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ON TENSOR PRODUCTS OF OPERATOR MODULES

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ABSTRACT. The injective tensor product of normal representable bimodules over von Neumann algebras is shown to be normal. The usual Banach module projective tensor product of central representable bimodules over an Abelian C^*-algebra is shown to be representable. A normal version of the projective tensor product is introduced for central normal bimodules.

KEYWORDS: *Operator bimodule, von Neumann algebra, tensor norms.*

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1. INTRODUCTION

A Banach bimodule X over C^*-algebras A and B is called *representable* [1], [20] if there exist Hilbert modules $\mathcal{H}$ and $\mathcal{K}$ over A and B, respectively (that is, Hilbert spaces with $*$-representations $\pi : A \to \mathrm{B}(\mathcal{H})$ and $\sigma : B \to \mathrm{B}(\mathcal{K})$) and an isometric A, B-bimodule homomorphism $X \to \mathrm{B}(\mathcal{K}, \mathcal{H})$. We denote the class of all such bimodules by ${}_A\mathrm{RM}_B$, and by $\mathrm{B}_A(X, Y)_B$ the space of all bounded A, B-bimodule maps from X into Y. If, in addition, A and B are von Neumann algebras and $\mathcal{H}$ and $\mathcal{K}$ are normal (that is, the representations π and σ are normal), then we say that X is a *normal representable* A, B-bimodule, which we shall write as $X \in {}_A\mathrm{NRM}_B$. In [1] the fundamentals of the tensor products of representable bimodules are studied. In particular the *projective tensor seminorm* on the algebraic tensor product $X \otimes_B Y$ of two bimodules $X \in {}_A\mathrm{RM}_B$ and $Y \in {}_B\mathrm{RM}_C$ is defined by

$$\gamma^B_{A,C}(w) = \inf\Big\{ \Big\| \sum_{j=1}^n a_j a_j^* \Big\|^{1/2} \Big\| \sum_{j=1}^n b_j^* b_j \Big\|^{1/2} : \ w = \sum_{j=1}^n a_j x_j \otimes_B y_j b_j, \\ a_j \in A,\ b_j \in B,\ x_j \in X,\ y_j \in Y,\ \|x_j\| \leqslant 1,\ \|y_j\| \leqslant 1 \Big\}. \tag{1.1}$$

Taking the quotient of $X \otimes_B Y$ by the zero space of this seminorm and completing, we obtain a representable A, C-bimodule, denoted by ${}_AX \overset{\gamma}{\otimes}_B Y_C$, and the induced

norm on this bimodule is denoted by $\gamma^B_{A,C}$ again. In the case $A = B = C = \mathbb{C}$ this reduces to the usual projective tensor product of Banach spaces, denoted simply by $X \overset{\gamma}{\otimes} Y$. As shown in [1], this seminorm can also be expressed by

$$\gamma^B_{A,C}\Big(\sum_{j=1}^n x_j \otimes_B y_j\Big) = \sup \sum_{j=1}^n \theta(x_j, y_j), \tag{1.2}$$

where the supremum is over all contractive bilinear maps θ from $X \times Y$ into $\mathrm{B}(\mathcal{L}, \mathcal{H})$, with $\mathcal{H}$ and $\mathcal{L}$ cyclic Hilbert modules over A and C (respectively), such that

$$\theta(axb, yc) = a\theta(x, by)c \quad \text{for all } a \in A,\ b \in B,\ c \in C,\ x \in X,\ y \in Y.$$

Further [1], the *injective tensor seminorm* is defined on $X \otimes_B Y$ by

$$\Lambda^B_{A,C}\Big(\sum_{j=1}^n x_j \otimes_B y_j\Big) = \sup \Big\| \sum_{j=1}^n \phi(x_j)\psi(y_j) \Big\|, \tag{1.3}$$

where the supremum is over all contractions $\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ and $\psi \in \mathrm{B}_B(Y, \mathrm{B}(\mathcal{L}, \mathcal{K}))_C$, with $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ cyclic Hilbert modules over A, B and C (respectively).

REMARK 1.1. The restriction that $\mathcal{H}$ and $\mathcal{K}$ in the above formulas are cyclic over A and B (respectively) implies by an argument of Smith ([22],Theorem 2.1) that each bounded A, B-bimodule homomorphism ϕ from an operator A, B-bimodule into $\mathrm{B}(\mathcal{K}, \mathcal{H})$ is completely bounded with $\|\phi\|_{\mathrm{cb}} = \|\phi\|$. Applying this to a pair $Y \subseteq X$ of representable A, B-bimodules and using the extension theorem for completely bounded bimodule maps [18], [24], it follows that each map $\phi \in \mathrm{B}_A(Y, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ can be extended to a map $\psi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ with $\|\psi\| = \|\phi\|$. Thus in this respect such maps behave like linear functionals.

Clearly there are similar definitions of the "projective" and the "injective" tensor seminorms (which turn out to be norms) in the category ${}_A\mathrm{NRM}_B$ for von Neumann algebras A and B; the only difference with the above definitions is that we require the cyclic Hilbert modules $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ to be normal. Now the natural question is if these new norms are different from the above ones. In Section 2 we shall show that the two injective norms are equal. Following the observation that the norm $\Lambda^B_{A,C}$ is in fact independent of A and C, the proof of equality of the two injective norms will be essentially a reduction to a density question concerning certain sets of normal states. Contrary to the injective, the two projective norms are not the same even if $A = B = C$ is Abelian and the bimodules are central. Here a C-bimodule X is called *central* if $cx = xc$ for all $c \in C$ and $x \in X$. We denote by CRM_C the class of all central representable C-bimodules and (if C is a von Neumann algebra) by CNRM_C the subclass of all central normal representable bimodules.

In Section 3 we show that ${}_C X \overset{\gamma}{\otimes}_C Y_C = {}_{\mathbb{C}} X \overset{\gamma}{\otimes}_C Y_{\mathbb{C}}$ for all bimodules $X, Y \in \mathrm{CRM}_C$. (Note that ${}_{\mathbb{C}} X \overset{\gamma}{\otimes}_C Y_{\mathbb{C}}$ is just $X \overset{\gamma}{\otimes}_C Y$, the quotient of the usual Banach space tensor product $X \overset{\gamma}{\otimes} Y$ by the closed subspace generated by all elements of the form $xc \otimes y - x \otimes cy$ ($x \in X$, $y \in Y$, $c \in C$) [21].) The main step of the proof will be to show that the central C-bimodule $X \overset{\gamma}{\otimes}_C Y$ is representable, which in the more traditional terminology (see [13]) means that the usual Banach space projective tensor product of C-locally convex modules over C is already C-locally convex. This simplifies the corresponding definition of such tensor product in [13].

If C is an Abelian von Neumann algebra and $X, Y \in \mathrm{CNRM}_C$, the bimodule $Z = X \overset{\gamma}{\otimes}_C Y$ is not necessarily normal. Therefore we introduce in Section 4 a new tensor product $X \overset{\nu}{\otimes}_C Y$, which plays the role of the projective tensor product in the category CNRM_C. We show that $Z_{\mathrm{n}} := X \overset{\nu}{\otimes}_C Y$ is just the normal part of Z in the sense that each bounded C-bimodule map ϕ from Z into a bimodule $V \in {}_C\mathrm{NRM}_C$ factors uniquely through Z_{n}. Further, the norm of elements in Z_{n} can be expressed by a formula similar to (1.1), but involving infinite sums that are not necessarily norm convergent. We do not know if there is an analogous formula in the case of non-central bimodules.

The background concerning operator spaces used implicitly in this article can be found in any of the books [8], [18], [19].

2. NORMALITY OF THE INJECTIVE OPERATOR BIMODULE TENSOR PRODUCT

If A, B and C are von Neumann algebras and $X \in {}_A\mathrm{NRM}_B$, $Y \in {}_B\mathrm{NRM}_C$, we define a norm on $X \otimes_B Y$ by

$$\lambda^B_{A,C}\Big(\sum_{j=1}^n x_j \otimes_B y_j\Big) = \sup\Big\|\sum_{j=1}^n \phi(x_j)\psi(y_j)\Big\|, \tag{2.1}$$

where the supremum is over all contractions $\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$ and $\psi \in \mathrm{B}_B(Y, \mathrm{B}(\mathcal{L}, \mathcal{K}))_C$ with $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ *normal* cyclic Hilbert modules over A, B and C (respectively). Except for the normality requirement on Hilbert modules, this is the same formula as (1.3), hence $\lambda^B_{A,C} \leqslant \Lambda^B_{A,C}$.

REMARK 2.1. To show that $\lambda^B_{A,C}$ is definite, suppose that $w = \sum_{j=1}^n x_j \otimes y_j \in X \otimes_B Y$ is such that $\sum_{j=1}^n \phi(x_j)\psi(y_j) = 0$ for all ϕ and ψ as in the definition of $\lambda^B_{A,C}$. We may assume that $X \subseteq \mathrm{B}(\mathcal{H}_B, \mathcal{H}_A)$ and $Y \subseteq \mathrm{B}(\mathcal{H}_C, \mathcal{H}_B)$ for some normal (faithful) Hilbert modules $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ over A, B and C, respectively.

Decomposing $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ into direct sums of cyclic submodules,

$$\mathcal{H}_A = \bigoplus_i e_i'\mathcal{H}_A, \quad \mathcal{H}_B = \bigoplus_j f_j'\mathcal{H}_B, \quad \mathcal{H}_C = \bigoplus_k g_k'\mathcal{H}_C,$$

where $e_i' \in A'$, $f_j' \in B'$ and $g_k' \in C'$ are projections, and considering the maps of the form $\phi(x) = e_i'xb'f_j'$ and $\psi(y) = f_j'b'yg_k'$, where $b' \in B'$, it follows that $[x_1, \dots, x_n]B'[y_1, \dots, y_n]^{\mathrm{t}} = 0$, which implies that $\sum_{j=1}^{n} x_j \otimes y_j = 0$ (see e.g. Lemma 1.1 of [14]).

We would like to show that $\Lambda^B_{A,C} = \lambda^B_{A,C}$, but first we shall show that $\Lambda^B_{A,C}$ and $\lambda^B_{A,C}$ are independent of A and C. We simplify the notation by $\lambda_B := \lambda^B_{\mathbb{C},\mathbb{C}}$ and $\Lambda_B := \Lambda^B_{\mathbb{C},\mathbb{C}}$. Note that Remark 1.1 implies that both norms $\Lambda^B_{A,C}$ and $\lambda^B_{A,C}$ are preserved under isometric embeddings of bimodules.

The conjugate (that is, the dual) space $\mathcal{H}^*$ of a (left) Hilbert A-module $\mathcal{H}$ is regarded below as a right A-module by $\xi^* a = (a^*\xi)^*$ ($\xi \in \mathcal{H}$, $a \in A$), where ξ^* denotes ξ regarded as an element of $\mathcal{H}^*$.

PROPOSITION 2.2. *The seminorms $\Lambda^B_{A,C}$ (for representable bimodules over C^*-algebras) and $\lambda^B_{A,C}$ (for normal representable bimodules over von Neumann algebras) do not depend on A and C.*

Proof. Choose $\varepsilon > 0$. Given $w = \sum_{j=1}^{n} x_j \otimes_B y_j \in X \otimes_B Y$ and contractions $\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}))_B$, $\psi \in \mathrm{B}_B(Y, \mathrm{B}(\mathcal{L}, \mathcal{K}))_C$ as in (1.3) or (2.1), we choose unit vectors $\xi \in \mathcal{H}$ and $\eta \in \mathcal{L}$ such that

$$\Big|\Big\langle \sum_{j=1}^{n} \phi(x_j)\psi(y_j)\eta, \xi \Big\rangle\Big| > \Big\| \sum_{j=1}^{n} \phi(x_j)\psi(y_j) \Big\| - \varepsilon.$$

Then

$$\alpha : X \to \mathcal{K}^*,\ \alpha(x) = (\phi(x)^*\xi)^* \quad \text{and} \quad \beta : Y \to \mathcal{K},\ \beta(y) = \psi(y)\eta$$

are contractive homomorphisms of modules over B such that

$$\Big| \sum_{j=1}^{n} \langle \beta(y_j), \alpha(x_j)^* \rangle \Big| > \Big\| \sum_{j=1}^{n} \phi(x_j)\psi(y_j) \Big\| - \varepsilon.$$

This implies that $\Lambda_B(w) \geqslant \Lambda^B_{A,C}(w)$ and $\lambda_B(w) \geqslant \lambda^B_{A,C}(w)$.

To prove the inequality $\Lambda_B(w) \leqslant \Lambda^B_{A,C}(w)$, let $\pi : B \to \mathrm{B}(\mathcal{K})$ be a cyclic representation and let $\alpha \in \mathrm{B}(X, \mathcal{K}^*)_B$, $\beta \in \mathrm{B}_B(Y, \mathcal{K})$ be contractions such that

$$\Big| \sum_{j=1}^{n} \langle \beta(y_j), \alpha(x_j)^* \rangle \Big| > \Lambda_B(w) - \varepsilon. \tag{2.2}$$

Since $\Lambda^B_{A,C}$ is preserved by inclusions we may assume that X and Y are C^*-algebras containing $A \cup B$ and $B \cup C$ (respectively). Then, since α and β are

complete contractions by a result of Smith quoted in Remark 1.1, it follows by the representation theorem for such mappings (see p. 102 of [18]) that there exist Hilbert spaces $\mathcal{H}$ and $\mathcal{L}$, $*$-representations $\Phi : X \to \mathrm{B}(\mathcal{H})$ and $\Psi : Y \to \mathrm{B}(\mathcal{L})$, unit vectors $\xi \in \mathcal{H}$ and $\eta \in \mathcal{L}$ and contractions $S \in \mathrm{B}(\mathcal{K}, \mathcal{H})$, $T \in \mathrm{B}(\mathcal{L}, \mathcal{K})$ such that

$$\alpha(x) = \xi^* \Phi(x) S \quad \text{and} \quad \beta(y) = T\Psi(y)\eta. \tag{2.3}$$

Clearly we may adjust $\mathcal{H}$, $\mathcal{K}$, S and T so that $[\Phi(X)\xi] = \mathcal{H}$ and $[\Psi(Y)\eta] = \mathcal{L}$. Then it follows from (2.3) (since α and β are B-module maps) that

$$\Phi(b)S = S\pi(b) \quad \text{and} \quad T\Psi(b) = \pi(b)T \quad (b \in B). \tag{2.4}$$

Replace $\mathcal{H}$ with the subspace $\mathcal{H}_1 = [\Phi(A)\xi]$ and $\mathcal{L}$ with $\mathcal{L}_1 = [\Psi(C)\eta]$ and define

$$\psi : Y \to \mathrm{B}(\mathcal{L}_1, \mathcal{K}), \quad \text{by} \quad \psi(y) = T\Psi(y)|\mathcal{L}_1$$

and

$$\phi : X \to \mathrm{B}(\mathcal{K}, \mathcal{H}_1), \quad \text{by} \quad \phi(x) = P\Phi(x)S,$$

where $P \in \mathrm{B}(\mathcal{H})$ is the orthogonal projection onto $\mathcal{H}_1$. Then $\eta \in \mathcal{L}_1$, $\xi \in \mathcal{H}_1$ and by (2.3)

$$\alpha(x) = \xi^* \phi(x) \quad (x \in X) \quad \text{and} \quad \beta(y) = \psi(y)\eta \quad (y \in Y). \tag{2.5}$$

Moreover, $\mathcal{H}_1$, $\mathcal{K}$ and $\mathcal{L}_1$ are cyclic over A, B and C (respectively) and (2.4) (together with the fact that $\mathcal{H}_1$ and $\mathcal{L}_1$ are invariant under $\Phi(A)$ and $\Psi(C)$, respectively) implies that $\phi(axb) = \Phi(a)\phi(x)\pi(b)$ and $\psi(byc) = \pi(b)\psi(y)\Psi(c)$, thus $\phi \in \mathrm{B}_A(X, \mathrm{B}(\mathcal{K}, \mathcal{H}_1))_B$ and $\psi \in \mathrm{B}_B(X, \mathrm{B}(\mathcal{L}_1, \mathcal{K}))_C$ are of the type required in the definition of the norm $\Lambda^B_{A,C}$. Since from (2.5) and (2.2) we have that

$$\Big\| \sum_{j=1}^n \phi(x_j)\psi(y_j) \Big\| \geqslant \Big| \Big\langle \sum_{j=1}^n \phi(x_j)\psi(y_j)\eta, \xi \Big\rangle \Big| = \Big| \sum_{j=1}^n \langle \beta(y_j), \alpha(x_j)^* \rangle \Big| > \Lambda_B(w) - \varepsilon,$$

it follows that $\Lambda^B_{A,C}(w) \geqslant \Lambda_B(w)$.

The proof of the inequality $\lambda_B(w) \leqslant \lambda^B_{A,C}(w)$ is same as the proof in the previous paragraph, with the addition that we must achieve that the modules $\mathcal{K}$, $\mathcal{H}_1$ and $\mathcal{L}_1$ are normal. First, since $X \in {}_A\mathrm{NRM}_B$ and $Y \in {}_B\mathrm{NRM}_C$ we may assume (by standard arguments) that (up to isometric isomorphisms) $A, X, B \subseteq \mathrm{B}(\mathcal{H}_0)$ and $B, Y, C \subseteq \mathrm{B}(\mathcal{L}_0)$ for some Hilbert spaces $\mathcal{H}_0$ and $\mathcal{L}_0$ (with the module multiplications just the products of operators). Then (by Remark 1.1 again) we may assume that $X = \mathrm{B}(\mathcal{H}_0)$ and $Y = \mathrm{B}(\mathcal{L}_0)$. By the definition of the norm λ_B we can choose a normal cyclic representation $\pi : B \to \mathcal{K}$ and $\alpha \in \mathrm{B}(X, \mathcal{K}^*)_B$, $\beta \in \mathrm{B}_B(Y, \mathcal{K})$ such that

$$\Big| \sum_{j=1}^n \langle \beta(y_j), \alpha(x_j)^* \rangle \Big| > \lambda_B(w) - \varepsilon. \tag{2.6}$$

Let U be the unit ball of $\mathrm{B}_B(Y, \mathcal{K}) = \mathrm{CB}_B(Y, \mathcal{K})$ (Remark 1.1, note that $\mathcal{K} = \mathrm{B}(\mathbb{C}, \mathcal{K})$) and U_σ the weak* continuous maps in U. Since $Y = \mathrm{B}(\mathcal{L}_0)$, it follows

from a variant of 2.5 in [7] that U_σ is dense in U in the point weak* topology; but since $\mathcal{K}$ is reflexive, this topology has the same continuous linear functionals as the point norm topology, hence by convexity U_σ is dense in U in the point norm topology. With a similar result for $\mathrm{B}(X, \mathcal{K}^*)_B$, it follows that we may assume that the maps α and β in (2.6) are weak* continuous. But then the proof of the representation theorem for completely bounded mappings ([18], Theorem 8.4) (together with Stinespring's construction) shows that the representations Φ and Ψ constructed in the previous paragraph are normal, which implies that the Hilbert modules $\mathcal{H}_1$ and $\mathcal{L}_1$ over A and C are also normal. (Alternatively, we could just take the normal parts of Φ and Ψ...) ∎

Note that the analogue of Proposition 2.2 for the projective norm does not hold, namely for a C^*-algebra A the norm $\gamma^{\mathbb{C}}_{A,A}$ on $A \otimes A$ coincides with the Haagerup norm, while the norm $\gamma^{\mathbb{C}}_{\mathbb{C},\mathbb{C}}$ is the usual Banach space projective tensor norm.

A subset K of an A, B-bimodule X is called *A, B-absolutely convex* if

$$\sum_{j=1}^{n} a_j x_j b_j \in K$$

for all $x_j \in K$ and $a_j \in A$, $b_j \in B$ satisfying $\sum\limits_{j=1}^{n} a_j a_j^* \leqslant 1$, $\sum\limits_{j=1}^{n} b_j^* b_j \leqslant 1$.

LEMMA 2.3. *If K is a $B, \mathbb{C}$-absolutely convex weak* compact subset of a von Neumann algebra B, then the set $L = \{x^*x : x \in K\}$ is convex and weak* compact.*

Proof. Given $x, y \in K$ and $t \in [0, 1]$, consider the polar decomposition

$$\begin{bmatrix} \sqrt{t}x \\ \sqrt{1-t}y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix} z,$$

where $z = \sqrt{tx^*x + (1-t)y^*y}$ and $[u, v]^{\mathrm{t}}$ is the partial isometric part. Since

$$z = [u^*\ v^*] \begin{bmatrix} \sqrt{t}x \\ \sqrt{1-t}y \end{bmatrix} = u^*x\sqrt{t} + v^*y\sqrt{1-t}$$

and K is $B, \mathbb{C}$-absolutely convex, $z \in K$. It follows that $tx^*x + (1-t)y^*y = z^*z \in L$, proving that L is convex.

Since K (hence also L) is bounded, it suffices now to prove that L is closed in the strong operator topology (SOT). Let y be in the closure of L and (x_j) a net in K such that $(x_j^* x_j)$ converges to y in the SOT. Since the function $x \mapsto \sqrt{x}$ is SOT continuous on bounded subsets of B^+, the net $(|x_j|)$ converges to $\sqrt{y}$. Since K is $B, \mathbb{C}$-absolutely convex, the polar decomposition shows that $|x_j| \in K$. Since K is weak* closed, it follows that $\sqrt{y} \in K$, hence $y \in L$. ∎

We denote by $\mathrm{R}_n(B)$ and $\mathrm{C}_n(B)$ the set of all $1 \times n$ and $n \times 1$ matrices (respectively) with the entries in a set B.

THEOREM 2.4. *For all $X \in \mathrm{NRM}_B$ and $Y \in {}_B\mathrm{NRM}$, $\Lambda_B = \lambda_B$ on $X \otimes_B Y$.*

Proof. The theorem will be proved first for free modules by translating the problem to states on B and approximating states by normal states. Then elements of general modules will be approximated by elements of free modules.

First assume that X and Y are free with basis $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$, respectively. More precisely, set

$$x = [x_1, \dots, x_n], \quad y = [y_1, \dots, y_n]^{\mathrm{t}}$$

and assume that the two maps

$$f : \mathrm{C}_n(B) \to X,\ f(b) = xb \quad \text{and} \quad g : \mathrm{R}_n(B) \to Y,\ g(b) = by$$

are invertible (with bounded inverses by the open mapping theorem). Set

$$\mathcal{S} = \{b \in \mathrm{C}_n(B) :\ \|xb\| \leqslant 1\}, \quad \mathcal{T} = \{b \in \mathrm{R}_n(B) :\ \|by\| \leqslant 1\}$$

and

$$\alpha = \sup\{\|b\| :\ b \in \mathcal{S} \cup \mathcal{T}\}.$$

Let $0 < \varepsilon < 1$. Choose $w \in X \otimes_B Y$ and note that w can be written as

$$w = \sum_{i,j=1}^{n} x_i \otimes_B d_{ij} y_j \quad (d_{ij} \in B). \tag{2.7}$$

By the definition of Λ_B there exist a cyclic representation $\pi : B \to \mathcal{K}$ and contractions $\phi \in \mathrm{B}(X, \mathcal{K}^*)_B$, $\psi \in \mathrm{B}_B(Y, \mathcal{K})$ such that

$$\Big| \sum_{i,j=1}^{n} \langle \pi(d_{ij})\psi(y_j), \phi(x_i)^* \rangle \Big| > \Lambda_B(w) - \varepsilon. \tag{2.8}$$

Let $\xi_0 \in \mathcal{K}$ be a unit cyclic vector for $\pi(B)$, ρ the state $\rho(b) = \langle \pi(b)\xi_0, \xi_0 \rangle$ on B, and choose $a_i, c_i \in B$ so that

$$\|\phi(x_i)^* - \pi(a_i^*)\xi_0\| < \varepsilon \quad \text{and} \quad \|\psi(y_i) - \pi(c_i)\xi_0\| < \varepsilon \quad (i = 1, \dots, n). \tag{2.9}$$

For $b = [b_{ij}] \in \mathrm{M}_{m,n}(B)$ denote the matrix $[\pi(b_{ij})]$ simply by $\pi(b)$. Set

$$\begin{aligned} &\xi = [\phi(x_1)^*, \dots, \phi(x_n)^*]^{\mathrm{t}}\ \ (\in \mathcal{K}^n), \quad \eta = [\psi(y_1), \dots, \psi(y_n)]^{\mathrm{t}}\ \ (\in \mathcal{K}^n), \\ &a = [a_1, \dots, a_n] \quad \text{and} \quad c = [c_1, \dots, c_n]^{\mathrm{t}}. \end{aligned} \tag{2.10}$$

Then from (2.9)

$$\|\xi - \pi(a)^*\xi_0\| < \varepsilon\sqrt{n} \quad \text{and} \quad \|\eta - \pi(c)\xi_0\| < \varepsilon\sqrt{n}. \tag{2.11}$$

Since ψ is a contractive B-module map, we have

$$\Big\| \sum_{j=1}^{n} \pi(b_j)\eta_j \Big\| = \Big\| \sum_{j=1}^{n} \pi(b_j)\psi(y_j) \Big\| = \Big\| \psi\Big(\sum_{j=1}^{n} b_j y_j \Big) \Big\| \leqslant \Big\| \sum_{j=1}^{n} b_j y_j \Big\|,$$

hence (and similarly)

$$\|\pi(b)^*\xi\| \leqslant \|xb\| \quad (b \in \mathrm{C}_n(B)) \quad \text{and} \quad \|\pi(b)\eta\| \leqslant \|by\| \quad (b \in \mathrm{R}_n(B)). \tag{2.12}$$

Thus, if $b \in \mathcal{S}$, then

$$\begin{aligned}\rho(abb^*a^*) &= \|\pi(b^*a^*)\xi_0\|^2 \\ &\leqslant (\|\pi(b)^*\xi\| + \|\pi(b)^*(\pi(a)^*\xi_0 - \xi)\|)^2 \\ &\leqslant \big(\|xb\| + \|\pi(b)\|\varepsilon\sqrt{n}\big)^2 \quad \text{(by (2.12) and (2.11))} \\ &\leqslant (1 + \alpha\varepsilon\sqrt{n})^2 \quad \text{(by definition of } \mathcal{S} \text{ and } \alpha) \\ &=: \beta.\end{aligned}$$

Similar arguments are valid for $b \in \mathcal{T}$, hence

$$\rho(abb^*a^*) \leqslant \beta \quad (b \in \mathcal{S}) \quad \text{and} \quad \rho(c^*b^*bc) \leqslant \beta \quad (b \in \mathcal{T}). \tag{2.13}$$

Set

$$K_1 = \{b^*a^* : \ b \in \mathcal{S}\}, \qquad K_2 = \{bc : \ b \in \mathcal{T}\},$$
$$L_1 = \{v^*v : \ v \in K_1\}, \qquad L_2 = \{v^*v : \ v \in K_2\}.$$

Since X and Y are normal modules over B, $\mathcal{S}$ and $\mathcal{T}$ are weak* closed; moreover, since f and g are invertible, $\mathcal{S}$ and $\mathcal{T}$ are bounded, hence weak* compact. Thus, K_1 and K_2 are also weak* compact. To verify that the subset $\mathcal{T}$ of $\mathrm{R}_n(B)$ is $B,\mathbb{C}$-absolutely convex, let $b_j \in \mathcal{T}$ $(j = 1,\dots,n)$ and let $\lambda_j \in \mathbb{C}$ and $d_j \in B$ satisfy $\sum|\lambda_j|^2 \leqslant 1$ and $\sum d_j d_j^* \leqslant 1$. Then to show that $\sum(d_j b_j \lambda_j)$ is in $\mathcal{T}$, just note that $\|(\sum d_j b_j \lambda_j)y\| = \|\sum d_j(b_j y)\lambda_j\| \leqslant \max_j \|b_j y\| \leqslant 1$. Similarly $\mathcal{S}$ is $\mathbb{C},B$-absolutely convex and it follows that K_1 and K_2 are $B,\mathbb{C}$-absolutely convex.

Now we deduce by Lemma 2.3 that L_1 and L_2 are convex weak* compact subsets of B_h (the self-adjoint part of B), hence the same holds for the convex hull $\mathrm{co}(L_1 \cup L_2)$ and therefore the set

$$L = \mathrm{co}(L_1 \cup L_2) - B^+$$

is weak* closed since B^+ (the positive part of B) is weak* closed. Set

$$L^\circ = \{\theta \in B^\sharp : \ \mathrm{Re}(\theta(v)) \leqslant 1 \ \forall v \in L\} \quad \text{and} \quad L_\circ = L^\circ \cap B_\sharp.$$

Since L is weak* closed and convex, $L_\circ$ is weak* dense in L° by a variant of the bipolar theorem. From (2.13) we have that $\rho \in \beta(L_1^\circ \cap L_2^\circ) = \beta(\mathrm{co}(L_1 \cup L_2))^\circ$, hence (since ρ is positive) $\rho \in \beta L^\circ$. Since $L_\circ$ is weak* dense in L°, there exists an $\omega_0 \in \beta L_\circ$ such that

$$\Big|(\omega_0 - \rho)\Big(\sum_{i,j=1}^n a_i d_{ij} c_j\Big)\Big| < \varepsilon \quad \text{and} \quad |(\omega_0 - \rho)(1)| < \varepsilon. \tag{2.14}$$

(Here d_{ij} are as in (2.7), thus d_{ij}, a_i and c_j are fixed.) Since $L \supseteq -B^+$ and $\omega_0 \in \beta L_\circ$, ω_0 is positive, hence $\omega = \omega_0/\omega_0(1)$ is a state. Since $\|\omega - \omega_0\| = \|(1 - \omega_0(1))\omega\| = |1 - \omega_0(1)| < \varepsilon$, we have from (2.14) that

$$\Big|(\omega - \rho)\Big(\sum_{i,j=1}^n a_i d_{ij} c_j\Big)\Big| < D\varepsilon, \tag{2.15}$$

where $D = 1 + \Big\| \sum_{i,j=1}^{n} a_i d_{ij} c_j \Big\|$. Let $\sigma : B \to \mathrm{B}(\mathcal{H})$ be the normal representation constructed from ω by the GNS construction and let $\eta_0 \in \mathcal{H}$ be the corresponding unit cyclic vector. From (2.15), (2.9) and (2.8) we deduce that

$$\begin{aligned}
&\Big| \sum_{i,j=1}^{n} \langle \sigma(a_i d_{ij} c_j)\eta_0, \eta_0 \rangle \Big| \\
&\qquad = \Big| \omega\Big(\sum_{i,j=1}^{n} a_i d_{ij} c_j \Big) \Big| > \Big| \rho\Big(\sum_{i,j=1}^{n} a_i d_{ij} c_j \Big) \Big| - D\varepsilon \\
&\qquad = \Big| \sum_{i,j=1}^{n} \langle \pi(a_i d_{ij} c_j)\xi_0, \xi_0 \rangle \Big| - D\varepsilon \qquad (2.16) \\
&\qquad > \Big| \sum_{i,j=1}^{n} \langle \pi(d_{ij})\psi(y_j), \phi(x_i)^* \rangle \Big| - D\varepsilon - n^2 \varepsilon \max_{i,j} \|d_{ij}\| (\|x\| + \|y\| + \varepsilon) \\
&\qquad > \Lambda_B(w) - r(\varepsilon),
\end{aligned}$$

where $r(\varepsilon)$ tends to 0 as $\varepsilon \to 0$.

Define $\Phi_0 \in \mathrm{B}(X, \mathcal{H}^*)_B$ and $\Psi_0 \in \mathrm{B}_B(Y, \mathcal{H})$, for all $b_j \in B$, by

$$\Phi_0\Big(\sum_{j=1}^{n} x_j b_j \Big) = \Big(\sum_{j=1}^{n} \sigma(b_j^* a_j^*)\eta_0 \Big)^*, \quad \Psi_0\Big(\sum_{j=1}^{n} b_j y_j \Big) = \sum_{j=1}^{n} \sigma(b_j c_j)\eta_0. \tag{2.17}$$

Since $\omega_0 \in \beta L_\circ$, $\omega = \omega_0/\omega_0(1)$ and $\|\omega - \omega_0\| < \varepsilon$, we have that $\omega \in \omega_0(1)^{-1}\beta L_\circ \subseteq (1-\varepsilon)^{-1}\beta L_\circ$, hence it follows from (2.17) (noting that $abb^*a^* \in L$ if $b \in \mathcal{S} \subseteq \mathrm{C}_n(B)$) that

$$\|\Phi_0(xb)\|^2 = \|\sigma(b^*a^*)\eta_0\|^2 = \omega(abb^*a^*) \leqslant (1-\varepsilon)^{-1}\beta \quad (b \in \mathcal{S})$$

and similarly

$$\|\Psi_0(by)\|^2 \leqslant (1-\varepsilon)^{-1}\beta \quad (b \in \mathcal{T}).$$

Thus, with $\delta = (1-\varepsilon)^{-1/2}\beta^{1/2} = (1-\varepsilon)^{-1/2}(1 + \alpha\varepsilon\sqrt{n})$, we have (recalling the definitions of $\mathcal{S}$ and $\mathcal{T}$) that $\|\Phi_0\| \leqslant \delta$ and $\|\Psi_0\| \leqslant \delta$. From (2.17), $\Phi_0(x_j) = (\sigma(a_j^*)\eta_0)^*$ and $\Psi_0(y_j) = \sigma(c_j)\eta_0$, hence we may rewrite (2.16) as

$$\Big| \sum_{i,j=1}^{n} \langle \sigma(d_{ij})\Psi_0(y_j), \Phi_0(x_i)^* \rangle \Big| > \Lambda_B(w) - r(\varepsilon).$$

Finally, setting $\Phi = (1/\delta)\Phi_0$ and $\Psi = (1/\delta)\Psi_0$, we have a normal cyclic Hilbert module $\mathcal{H}$ and we have contractions $\Phi \in \mathrm{B}(X, \mathcal{H}^*)_B$, $\Psi \in \mathrm{B}_B(Y, \mathcal{H})$ such that $|\sum \langle \sigma(d_{ij})\Psi(y_j), \Phi(x_i)^* \rangle|$ approaches $\Lambda_B(w)$ as ε tends to 0 since $r(\varepsilon) \to 0$ and $\delta \to 1$. Thus $\Lambda_B(w) = \lambda_B(w)$.

In general, when X and Y are not free, let $w = \sum_{j=1}^{n} x_j \otimes_B y_j \in X \otimes_B Y$ and

$$X_1 = X \oplus \mathrm{R}_n(B) \quad \text{and} \quad Y_1 = Y \oplus \mathrm{C}_n(B).$$

Since both norms Λ_B and λ_B respect isometric embeddings, it suffices to prove that $\Lambda_B(w) \leqslant \lambda_B(w)$ in $X_1 \otimes_B Y_1$. For each real $t > 0$ put

$$w(t) = \sum_{j=1}^{n} (x_j, te_j^{\mathrm{t}}) \otimes_B (y_j, te_j),$$

where $e_j = (0, \dots, 1, \dots, 0) \in \mathrm{C}_n(\mathbb{C}) \subseteq \mathrm{C}_n(B)$. Since the elements $x_j(t) := (x_j, te_j^{\mathrm{t}})$ $(j = 1, \dots, n)$ generate a free module in the above sense and similarly the $y_j(t) := (y_j, te_j)$, it follows that $\Lambda_B(w(t)) = \lambda_B(w(t))$. But, as t tends to 0, $\Lambda_B(w(t))$ tends to $\Lambda_B(w)$ (since $\Lambda_B(w(t) - w) \leqslant t \sum_{j=1}^{n} (\|x_j\| + \|y_j\| + t)$) and $\lambda_B(w(t))$ tends to $\lambda_B(w)$, hence $\Lambda_B(w) = \lambda_B(w)$. ∎

By Theorem 2.4 and Proposition 2.2 the injective norm is given by (2.1) where $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ are normal, hence using the condition for normality recalled in the last part of Theorem 4.2 below we conclude:

COROLLARY 2.5. *If $X \in {}_A\mathrm{NRM}_B$ and $Y \in {}_B\mathrm{NRM}_C$, then $X \overset{\Lambda}{\otimes}_B Y \in {}_A\mathrm{NRM}_C$.*

3. THE PROJECTIVE TENSOR PRODUCT OF CENTRAL BIMODULES

Throughout this section C is a unital Abelian C^*-algebra, $\widetilde{C}$ the universal von Neumann envelope of C in the standard form and $X, Y \in \mathrm{CRM}_C$.

REMARK 3.1. For an Abelian C^*-algebra C we denote by Δ the spectrum of C and by C_t the kernel of a character $t \in \Delta$. For a bimodule $X \in \mathrm{CRM}_C$ we consider the quotients $X(t) = X/[C_t X]$. Given $x \in X$ we denote by $x(t)$ the coset of x in $X(t)$. It is known (see p. 37, 41 of [6] and p. 71 of [20] or [17]) that the function

$$\Delta \ni t \mapsto \|x(t)\| \tag{3.1}$$

is upper semicontinuous and that

$$\|x\| = \sup_{t \in \Delta} \|x(t)\|. \tag{3.2}$$

We shall call the embedding

$$X \to \bigoplus_{t \in \Delta} X(t), \quad x \mapsto (x(t))_{t \in \Delta}$$

the *canonical decomposition* of X.

Let $X \overset{\gamma}{\otimes}_C Y$ be the quotient of the Banach space projective tensor product $X \overset{\gamma}{\otimes} Y$ by the closed subspace generated by all elements of the form $xc \otimes y - x \otimes$

cy ($x \in X$, $y \in Y$, $c \in C$). First we shall prove that $X \overset{\gamma}{\otimes}_C Y$ is a representable C-bimodule. In classical terminology, this means that $X \overset{\gamma}{\otimes}_C Y$ is C-locally convex, which simplifies the definition of the tensor product of C-locally convex modules ([13], p. 445) since it eliminates the need for Banach bundles.

Consider the canonical decompositions $X \to \bigoplus_{t\in\Delta} X(t)$ and $Y \to \bigoplus_{t\in\Delta} Y(t)$ along the spectrum Δ of C (see Remark 3.1). For each $t \in \Delta$ the C-balanced bilinear map

$$\kappa_t : X \times Y \to X(t) \overset{\gamma}{\otimes} Y(t), \quad \kappa_t(x,y) = x(t) \otimes y(t)$$

induces a contraction $\widetilde{\kappa}_t : X \overset{\gamma}{\otimes}_C Y \to X(t) \overset{\gamma}{\otimes} Y(t)$. Since the kernel of $\widetilde{\kappa}_t$ contains the submodule $C_t(X \overset{\gamma}{\otimes}_C Y)$ (where $C_t = \ker t$), $\widetilde{\kappa}_t$ induces a contraction

$$\mu_t : (X \overset{\gamma}{\otimes}_C Y)(t) \to X(t) \overset{\gamma}{\otimes} Y(t).$$

On the other hand, the natural bilinear map $X \times Y \to X \otimes Y \to (X \overset{\gamma}{\otimes}_C Y)(t)$ annihilates $C_tX \times Y$ and $X \times C_tY$, hence it induces a bilinear map $X(t) \times Y(t) \to (X \overset{\gamma}{\otimes}_C Y)(t)$ and therefore a linear map $\sigma_t : X(t) \overset{\gamma}{\otimes} Y(t) \to (X \overset{\gamma}{\otimes}_C Y)(t)$, which must be a contraction by the maximality of the cross norm γ. Clearly σ_t is inverse to μ_t and since both are contractions, they must be isometries. Thus, we have the isometric identification

$$(X \overset{\gamma}{\otimes}_C Y)(t) = X(t) \overset{\gamma}{\otimes} Y(t) \quad (t \in \Delta). \tag{3.3}$$

For each $w \in X \overset{\gamma}{\otimes}_C Y$ we denote by $w(t)$ the corresponding class in $X(t) \overset{\gamma}{\otimes} Y(t)$.

We begin with the following result.

THEOREM 3.2. *The natural contraction*

$$\kappa : X \overset{\gamma}{\otimes}_C Y \to \bigoplus_{t\in\Delta}(X(t) \overset{\gamma}{\otimes} Y(t)), \quad \kappa(x \otimes_C y) = (x(t) \otimes y(t))_{t\in\Delta} \tag{3.4}$$

is isometric, hence $X \overset{\gamma}{\otimes}_C Y$ is a representable C-bimodule.

For the proof we need some preparation. Set $Z = X \overset{\gamma}{\otimes}_C Y$. Since the C-bimodule $\bigoplus_{t\in\Delta} Z(t)$ is clearly representable and $Z(t) = X(t) \overset{\gamma}{\otimes} Y(t)$ by (3.3), it will suffice to prove that the map (3.4) is isometric. Further, since for each element $w \in X \overset{\gamma}{\otimes}_C Y$ its norm is equal to

$$\gamma_C(w) = \sup\{|\theta(w)| : \theta \in (X \overset{\gamma}{\otimes}_C Y)^\sharp, \ \|\theta\| \leqslant 1\},$$

it will suffice to show that

$$|\theta(w)| \leqslant \sup_{t\in\Delta} \|w(t)\| \tag{3.5}$$

for each θ in the unit ball of $X \overset{\gamma}{\otimes}_C Y$.

REMARK 3.3 (Definition). Given $\theta \in (X \overset{\gamma}{\otimes}_C Y)^\sharp$ (regarded as a bilinear form) and an open subset Λ of Δ, let us define that

$$\theta|\Lambda = 0 \iff \theta(x, cy) = 0 \ \forall c \in C = C(\Delta) \text{ with } \operatorname{supp} c \subseteq \Lambda \text{ and } \forall x \in X, \forall y \in Y.$$

If (Λ_j) is a family of open subsets of Δ with the union Λ and if $\theta|\Lambda_j = 0$ for all j, then a standard partition of unity argument shows that $\theta|\Lambda = 0$. It follows that there exists the largest open subset Λ of Δ such that $\theta|\Lambda = 0$; then $\Delta \setminus \Lambda$ is called the *support of* θ, denoted by $\operatorname{supp}\theta$.

LEMMA 3.4. *If θ is an extreme point of the unit ball of $(X \overset{\gamma}{\otimes}_C Y)^\sharp$ then* $\operatorname{supp}\theta$ *is a singleton.*

Proof. We can extend θ to a contractive bilinear form on $X^{\sharp\sharp} \times Y^{\sharp\sharp}$, denoted by θ again, such that the maps

$$X^{\sharp\sharp} \ni F \mapsto \theta(F, y) \quad (y \in Y) \quad \text{and} \quad Y^{\sharp\sharp} \ni G \mapsto \theta(x, G) \quad (x \in X) \tag{3.6}$$

are weak* continuous (see p. 12 of [5] if necessary). Since X and Y are representable, we may regard $X^{\sharp\sharp}$ and $Y^{\sharp\sharp}$ as normal dual bimodules over $\widetilde{C} = C^{\sharp\sharp}$ by [17] (this is explained in more detail also in the beginning of Section 4). In particular, for each bounded Borel function f on Δ and each $y \in Y$, fy is defined as an element of $Y^{\sharp\sharp}$. Thus, we may define a bilinear form $f\theta$ on $X \times Y$ by

$$(f\theta)(x, y) = \theta(x, fy),$$

which satisfies

$$(cf)\theta = c(f\theta) \quad (c \in C). \tag{3.7}$$

Using the separate weak* continuity of the maps (3.6) and the fact that the $\widetilde{C}$-bimodules $X^{\sharp\sharp}$ and $Y^{\sharp\sharp}$ are normal, it also follows that

$$\theta(xc, y) = \theta(x, cy) \quad (c \in \widetilde{C},\ x \in X,\ y \in Y). \tag{3.8}$$

Suppose that there exist two different points $t_1, t_2 \in \operatorname{supp}\theta$. Choose an open neighborhood Δ_1 of t_1 such that $t_2 \notin \overline{\Delta}_1$ and let χ be the characteristic function of Δ_1. Then $\chi\theta \neq 0$. (Indeed, $\chi\theta = 0$ would imply for all $c \in C$ with support in Δ_1 that $c\theta = (c\chi)\theta = c(\chi\theta) = 0$ by (3.7), hence $\theta(x, cy) = (c\theta)(x, y) = 0$ for all x, y, thus $\theta|\Delta_1 = 0$, but this would contradict the fact that $t_1 \in \operatorname{supp}\theta$.) Similarly $(1 - \chi)\theta \neq 0$. Further,

$$\|\chi\theta\| + \|(1 - \chi)\theta\| = \|\theta\| = 1. \tag{3.9}$$

Indeed, given $x, u \in X$ and $y, v \in Y$, for suitable $\alpha, \beta \in \mathbb{C}$ of modules 1 we compute by using the property (3.8) that

$$\begin{aligned}|(\chi\theta)(x,y)| + |((1-\chi)\theta)(u,v)| &= \alpha(\chi\theta)(x,y) + \beta((1-\chi)\theta)(u,v)\\ &= \theta(x\chi, \alpha\chi y) + \theta(u(1-\chi), \beta(1-\chi)v)\\ &= \theta(x\chi + u(1-\chi), \alpha\chi y + \beta(1-\chi)v)\\ &\leqslant \|x\chi + u(1-\chi)\|\,\|\alpha\chi y + \beta(1-\chi)v\|\\ &\leqslant \max\{\|x\|, \|u\|\}\max\{\|y\|, \|v\|\}.\end{aligned}$$

This implies that $\|\chi\theta\| + \|(1-\chi)\theta\| \leqslant 1\ (= \|\theta\|)$, while the reverse inequality is immediate from $\theta = \chi\theta + (1-\chi)\theta$.

Setting $s = \|\chi\theta\|$, it follows that θ is the convex combination $\theta = s(s^{-1}\chi\theta) + (1-s)((1-s)^{-1}(1-\chi)\theta)$, where $s^{-1}\chi\theta$ and (by (3.9)) $(1-s)^{-1}(1-\chi)\theta$ are in the unit ball of $(X \overset{\gamma}{\otimes}_C Y)^\sharp$. This is a contradiction since θ is an extreme point. ∎

Proof of Theorem 3.2. As we have already noted, it suffices to prove (3.5). By the Krein-Milman theorem we may assume that θ is an extreme point in the unit ball of $X \overset{\gamma}{\otimes}_C Y$. Then by Lemma 3.4 $\operatorname{supp}\theta = \{t\}$ for some $t \in \Delta$. This implies that $\theta(XC_t, Y) = 0 = \theta(X, C_tY)$ since each $c \in C_t$ can be approximated by functions with supports in $\Delta \setminus \{t\}$. Consequently θ can be factored through $X(t) \times Y(t)$, in other words, there exists a contraction $\theta_t \in (X(t) \overset{\gamma}{\otimes} Y(t))^\sharp$ such that $\theta = \theta_t \circ \widetilde{\kappa}_t$. It follows that $|\theta(w)| \leqslant \|w(t)\|$ for each $w \in X \overset{\gamma}{\otimes}_C Y$. ∎

REMARK 3.5. If $Z \in \mathrm{CRM}_C$, then $\|w\| = \sup\{\|\phi(w)\| : \phi \in \mathrm{B}_C(Z, \widetilde{C}),\ \|\phi\| \leqslant 1\}$ (this is known, [20]); moreover, if $Z \in \mathrm{CNRM}_C$, then we may replace in this formula $\widetilde{C}$ by C. The later fact can be deduced from [17] by identifying the proper bimodule dual of Z with $\mathrm{B}_C(Z, C)$, but can also be deduced from an earlier result of Halpern ([10], Theorem 3) by representing Z (and C) in some $\mathrm{B}(\mathcal{H})$ and noting that then $Z \subseteq C'$ since Z is central.

COROLLARY 3.6. *For each* $w \in X \otimes_C Y$

$$\gamma_C(w) = \inf\Big\{\Big\|\sum_{j=1}^n c_j\Big\| :\ w = \sum_{j=1}^n c_j x_j \otimes_C y_j,\ c_j \in C^+,\ x_j \in B_X,\ y_j \in B_Y\Big\}, \tag{3.10}$$

hence ${}_{\mathbb{C}}X \overset{\gamma}{\otimes}_C Y_{\mathbb{C}} = {}_CX \overset{\gamma}{\otimes}_C Y_C$ *and this is just the usual projective tensor product* $X \overset{\gamma}{\otimes}_C Y$ *of Banach* C*-modules.*

Proof. Since by Theorem 3.2 $X \overset{\gamma}{\otimes}_C Y \in \mathrm{CRM}_C$, by Remark 3.5 the norm of $w \in X \overset{\gamma}{\otimes}_C Y$ is $\gamma_C(w) = \sup\{\|\phi(w)\| : \phi \in \mathrm{B}_C(X \overset{\gamma}{\otimes}_C Y, \widetilde{C}),\ \|\phi\| \leqslant 1\}$. For w of the form $w = \sum_{j=1}^{n} c_j x_j \otimes_C y_j$, where $c_j \in C^+$, $\|x_j\| \leqslant 1$, $\|y_j\| \leqslant 1$, and a contraction $\phi \in \mathrm{B}_C(X \overset{\gamma}{\otimes}_C Y, \widetilde{C})$ we have

$$\begin{aligned}\|\phi(w)\| &= \Big\| \sum_{j=1}^{n} c_j^{1/2} \phi(x_j \otimes_C y_j) c_j^{1/2} \Big\| \\ &\leqslant \|[c_1^{1/2}, \dots, c_n^{1/2}]\| \max_j \|\phi(x_j \otimes_C y_j)\| \left\| \begin{bmatrix} c_1^{1/2} \\ \vdots \\ c_n^{1/2} \end{bmatrix} \right\| \\ &\leqslant \Big\| \sum_{j=1}^{n} c_j \Big\| \max_j \|x_j \otimes_C y_j\| \leqslant \Big\| \sum_{j=1}^{n} c_j \Big\|.\end{aligned}$$

This implies that $\gamma_C(w)$ is dominated by the right side of (3.10). But, by definition

$$\gamma_C(w) = \inf\Big\{ \sum_{j=1}^{n} \lambda_j : \ w = \sum_{j=1}^{n} \lambda_j x_j \otimes_C y_j,\ \lambda_j \in \mathbb{R}^+,\ x_j \in B_X,\ y_j \in B_Y,\ n \in \mathbb{N} \Big\},$$

which clearly dominates the right side of (3.10) since $\mathbb{C} \subseteq C$. The conclusions of the corollary follow now from definitions of the corresponding norms. ∎

EXAMPLE 3.7. If C is an Abelian von Neumann algebra and $X, Y \in \mathrm{CNRM}_C$, then the representable C-bimodule $X \overset{\gamma}{\otimes}_C Y$ is not necessarily normal. To show this, we modify an idea from Example 3.1 in [12]. Let $U_0 \subseteq U$ and V be Banach spaces such that the contraction $U_0 \overset{\gamma}{\otimes} V \to U \overset{\gamma}{\otimes} V$ is not isometric. Choose $t_0 \in \Delta$ and set $X = \{f \in C(\Delta, U) : \ f(t_0) \in U_0\}$, $Y = C(\Delta, V)$. Then

$$X(t) = \begin{cases} U & \text{if } t \neq t_0, \\ U_0 & \text{if } t = t_0, \end{cases} \qquad \text{and} \quad Y(t) = V \quad \text{for all } t \in \Delta.$$

Choose $w = \sum_{j=1}^{n} u_j \otimes v_j \in U_0 \otimes V$ so that $\|w\|_{U \overset{\gamma}{\otimes} V} < \|w\|_{U_0 \overset{\gamma}{\otimes} V}$, denote by $\widetilde{u}_j$ and $\widetilde{v}_j$ the constant functions $\widetilde{u}_j(t) = u_j$ and $\widetilde{v}_j(t) = v_j$ and set $\widetilde{w} = \sum_{j=1}^{n} \widetilde{u}_j \otimes_C \widetilde{v}_j$. Then the function $t \mapsto \|\widetilde{w}(t)\|$, where $\widetilde{w}(t) = \sum_{j=1}^{n} \widetilde{u}_j(t) \otimes \widetilde{v}_j(t) \in X(t) \overset{\gamma}{\otimes} Y(t)$, is not continuous since $\|\widetilde{w}(t_0)\| = \|w\|_{U_0 \overset{\gamma}{\otimes} V} > \|w\|_{U \overset{\gamma}{\otimes} V} = \|w(t)\|$ if $t \neq t_0$. By the last sentence of Theorem 4.3 below this discontinuity implies that $X \overset{\gamma}{\otimes}_C Y$ is not normal. (We have used only one direction of Theorem 4.3, which was deduced in [16] from a special case in Lemma 10 of [9].)

4. THE NORMAL PROJECTIVE TENSOR PRODUCT

Since for bimodules $X, Y \in \mathrm{CNRM}_C$ the bimodule $X \overset{\gamma}{\otimes}_C Y$ is not necessarily in CNRM_C, we introduce in this section a new tensor product in the category CNRM_C.

We first recall the definition and the construction of the normal part of a bimodule.

DEFINITION 4.1. Let A be von Neumann algebra. The *normal part* of a bimodule $X \in {}_A\mathrm{RM}_A$ is a bimodule $X_\mathrm{n} \in {}_A\mathrm{NRM}_A$ together with a contraction $\iota \in \mathrm{B}_A(X, X_\mathrm{n})_A$ such that for each bimodule $Y \in {}_A\mathrm{NRM}_A$ and each $T \in \mathrm{B}_A(X, Y)_A$ there exists a unique map $T_\mathrm{n} \in \mathrm{B}_A(X_\mathrm{n}, Y)_A$ such that $T_\mathrm{n}\iota = T$ and $\|T_\mathrm{n}\| \leqslant \|T\|$.

By elementary categorical arguments X_n is unique (up to an A-bimodule isometry) if it exists. To sketch a construction of X_n, let $\Phi : A \to \mathrm{B}(\mathcal{G})$ be the universal representation and $\widetilde{A} = \overline{\Phi(A)}$ the universal von Neumann envelope of A. Let $P \in \widetilde{A}$ be the central projection such that the unique weak* continuous extension of the $*$-homomorphism Φ^{-1} has the kernel $P^\perp\widetilde{A}$ (see Section 10.1 of [11] for more details, if necessary). Consider X as a subbimodule in its second dual $X^{\sharp\sharp}$ equipped with the canonical bidual A-bimodule structure. Since X is representable, $X^{\sharp\sharp}$ can be equipped with a structure of a dual operator A-bimodule and by [2] or 5.4, 5.7 of [3] the bimodule action of A is necessarily induced by a pair of $*$-homomorphisms $\pi : A \to A_\mathrm{l}(X^{\sharp\sharp})$ and $\sigma : A \to A_\mathrm{r}(X^{\sharp\sharp})$, where $A_\mathrm{l}(X^{\sharp\sharp})$ and $A_\mathrm{r}(X^{\sharp\sharp})$ are certain fixed von Neumann algebras associated to the dual operator space $X^{\sharp\sharp}$ such that $X^{\sharp\sharp}$ is a normal dual operator $A_\mathrm{l}(X^{\sharp\sharp}), A_\mathrm{r}(X^{\sharp\sharp})$-bimodule. Then we may regard $X^{\sharp\sharp}$ as a normal dual operator $\widetilde{A}$-bimodule through the normal extensions of π and σ to $\widetilde{A}$. Now PXP is an A-subbimodule in $X^{\sharp\sharp}$, hence so is its norm closure $X_\mathrm{n} = \overline{\overline{PXP}}$. Finally, define $\iota : X \to X_\mathrm{n}$ by $\iota(x) = PxP$. If $T \in \mathrm{B}_A(X, Y)_A$, then $T^{\sharp\sharp} : X^{\sharp\sharp} \to Y^{\sharp\sharp}$ is an $\widetilde{A}$-bimodule map, hence it maps PXP into PYP. It can be proved [17] that for a normal bimodule $Y \in {}_A\mathrm{NRM}_A$ the map

$$\iota_Y : Y \to PYP \quad \iota_Y(y) = PyP,$$

is isometric, hence we have the factorization $T = T_\mathrm{n}\iota_X$, where $T_\mathrm{n} = \iota_Y^{-1}T^{\sharp\sharp}|\overline{\overline{PXP}}$. We summarize the discussion in the following theorem, which is proved in more details in [17].

THEOREM 4.2. ([17]) *Let A be a von Neumann algebra, $X \in {}_A\mathrm{RM}_A$ and regard X as an A-subbimodule in $X^{\sharp\sharp}$. Then $X^{\sharp\sharp}$ is a normal dual (representable) Banach $\widetilde{A}$-bimodule and the normal part of X is $X_\mathrm{n} = \overline{\overline{PXP}} \subseteq X^{\sharp\sharp}$ with $\iota : X \to X_\mathrm{n}$ the map $\iota(x) = PxP$. Moreover,*

$$\|\iota(x)\| = \inf\Big(\sup_j \|e_j x f_j\|\Big), \quad (x \in X) \tag{4.1}$$

where the infimum is taken over all nets (e_j) and (f_j) of projections in A converging to 1.

In particular $X \in {}_A\mathrm{NRM}_A$ if and only if for all nets of projections (e_j) in A and (f_j) in B converging to 1 *we have that $\lim_j \|e_j x\| = \|x\| = \lim_j \|x f_j\|$. If A is σ-finite it suffices to consider increasing sequences of projections instead of nets.*

We recall that a von Neumann algebra A is σ-finite if each orthogonal family of nonzero projections in A is countable. The last part of Theorem 4.2 was proved for one sided modules in Theorem 3.3 of [16] and this will suffice for our application here since we will consider central bimodules only.

Now we consider briefly the special case of central bimodules. For a function $f : \Delta \to \mathbb{R}$, let essup f be the infimum of all $c \in \mathbb{R}$ such that the set $\{t \in \Delta : f(t) > c\}$ is meager (= contained in a countable union of closed sets with empty interiors). Define the *essential direct sum*, $\operatorname{ess}\bigoplus_{t\in\Delta} X(t)$, of a family of Banach spaces $(X(t))_{t\in\Delta}$ as the quotient of the ℓ_∞-direct sum $\oplus_{t\in\Delta} X(t)$ by the zero space of the seminorm $x \mapsto \operatorname{essup} \|x(t)\|$. Then $\operatorname{ess}\bigoplus_{t\in\Delta} X(t)$ with the norm $\dot{x} \mapsto \operatorname{essup} \|x(t)\|$ is a Banach space and we denote by $e : \bigoplus_{t\in\Delta} X(t) \to \operatorname{ess}\bigoplus_{t\in\Delta} X(t)$ the quotient map.

THEOREM 4.3. ([17]) *Given a bimodule $X \in \mathrm{CRM}_C$ with the canonical decomposition $\kappa : X \to \bigoplus_{t\in\Delta} X(t)$ (see Remark* 3.1*), its normal part X_{n} is just the closure of $e\kappa(X)$ in* $\operatorname{ess}\bigoplus_{t\in\Delta} X(t)$. *Moreover, $X \in \mathrm{CNRM}_C$ if and only if for each $x \in X$ the function $\Delta \ni t \mapsto \|x(t)\|$ is continuous.*

DEFINITION 4.4. If $X, Y \in \mathrm{CNRM}_C$, let $X \overset{\nu}{\otimes}_C Y$ be the completion of $X \otimes_C Y$ with the norm

$$\nu_C(w) = \sup \|\phi(w)\|, \quad (w \in X \otimes_C Y),$$

where the supremum is over all C-bilinear contractions ϕ from $X \times Y$ into normal representable C-bimodules.

That ν_C is indeed a norm (not just a seminorm) follows since it dominates the Haagerup norm on $\mathrm{MIN}(X) \otimes_C \mathrm{MIN}(Y)$. (Namely, each completely contractive bilinear map is contractive. The definiteness of the Haagerup norm on $X \otimes_C Y$ follows from 1.1, 2.3 of [14]). We shall omit the easy proof of the following proposition (the last part of Theorem 4.2 may be used).

PROPOSITION 4.5. *If $X, Y \in \mathrm{CNRM}_C$, then $X \overset{\nu}{\otimes}_C Y \in \mathrm{CNRM}_C$ and for each bounded C-bilinear map $\psi : X \times Y \to Z \in \mathrm{CNRM}_C$ there exists a unique $\widetilde{\psi} \in \mathrm{B}_C(X \overset{\nu}{\otimes}_C Y, Z)$ such that $\widetilde{\psi}(x \otimes_C y) = \psi(x, y)$ for all $x \in X$, $y \in Y$, and $\|\widetilde{\psi}\| = \|\psi\|$.*

In particular, ν_C is the largest among the norms on $X \otimes_C Y$ such that $\|x \otimes_C y\| \leqslant \|x\|\|y\|$ and that (the completion of) $X \otimes_C Y$ with the norm $\|\cdot\|$ is a normal representable C-bimodule.

PROPOSITION 4.6. (i) $X \overset{\nu}{\otimes}_C Y = (X \overset{\gamma}{\otimes}_C Y)_{\mathrm{n}}$, *hence the canonical map*

$$X \overset{\nu}{\otimes}_C Y \to \operatorname{ess}\bigoplus_{t\in\Delta}(X(t) \overset{\gamma}{\otimes} Y(t))$$

is isometric.

(ii) $\nu_C\Big(\sum_{j=1}^n x_j \otimes_C y_j\Big) = \sup\Big\|\sum_{j=1}^n \theta(x_j, y_j)\Big\|$, *where the supremum is over all C-bilinear contractions from $X \times Y$ to C.*

(iii) $\nu_C\Big(\sum_{j=1}^n x_j \otimes_C y_j\Big) = \sup\Big\|\sum_{j=1}^n \theta(x_j, y_j)\Big\|$, *where the supremum is over all $\mathbb{C}$-bilinear C-balanced contractions $\theta : X \times Y \to \mathbb{C}$ such that the map $C \ni c \mapsto \theta(x, cy)$ is weak* continuous for all $x \in X$, $y \in Y$.*

Proof. (i) From Proposition 4.5 $X \overset{\nu}{\otimes}_C Y$ has the same universal property as the normal part of $X \overset{\gamma}{\otimes}_C Y$, hence they must be isometric as C-bimodules. Then the rest of (i) follows from Theorem 4.3 since $(X \overset{\gamma}{\otimes}_C Y)(t) = X(t) \overset{\gamma}{\otimes} Y(t)$ by (3.3).

(ii) This is a consequence of the fact that the norm of an element w in a bimodule $Z \in \mathrm{CNRM}_C$ is equal to $\sup\{\|\phi(w)\| : \phi \in \mathrm{B}(Z, C),\ \|\phi\| \leqslant 1\}$ (Remark 3.5).

(iii) For each $w = \sum_{j=1}^n x_j \otimes_C y_j \in X \otimes_C Y$ set

$$\widetilde{\nu}_C(w) = \sup\Big\|\sum_{j=1}^n \theta(x_j, y_j)\Big\|,$$

where the supremum is over all θ as in the statement (iii). Since for each $\rho \in C_\sharp$ of norm 1 and each C-bilinear contraction $\phi : X \times Y \to C$ the contraction $\theta = \rho \circ \phi$ is $\mathbb{C}$-bilinear and C-balanced, it follows that $\widetilde{\nu}_C(w) \geqslant \nu_C(w)$. To prove the reverse inequality, regard a C-balanced contraction $\theta : X \times Y \to \mathbb{C}$ as a linear functional on $V := X \overset{\gamma}{\otimes}_C Y$. If the functionals $c \mapsto \theta_{x,y}(c) = \theta(x, cy)$ are normal, then $\theta(w) = \lim_j \theta(e_j w)$ for each $w \in X \overset{\gamma}{\otimes}_C Y$ and each net of projections $e_j \in C$ converging to 1. Thus by (4.1) $\widetilde{\nu}_C(w) \leqslant \|\iota(w)\|$, where ι is the canonical map from $X \overset{\gamma}{\otimes}_C Y$ into $(X \overset{\gamma}{\otimes}_C Y)_{\mathrm{n}}$. But $\|\iota(w)\| = \nu_C(w)$ by (i), hence $\widetilde{\nu}_C(w) \leqslant \nu_C(w)$. ∎

We call a bimodule $Z \in \mathrm{CNRM}_C$ *strong* if $\sum_{j\in\mathbb{J}} p_j z_j \in Z$ for all bounded sets $(z_j) \subseteq Z$ and orthogonal families of projections $(p_j) \subseteq C$. (Note that the sum weak* converges in each $\mathrm{B}(\mathcal{H})$ containing Z as a normal operator C-bimodule. Since Z is central, this agrees with the definition of general strong bimodules

in [15].) Strong modules are characterized as closed in a certain topology [15], but here we shall only need that each bimodule $Z \in \mathrm{CNRM}_C$ is contained in a smallest strong bimodule, which follows from 2.2 of [15].

REMARK 4.7. Denote by B_X the closed unit ball of a normed space X. Let $X, Y \in \mathrm{CNRM}_C$. If $(x_j)_{j\in\mathbb{J}} \subseteq B_X$, $(y_j)_{j\in\mathbb{J}} \subseteq B_Y$ and $(c_j)_{j\subseteq\mathbb{J}} \subseteq C^+$ are such that $\sum_{j\in\mathbb{J}} c_j$ weak* converges, then the sum $\sum_{j\in\mathbb{J}} c_j x_j \otimes_C y_j$ weak* converges in every $\mathrm{B}(\mathcal{L})$ containing $X \overset{\nu}{\otimes}_C Y$ as a normal C-subbimodule since the sum is just the product of bounded operator matrices

$$\sum_{j\in\mathbb{J}} c_j x_j \otimes_C y_j = [c_j]_{j\in\mathbb{J}}^{1/2} \mathrm{diag}(x_j \otimes_C y_j)(c_j^{1/2})_{j\in\mathbb{J}}. \tag{4.2}$$

THEOREM 4.8. *Given $X, Y \in \mathrm{CNRM}_C$, let $X \overset{\nu}{\widetilde{\otimes}} Y$ be the smallest strong C-bimodule containing $X \overset{\nu}{\otimes}_C Y$. Then every $w \in X \overset{\nu}{\widetilde{\otimes}} Y$ can be represented in the form*

$$w = \sum_{j\in\mathbb{J}} c_j x_j \otimes_C y_j, \quad x_j \in B_X, y_j \in B_Y, c_j \in C^+, \tag{4.3}$$

where the sum $\sum_{j\in\mathbb{J}} c_j$ weak converges. The norm of w is equal to $\inf \left\| \sum_{j\in\mathbb{J}} c_j \right\|$ over all such representations.*

Proof. For $w \in X \overset{\nu}{\otimes}_C Y$ set $g(w) = \inf \left\| \sum_{j\in\mathbb{J}} c_j \right\|$, where the infimum is over all representations of w as in (4.3). (Since $X \overset{\nu}{\otimes}_C Y$ is just the norm completion of $X \otimes Y$, a representation of w of the form (4.3) is possible with the norm convergent series $\sum c_j$.) The inequality $\nu_C(w) \leqslant g(w)$ is proved by essentially the same computation as in the proof of Corollary 3.6. The reverse inequality follows from the maximality of ν_C (Proposition 4.5) since the completion W of $X \otimes_C Y$ with the norm g is a representable normal C-bimodule. The representability can be verified by using the characterization of representable bimodules ([16], Theorem 2.1 and [20]) and will be omitted here. To prove normality we may assume that C is σ-finite for in general C is a direct sum of σ finite algebras and X, Y and $X \otimes_C Y$ also decompose into the corresponding direct sums since these are central C-bimodules. If W is not normal, then by the last part of Theorem 4.2 there exist a sequence of projections $p_j \in C$ increasing to 1, an element $w \in W$ and a constant M such that $g(p_j w) < M < g(w)$ for all j. Setting $q_0 = p_0$ and $q_j = p_j - p_{j-1}$ if $j \geqslant 1$, we obtain an orthogonal sequence of projections q_j in C with the sum 1 such that $g(q_j w) < M$ for all j. Thus, for each j we can choose $x_{ij} \in B_X$ and $y_{ij} \in B_Y$ and positive elements $c_{ij} \in C$ such that

$$q_j w = \sum_{i\in\mathbb{I}} c_{ij} x_{ij} \otimes_C y_{ij} \quad \text{and} \quad \left\| \sum_{i\in\mathbb{I}} c_{ij} \right\| < M, \tag{4.4}$$

where $\mathbb{I}$ is a sufficiently large index set. Then $w = \sum_j q_j w = \sum_j \sum_{i\in\mathbb{I}} q_j c_{ij} x_{ij} \otimes_C y_{ij}$, hence (since the projections q_j are central and mutually orthogonal) $g(w) \leqslant \left\| \sum_j q_j \sum_{i\in\mathbb{I}} c_{ij} \right\| = \sup_j \left\| \sum_{i\in\mathbb{I}} c_{ij} \right\| \leqslant M$. But this contradicts the choice of M.

To prove that $X \overset{\nu}{\widetilde{\otimes}} Y$ consists of elements of the form (4.3), we may assume (by a direct sum decomposition argument again) that C is σ finite. Then the index set $\mathbb{J}$ in (4.3) may be taken to be countable. Given w as in (4.3), it follows by Egoroff's theorem ([23], p. 85) that there exists an orthogonal sequence of projections $p_k \in C$ with the sum 1 such that the sum $\sum_{j\in\mathbb{J}} c_j p_k$ is norm convergent for each k. Then the sum $w_k := \sum_{j\in\mathbb{J}} c_j p_k x_j \otimes_C y_j$ is also norm convergent (to see this, write w_k in the form similar to (4.2)), hence $w_k \in X \overset{\nu}{\otimes}_C Y$ and $w = \sum_k w_k p_k \in X \overset{\nu}{\widetilde{\otimes}} Y$. Conversely, for each $w \in X \overset{\nu}{\widetilde{\otimes}} Y$ there exists an orthogonal sequence of projections $p_k \in C$ such that $wp_k \in X \overset{\nu}{\otimes}_C Y$ by Proposition 2.2 of [15]. By the first paragraph of the proof $wp_k = \sum_j c_{jk} x_{jk} \otimes_C y_{jk}$ for some elements $x_{jk} \in B_X$, $y_{jk} \in B_Y$ and $c_{jk} = c_{jk} p_k \in C^+$ such that $\left\| \sum_j c_{jk} \right\| < \|wp_k\| + \varepsilon$, where $\varepsilon > 0$. Then $\left\| \sum_{j,k} c_{jk} \right\| \leqslant \|w\| + \varepsilon$ and $w = \sum_{j,k} c_{jk} x_{jk} \otimes_C y_{jk}$. This also proves that $g(w) \leqslant \|w\|$; the reverse inequality is clear from (4.2) by representing $X \overset{\nu}{\widetilde{\otimes}} Y$ as a normal operator C-bimodule. ∎

Since the quotient of a strong bimodule $X \in \mathrm{CNRM}_C$ by a strong subbimodule X_0 is a strong bimodule in CNRM_C by [17], we can state the following:

COROLLARY 4.9. *If $X_0 \subseteq X$ and $Y_0 \subseteq Y$ in CNRM_C are strong, then the canonical map $X \overset{\nu}{\widetilde{\otimes}} Y \to (X/X_0) \overset{\nu}{\widetilde{\otimes}} (Y/Y_0)$ maps the open unit ball onto the open unit ball.*

To conclude, we note without presenting the details that results analogous to the above ones also hold for the operator module versions of tensor products (that is, the module versions of tensor products of operator spaces studied in [4]).

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