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On weakly central C*-algebras

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Abstract

A unital C*-algebra A is weakly central if and only if for every $x \in A$ there exists a sequence of elementary unital completely positive maps α_n on A such that the sequence $(\alpha_n(x))$ converges to a central element.
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1. Introduction and the main result

A unital C*-algebra A has the *Dixmier property* if for each $x \in A$ the closure of the convex hull of the unitary orbit of x intersects the center of A. The class of such algebras includes all von Neumann algebras (see [6], [11, 8.3.5]; see also e.g. [9,15,16] for more on this and related questions) and several important examples of simple C*-algebras (see e.g. [3] and [5, VII.7.4]). However, in a general C*-algebra A, the existence of too many traces may be an obstruction to the Dixmier property: if A is simple and unital, then by [8] A has the Dixmier property if and only if A has at most one tracial state. In this note we shall show how to circumvent this multiple trace obstruction, by considering slightly more general maps than just the convex combinations of unitary similarities, to get a Dixmier type theorem valid for a class of C*-algebras as large as possible.

Throughout the paper A is a unital C-algebra.* An *elementary unital completely positive map* on A is a map of the form

$$\alpha(x) = \sum_{i=1}^{\ell} a_i^* x a_i \quad (x \in A),$$

where a_i are fixed elements of A such that $\sum_{i=1}^{\ell} a_i^* a_i = 1$. (For characterizations of completely positive maps among general elementary operators we refer to [1, Section 5.2].) The set of all such maps on A is denoted by EUCP(A). Clearly maps in EUCP(A) are contractive, preserve all ideals in A and the positive part A^+ of A, and act as the identity on the center Z of A. Moreover, EUCP(A) is a semigroup under composition of maps. By an *ideal* we shall mean a closed two-sided ideal.

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A unital C*-algebra A is called *weakly central* [14] if the map $\zeta : M \mapsto M \cap Z$ from the space $\operatorname{Max} A$ of all maximal ideals of A to the spectrum $\check{Z} := \operatorname{Max} Z$ of the center Z of A is injective. Examples include von Neumann algebras (and their quotients) [14], [11, 8.6.16], and unital C*-algebras with Hausdorff primitive spectrum (the latter are even central, a consequence of the Dauns–Hofmann theorem).

Theorem 1.1. *A unital C^*-algebra A is weakly central if and only if for each $x \in A$ there exists a sequence of maps $\alpha_n \in \mathrm{EUCP}(A)$ such that $\alpha_n(x)$ converge to an element $z \in Z$. Moreover, in this case, given a sequence (x_k) in A, the maps α_n can be chosen so that for each k the sequence $(\alpha_n(x_k))$ converges to a central element z_k.*

In some situations the above theorem can be used for C*-algebras in the same way as the Dixmier theorem is used for von Neumann algebras. For example, it implies Vesterstrøm's [17] equality $Z(A/J) = Z(A)/Z(J)$, where J is any ideal of a weakly central C*-algebra A and $Z(B)$ denotes the center of a C*-algebra B.

Corollary 1.2. *If A is separable, unital and weakly central then there exists a conditional expectation E from A onto the center Z of A such that $E(J) \subseteq J \cap Z$ for each ideal J of A. Moreover (if all Glimm ideals of A are primal) all conditional expectations from A to Z that preserve the ideals of A are in the point—norm closure of* $\mathrm{EUCP}(A)$.

Proof. Choose a dense sequence (x_k) in A. By Theorem 1.1 there is a sequence of maps $\alpha_n \in \mathrm{EUCP}(A)$ such that for each k the sequence $(\alpha_n(x_k))$ converges to an element $z_k \in Z$. Since the sequence (α_n) is bounded and (x_k) is dense in A, it follows that for each $x \in A$ the limit $E(x) := \lim_n \alpha_n(x)$ exists in Z. This defines a conditional expectation E from A onto Z (that is, a positive idempotent map on A with the range Z), which preserve ideals of A.

Let $E : A \to Z$ be a conditional expectation that preserves all ideals of A. Since the range Z of E is nuclear, by [13, 2.2] E can be approximated pointwise by a net (ψ_k) of complete contractions ψ_k on A of the form $\psi_k(x) = \sum_{i=1}^{\ell_k} a_{k,i}^* x b_{k,i}$, where $a_{k,i}, b_{k,i} \in A$. If the CB norm of every elementary operator on A coincides with the Haagerup norm of the corresponding tensor in $A \otimes_Z A$ (by [4] this is the case precisely when all Glimm ideals of A are primal), we may assume that for each k the two columns $a_k := (a_{k,1}, \ldots, a_{k,\ell_k})^T$ and $b_k := (b_{k,1}, \ldots, b_{k,\ell_k})^T$ have norm at most 1. (The same can be arranged in general, but for this we would need the observation that in the proofs of [13, 2.1 and 2.2] the unit ball $E_1(A)$ in the CB norm can be replaced by the unit ball in the Haagerup norm.) Then $a_k^* a_k \leqslant 1$ and $b_k^* b_k \leqslant 1$, hence (noting that $a_k^* b_k = \psi_k(1) \to E(1) = 1$)

$$\|a_k - b_k\|^2 = \left\|a_k^* a_k + b_k^* b_k - a_k^* b_k - b_k^* a_k\right\| \leqslant \left\|\big(1 - \psi_k(1)\big) + \big(1 - \psi_k(1)\big)^*\right\|$$

tends to 0, so that the net of maps $\phi_k(x) := a_k^* x a_k = \sum_i a_{k,i}^* x a_{k,i}$ on A also converges pointwise to E. In particular $c_k := \sqrt{1 - a_k^* a_k} = \sqrt{E(1) - \phi_k(1)} \to 0$, so that the maps $\theta_k(x) := \phi_k(x) + c_k x c_k$ converge pointwise to E, and $\theta_k \in \mathrm{EUCP}(A)$. $\square$

2. Proof of the theorem

In the easier direction the proof of Theorem 1.1 is essentially the same as Misonou's proof [14] that von Neumann algebras are weakly central, but let us nevertheless present the short argument for completeness. Assume for each $x \in A$ the existence of maps $\alpha_n \in \mathrm{EUCP}(A)$ such that the sequence $(\alpha_n(x))$ converges to a central element. Suppose that M_1 and M_2 were different maximal ideals of A such that $M_1 \cap Z = M_2 \cap Z$. By maximality, $1 = a_1 + a_2$, where $a_i \in M_i$. By hypothesis there exist $\alpha_n \in \mathrm{EUCP}(A)$ and $z_1 \in Z$ such that $z_1 = \lim_{n\to\infty} \alpha_n(a_1)$. Since $\alpha_n(a_1) \in M_1$, we have that $z_1 \in M_1 \cap Z$. Then the elements $\alpha_n(a_2) = 1 - \alpha_n(a_1)$ converge to $1 - z_1$ and, since $\alpha_n(a_2) \in M_2$, it follows that $1 - z_1 \in M_2 \cap Z$. But now $M_1 \cap Z = M_2 \cap Z$ contains both z_1 and $1 - z_1$, in contradiction with $1 \notin M_1$.

For the proof of the reverse direction we shall need the following variant for maps in $\mathrm{EUCP}(A)$ of an argument from the proof of Dixmier's theorem [11, 8.3.3, 8.3.4].

Lemma 2.1. *Suppose that for each $\varepsilon > 0$ and all $x \in A$ there exist $\alpha \in \mathrm{EUCP}(A)$ and $c \in Z$ such that $\|\alpha(x) - c\| < \varepsilon$. Then for each finite subset $\{x_1, \ldots, x_n\}$ of A there exist $\alpha \in \mathrm{EUCP}(A)$ and $c_j \in Z$ $(j = 1, \ldots, n)$ such that*

$$\left\|\alpha(x_j) - c_j\right\| < \varepsilon \quad \text{for all } j = 1, \ldots, n. \tag{2.1}$$

Moreover, for each sequence (x_k) in A there exists a sequence of maps $\alpha_n \in \mathrm{EUCP}(A)$ and elements $z_k \in Z$ such that $\lim_{n\to\infty} \alpha_n(x_k) = z_k$ for all k.

Proof. It follows by an induction on n that for each finite subset $\{x_1, \dots, x_n\}$ of A there exist maps $\beta_1, \dots, \beta_n$ in $\mathrm{EUCP}(A)$ and elements $c_j \in Z$ satisfying

$$\left\|\beta_n \dots \beta_1(x_j) - c_j\right\| < \varepsilon \quad \text{for all } j = 1, \dots, n. \tag{2.2}$$

The case $n = 1$ is just the hypothesis. Assuming (2.2) for some n and given $x_{n+1} \in A$, by hypothesis there exists $\beta_{n+1} \in \mathrm{EUCP}(A)$ and $c_{n+1} \in Z$ such that $\|\beta_{n+1}(\beta_n \dots \beta_1(x_{n+1})) - c_{n+1}\| < \varepsilon$. Since β_{n+1} is contractive and fixes elements of Z, we have that $\|\beta_{n+1}\beta_n \dots \beta_1(x_j) - c_j\| < \varepsilon$ also for all $j = 1, \dots, n$, which concludes the inductive step. To prove (2.1) we just put $\alpha = \beta_n \dots \beta_1$.

Given a sequence (x_k) in A, by applying what we have just proved, we can inductively find maps $\gamma_j \in \mathrm{EUCP}(A)$ and elements $c_{n,j} \in Z$ such that

$$\left\|\gamma_n \dots \gamma_1(x_j) - c_{n,j}\right\| < 2^{-n} \quad \text{for all } j = 1, \dots, n.$$

Assuming that for some n we already have γ_j and $c_{n,j}$ for all $j \leqslant n$, to find γ_{n+1} and $c_{n+1,j}$ $(j = 1, \dots, n+1)$, we apply the already proved part of the lemma to the subset $\{\gamma_n \dots \gamma_1(x_j)\colon\ j = 1, \dots, n+1\}$ and $\varepsilon = 2^{-n-1}$. Then for each fixed j the sequence $(c_{n,j})_{n \geqslant j}$ is Cauchy since

$$\|c_{n,j} - c_{n+1,j}\| = \left\|\gamma_{n+1}\big(c_{n,j} - \gamma_n \dots \gamma_1(x_j)\big) + \big(\gamma_{n+1}\gamma_n \dots \gamma_1(x_j) - c_{n+1,j}\big)\right\| < 2^{-n+1},$$

hence the limit $z_j := \lim_{n\to\infty} c_{n,j}$ exists, and $z_j \in Z$. With $\alpha_n := \gamma_n \dots \gamma_1 \in \mathrm{EUCP}(A)$, we have that $\lim_{n\to\infty} \alpha_n(x_j) = z_j$ for each j. □

Denote by $\check{A}$ the primitive spectrum of A (see [5,7]) and for each $x \in A$ and $P \in \check{A}$ let $x(P)$ be the coset of x in $A(P) := A/P$.

Lemma 2.2. *If A is weakly central, then for each $x \in A$ and $\varepsilon > 0$ there exist $\alpha \in \mathrm{EUCP}(A)$ and $z \in Z$ such that $\|\alpha(x) - z\| < \varepsilon$.*

Proof. Using an argument from the proof of Lemma 2.1 we see that it is sufficient to consider the case when $0 \leqslant x \leqslant 1$. We denote by $W(x)$ the (algebraic) numerical range of x, which in our case (since x is normal) is equal to the (closed) convex hull of the spectrum of x. It suffices to show that there exists an element $z \in Z$ such $W(z(P)) \subseteq W(x(P))$ for each primitive ideal P of A, for by [12, 4.1] this implies the existence of a map $\alpha \in \mathrm{EUCP}(A)$ satisfying $\|\alpha(x) - z\| < \varepsilon$. Note that each ideal P is contained in a maximal ideal M (since A is unital). Since $P \cap Z \subseteq M \cap Z$ and $P \cap Z$ is maximal in Z (because irreducible representations of A map Z to $\mathbb{C}$), it follows that $P \cap Z = M \cap Z$, hence $(Z + P)/P \cong Z/(Z \cap P) = Z/(Z \cap M) \cong (Z + M)/M$. This implies that $W(z(P)) = W(z(M))$ for each $z \in Z$. On the other hand, since A/M is a quotient of A/P, $W(x(M)) \subseteq W(x(P))$ for each $x \in A$. Thus, it suffices to find an element $z \in Z$ such that $W(z(M)) \subseteq W(x(M))$ for each maximal ideal M of A or, equivalently, that $z(t) \in W(x(M_t))$ for each $t \in \check{Z}$, where M_t denotes the unique maximal ideal of A such that $M_t \cap Z = t$ for each $t \in \check{Z}$. Note that we may identify $\operatorname{Max} A$ with $\check{Z}$ since the map $M \mapsto M \cap Z$ from $\operatorname{Max} A$ to $\check{Z}$ is continuous and, by our assumption about A, bijective, hence a homeomorphism (because $\check{Z}$ is Hausdorff and $\operatorname{Max} A$ compact). Now we could prove the existence of z by using the Michael selection theorem, but (as suggested by the referee) it is slightly easier to use the following argument from [2, p. 279]. Consider the functions $f, g : \operatorname{Max} A \to [0, \infty)$, defined by $f(M) = \|x(M)\|$ and $g(M) = 1 - \|1 - x(M)\|$ (the max and the min of the spectrum of $x(M)$, respectively, so that $W(x(M)) = [g(M), f(M)]$). Then f is lower semicontinuous and g is upper semicontinuous, hence by Katetov's theorem [10, p. 121] there exists a continuous function z such that $g \leqslant z \leqslant f$, which gives an element of Z with the desired property. □

Theorem 1.1 now follows from Lemmas 2.1 and 2.2.

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References

[1] P. Ara, M. Mathieu, Local Multipliers of C*-Algebras, Springer Monogr. Math., Springer-Verlag, Berlin, 2003.
[2] R.J. Archbold, On the norm of an inner derivation of a C*-algebra, Math. Proc. Cambridge Philos. Soc. 84 (1978) 273–291.
[3] R.J. Archbold, On the Dixmier property of certain algebras, Math. Proc. Cambridge Philos. Soc. 86 (1979) 251–259.
[4] R.J. Archbold, D.W.B. Somerset, R.M. Timoney, On the central Haagerup tensor product and completely bounded mappings of a C*-algebra, J. Funct. Anal. 226 (2005) 406–428.
[5] K.R. Davidson, C*-Algebras by Example, Fields Inst. Monogr., vol. 6, AMS, Providence, RI, 1996.
[6] J. Dixmier, Les anneaux d'opérateurs de classe finie, Ann. Sci. École Norm. Sup. 66 (1949) 209–261.
[7] J. Dixmier, Les C*-algèbres et leurs représentations, Gauthier–Villars, Paris, 1964.
[8] U. Haagerup, L. Zsidó, Sur la propriété de Dixmier pour les C^*-algèbres, C. R. Math. Acad. Sci. Paris Sér. I 298 (1984) 173–176.
[9] H. Halpern, Essential central spectrum and range for elements of a von Neumann algebra, Pacific J. Math. 43 (1972) 349–380.
[10] G.J.O. Jameson, Topology and Normed Spaces, Chapman and Hall, London, 1974.
[11] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras, vol. 2, Academic Press, London, 1986.
[12] B. Magajna, C*-convexity and the numerical range, Canad. Math. Bull. 43 (2000) 193–207.
[13] B. Magajna, Pointwise approximation by elementary complete contractions, preprint, http://www.fmf.uni-lj.si/magajna/Publications/osnovna.htm, October 2007.
[14] Y. Misonou, On a weakly central operator algebra, Tôhoku Math. J. 4 (1952) 194–202.
[15] S. Popa, On the relative Dixmier property for inclusions of C^*-algebras, J. Funct. Anal. 171 (2000) 139–154.
[16] J.R. Ringrose, On the Dixmier approximation theorem, Proc. London Math. Soc. 49 (1984) 27–57.
[17] J. Vesterstrøm, On the homomorphic image of the center of a C*-algebra, Math. Scand. 29 (1971) 134–136.