted simply by ω.) Indeed, assuming (as we may) that ω is positive and $\|\omega\| = 1$, we have for each vector $\xi \in \mathcal{H}^{\mathcal{I}}$ by the Schwarz inequality for completely positive maps that

$$\|\omega(x^j - x)\xi\|^2 = \langle \omega(x^j - x)^* \omega(x^j - x)\xi, \xi\rangle \leqslant \langle \omega((x^j - x)^*(x^j - x))\xi, \xi\rangle,$$

which converges to 0 since (the extension to $\mathcal{M}(R)$ of) ω is normal. Similarly, $\rho(y^j) \to \rho(y)$ for each $\rho \in R_\natural$. Hence $\omega(x^j)\rho(y^j) \to \omega(x)\rho(x)$ in the strong operator topology. Since the net $\{x^j \odot_C y^j\}$ is bounded, this implies weak* convergence in $R \bar{\otimes}_C^h R$.

LEMMA 8.6. *If Y is a subset of $\mathcal{C}(R)$, the weak* closed R-subbimodule W of $\mathcal{C}(R)$ generated by Y consists of all $w \in \mathcal{C}(R)$ such that for each $c \in \mathcal{R}(C)$ the condition $cY = 0$ implies $cw = 0$. Moreover, for each $w \in W$ there exists a net $\{w^j\}$ in the R-subbimodule W_0 generated algebraically by Y that converges to w in the strong operator topology and satisfies $\|w_j\| \leqslant \|w\|$ for all j.*

Proof. This lemma is an extension of a result from [**23**] and the proof is similar. As usual, we regard $\mathcal{C}(R)$ and $\mathcal{R}(R)$ as subsets of $\mathcal{M}(R)$ by embedding onto the first column and row. Given $w \in \mathcal{C}(R)$, suppose that for each $c \in \mathcal{R}(C)$ the condition $cY = 0$ implies $cw = 0$. The right ideal $J := W_0\mathcal{R}(R)$ of $\mathcal{M}(R)$ satisfies $R^{(\mathcal{I})}J \subseteq J$. Hence $\bar{J} = p\mathcal{M}(R)$ for some projection $p \in \mathcal{M}(R) \cap (R^{(\mathcal{I})})' = \mathcal{M}(C)$. Then $p^\perp Y = 0$, and hence $cY = 0$ for each row c of the matrix $p^\perp$. By the assumption of the lemma, this implies that $cw = 0$. Therefore $p^\perp w = 0$, which means that $w \in \bar{J}$. If $\{y^j\}$ is a net in J converging to p in the strong operator topology and $\|y^j\| \leqslant 1$ for all j, then the net $\{y^j w\}$ is contained in

$$Jw = W_0\mathcal{R}(R)w \subseteq W_0 R = W_0,$$

converges strongly to $pw = w$, and satisfies $\|y^j w\| \leqslant \|w\|$ for all j. This proves the lemma in one direction; the proof of the converse is trivial.

From Theorem 5.3 we deduce that the centre of $R \bar{\otimes}_C^h R$ is $C \bar{\otimes}_C^h C$, which is identified with C.

THEOREM 8.7. *Each weak* closed two-sided ideal J of $H = R \bar{\otimes}_C^h R$ is of the form $J = pH$ for some projection $p \in C$.*

Proof. Let p be the union of all projections in $J \cap C$; we have to prove that $\vartheta = p\vartheta$ for each $\vartheta \in J$. By Lemma 8.4, each $\vartheta \in J$ has a representation

$$\vartheta = x \odot_C y \quad (x \in \mathcal{R}(R),\ \ y \in \mathcal{C}(R))$$

such that the 0-projections $e^\perp$ and $f^\perp$ in $\mathcal{M}(C)$ of x and y are diagonal. Let $f_i \in C$ $(i \in \mathcal{I})$ be the diagonal entries of f and denote by $\mathbf{f}_i$ the ith column of f (thus, f_i is the only non-zero entry of $\mathbf{f}_i$). Let $c \in \mathcal{R}(C)$. If $cy = 0$, then $cf = 0$ (from the definition of 0-projections); hence $c\mathbf{f}_i = 0$ for all i. Since this holds for all $c \in \mathcal{R}(C)$, Lemma 8.6 shows that $\mathbf{f}_i$ is contained in the weak* closed R-subbimodule of $\mathcal{C}(R)$ generated by y. Hence by the same lemma there exists (for a fixed i) a bounded net $\{y^j\}$ in $R^{(\mathcal{I})}yR$ converging strongly to $\mathbf{f}_i$. Each y^j is a finite sum

$$y^j = \sum_k r_{jk}^{(\mathcal{I})} y s_{jk} \quad (r_{jk}, s_{jk} \in R).$$

Using (8.1) we see that

$$x \odot_C y^j = \sum_k (1 \otimes_C s_{jk})(x \odot_C y)(1 \otimes_C r_{jk}),$$

which is in J for each j since J is a two-sided ideal containing $x \odot_C y$. Since J is weak* closed and the net $\{x \odot_C y^j\}$ converges to $x \odot_C \mathbf{f}_i$ by Lemma 8.5, we see that $x \odot_C \mathbf{f}_i \in J$. Denoting by e_i and $\mathbf{e}_i$ the ith diagonal entry and the ith row of e, we see now by similar arguments that $\mathbf{e}_i \odot_C \mathbf{f}_i \in J$. But $\mathbf{e}_i \odot_C \mathbf{f}_i = e_i \otimes_C f_i$ and, after the identification of $C \bar{\otimes}_C^h C$ with C, this becomes $e_i f_i$. Thus $e_i f_i \in J$. Since p is the largest projection in $J \cap C$, we have $e_i f_i \leqslant p$ for all i. Hence

$$p\vartheta = pxe \odot_C fy = \sum_{i \in \mathcal{I}} px_i e_i \otimes_C f_i y_i = \sum_{i \in \mathcal{I}} x_i e_i \otimes_C f_i y_i = x \odot_C y = \vartheta.$$

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