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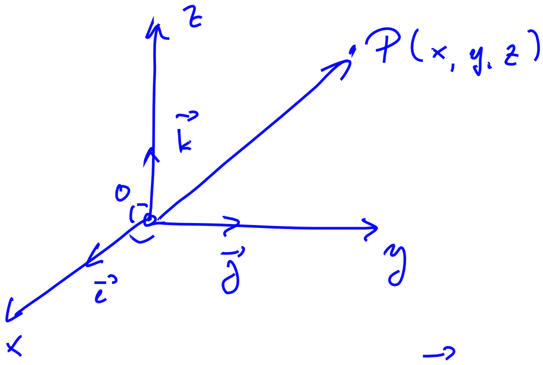
KINEMATIKA IN DINAMIKA

Kinematika

Položaj točke P, opazovalec O, kartezični koordinatni sistem x, y, z.



$$\vec{r} = \vec{OP} = P - O$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$ komponente

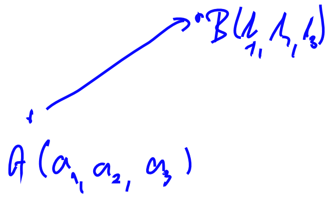
Koordinate točke $P(x, y, z)$, krajevni vektor $P - O = \vec{OP} = \vec{r}$ od izhodišča O do točke P.

V kartezičnem KS so komponente krajevnega vektorja $\vec{OP}$ enake koordinatam točke P:

$\vec e_1 = \vec i, \quad \vec e_2 = \vec j, \quad \vec e_3 = \vec k$

$\vec a = \sum_{i=1}^{3} a_i \vec e_i = a_i \vec e_i$

sumacijski dogovor



$\vec{AB} = B - A = (b_1 - a_1)\vec i + (b_2 - a_2)\vec j + (b_3 - a_3)\vec k$

$\vec a = a_1 \vec i + a_2 \vec j + a_3 \vec k = (a_1, a_2, a_3)$

$$\vec r = x\vec\imath + y\vec\jmath + z\vec k.$$

$\vec b = b_1 \vec i + b_2 \vec j + b_3 \vec k$

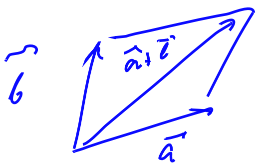
$\vec a + \vec b = (a_1 + b_1)\vec i + (a_2 + b_2)\vec j + (a_3 + b_3)\vec k$

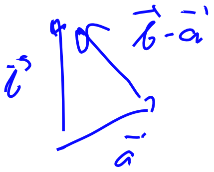
Osnovne vektorskega računa

1. bazni vektorji $\vec\imath$, $\vec\jmath$, $\vec k$;
2. seštevanje, odštevanje vektorjev, množenje vektorjev s skalarjem;
3. velikost vektorja;
4. skalarni produkt;
5. vektorski produkt.

$\vec a - \vec b = (a_1 - b_1)\vec i + \dots$

$\alpha \vec a = \alpha a_1 \vec i + \alpha a_2 \vec j + \alpha a_3 \vec k$





$\vec a + (\vec b - \vec a) = \vec a + \vec b - \vec a = \vec b$

$|\vec a| = |a_1 \vec i + a_2 \vec j + a_3 \vec k| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ $\qquad \vec a = \vec{OA}$

$\vec a$ je enotski vektor, če je $|\vec a| = 1$

$\vec b$ poljubni vektor; $\vec b \neq \vec 0$

$\frac{\vec b}{|\vec b|}$ enotski vektor v smeri $\vec b$

$\boxed{\vec b \mapsto \frac{\vec b}{|\vec b|} \ \text{normiranje}}$

$\left|\frac{\vec b}{|\vec b|}\right| = \sqrt{\frac{b_1^2}{|\vec b|^2} + \left(\frac{b_2}{|\vec b|}\right)^2 + \left(\frac{b_3}{|\vec b|}\right)^2} = \frac{1}{|\vec b|}\sqrt{b_1^2 + b_2^2 + b_3^2} = 1$

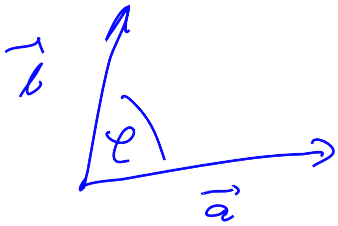
$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\vec{a}\cdot\vec{a} = a_1^2 + a_2^2 + a_3^2 = |\vec{a}|^2 \quad ; \quad |\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$$

$$\vec{a}\cdot(\vec{b}\pm\vec{c}) = \vec{a}\cdot\vec{b} \pm \vec{a}\cdot\vec{c}$$

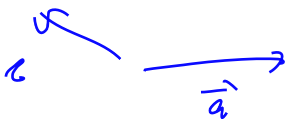
$$\vec{a}\cdot(\beta\vec{b}) = \beta(\vec{a}\cdot\vec{b})$$

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cdot\cos\varphi$$

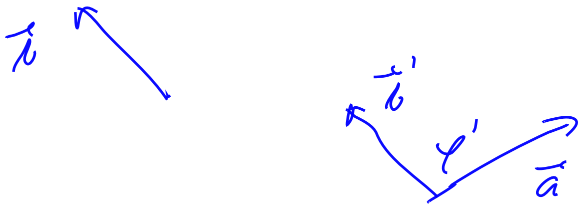


$\vec{a}\cdot\vec{b} > 0$; kot je ostri

$\vec{a}\cdot\vec{b} < 0$

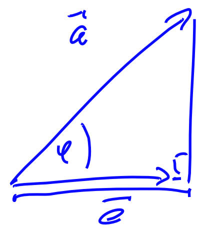


$\vec{b} \parallel \vec{b}'$



$|\vec{b}| = |\vec{b}'|$

$\varphi' = \varphi$



$|\vec{b}| = 1$ $\quad \vec{b} = \vec{e}$; $|\vec{e}|$ $\quad \vec{a}\cdot\vec{e} = |\vec{a}|\cdot\cos\varphi$ $\quad \cos\varphi = \dfrac{\vec{a}\cdot\vec{c}}{|\vec{a}|}$

$$\vec{a}\times\vec{b} = \vec{c} \qquad \vec{a}\times(\vec{b}_1+\vec{b}_2) = (\vec{a}\times\vec{b}_1) + (\vec{a}\times\vec{b}_2)$$

$$(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k}) =$$

$$a_1 b_1\, \vec{i}\times\vec{i} + a_1 b_2\, \vec{i}\times\vec{j} + a_1 b_3\, \vec{i}\times\vec{k} +$$

$$a_2 b_1\, \vec{j}\times\vec{i} + a_2 b_2\, \vec{j}\times\vec{j} + a_2 b_3\, \vec{j}\times\vec{k} +$$

$$a_3 b_1\, \vec{k}\times\vec{i} + a_3 b_2\, \vec{k}\times\vec{j} + a_3 b_3\, \vec{k}\times\vec{k}$$

$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0}$ ✓

$\vec{i}\times\vec{j} = -\vec{j}\times\vec{i}$

$\vec{i}\times\vec{k} = -\vec{k}\times\vec{i}$ ✓

$\vec{j}\times\vec{k} = -\vec{k}\times\vec{j}$

$\{\ \vec{i}\times\vec{j} = \vec{k},\ \vec{j}\times\vec{k} = \vec{i}\ ;\ \vec{k}\times\vec{i} = \vec{j}$ ✓

$$\vec{a}\times\vec{b} = (a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k} \ ✓$$

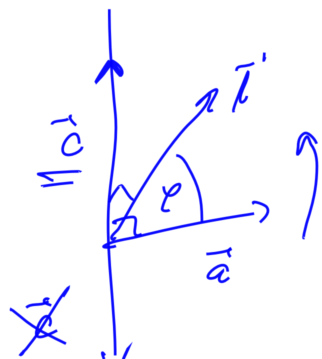
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\vec{i} + (a_3 b_1 - a_1 b_3)\vec{j} + (a_1 b_2 - a_2 b_1)\vec{k} \ ✓$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$\boxed{\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}}$$

$\{\ \vec{a}\times\vec{b} = \vec{c}$ $\qquad \vec{c}\perp\vec{a}$ in $\vec{c}\perp\vec{b}$

$|\vec{c}| = |\vec{a}||\vec{b}|\sin\varphi$



$\vec a \parallel \vec b \iff \vec a \times \vec b = \vec 0$

$\vec a \perp \vec b \iff \vec a \cdot \vec b = 0$

Gibanje točke

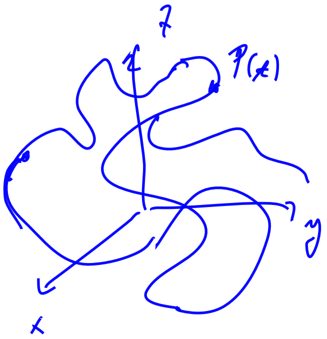
$\vec r = \vec{OP}$ $\quad P(x, y, z)$

Zapis $\vec r = \vec r(t)$.

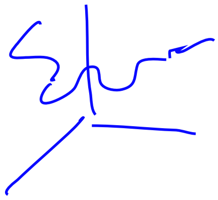
t čas

τ tam

$x = x(t), \quad y = y(t), \quad z = z(t)$



tir gibanja



$\vec r(t_1)$; $\vec r(t_2)$

$\frac{\vec r(t_2) - \vec r(t_1)}{t_2 - t_1} = \frac{\Delta \vec r}{\Delta t}$

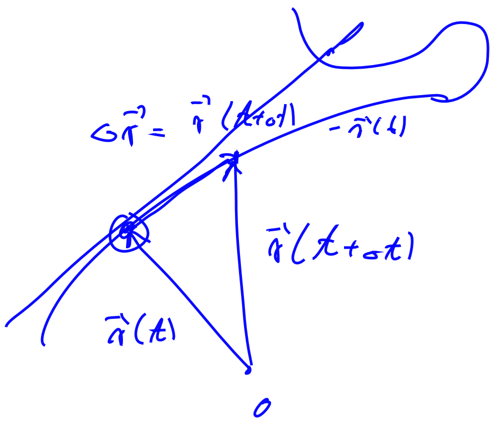
Vektor hitrosti: trenutna sprememba položaja po času, oziroma odvod krajevnega vektorja po času.

Kartezični zapis

$$\vec v = \dot x\vec\imath + \dot y\vec\jmath + \dot z\vec k.$$

$\lim_{\Delta t \to 0} \frac{\Delta \vec r}{\Delta t} = \vec v \qquad [\vec v] = \frac{m}{s}$

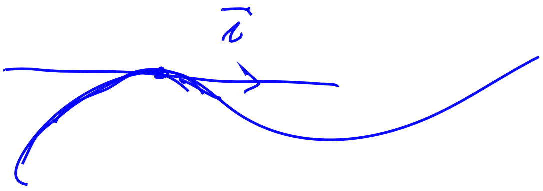
Geometrijski pomen vektorja hitrosti: vektor hitrosti je tangentni vektor na tir gibanja. Velikost vektorja hitrosti je brzina.



$\vec v$ se dotika tira

$\vec v$ ima smer tangente na tir

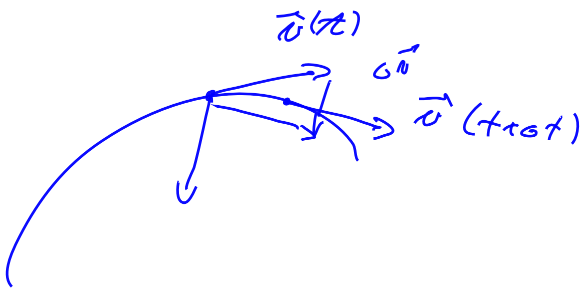
$|\vec v| = v$ brzina



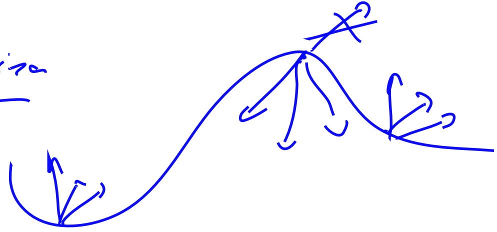
$\frac{\Delta \vec v}{\Delta t} = \frac{\vec v(t+\Delta t) - \vec v(t)}{\Delta t}$

$\lim_{\Delta t \to 0} \frac{\Delta \vec v}{\Delta t} = \vec a$ vektor pospeška = trenutna sprememba $\vec v$ po času

$\vec a = \frac{d\vec v}{dt} \qquad [\vec a] = \frac{m}{s^2}$



$\vec{a}$ kaže v smer zavijanja tira



$|\vec{a}| = a$

Vektor pospeška: trenutna sprememba vektorja hitrosti po času, oziroma odvod vektorja hitrosti po času.

Kartezični zapis

$$\vec{a} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}.$$

Geometrijski pomen vektorja pospeška: smer zavijanja.

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \;; \quad \vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}}$$

$$\dot{\vec{r}} = \frac{d}{dt}(x\vec{\imath} + y\vec{\jmath} + z\vec{k}) = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$$

$$\frac{d}{dt}\left(f(t)+g(t)\right)=\frac{df}{dt}+\frac{dg}{dt}=\dot{f}+\dot{g}$$

$$(fg)^{\circ}=\dot{f}g+f\dot{g}$$

$$\dot{f}=\lim_{\Delta t\to 0}\frac{\Delta f}{\Delta t}$$

$$(\alpha f)^{\circ}=\dot{\alpha}f+\alpha\dot{f}=\alpha\dot{f}$$

α konst; $f=f(t)$

Osnovne diferencialnega računa

1. $(f+g)^{\cdot}=\dot{f}+\dot{g}$;
2. $(fg)^{\cdot}=\dot{f}g+f\dot{g}$;
3. $f(t)=konst \iff \dot{f}=0$;
4. odvod afine funkcije: $f(t)=\alpha+\beta t+x_0 \Rightarrow \dot{f}=\beta$.
5. odvod kvadratne funkcije: $f(t)=\alpha+\beta t+\gamma t^2 \Rightarrow \dot{f}=\beta+2\gamma t$.
6. Odvod sinusa, kosinusa:
7. verižno pravilo $(f(g(t)))^{\cdot}=\dot{f}(g(t))\dot{g}$, $(\sin\omega t)^{\cdot}=\omega\cos\omega t$, $(\cos\omega t)^{\cdot}=-\omega\sin\omega t$.

$$(\alpha f+\beta g)^{\circ}=\alpha\dot{f}+\beta\dot{g}$$

odvod je linearna operacija

$f(t)=\alpha+\beta t$; α,β konstanti

$$\Delta f=f(t+\Delta t)-f(t)=\alpha+\beta(t+\Delta t)-(\alpha+\beta t)=\beta\Delta t$$

$$\dot{f}=\lim_{\Delta t\to 0}\frac{\Delta f}{\Delta t}=\lim_{\Delta t\to 0}\frac{\beta\Delta t}{\Delta t}=\lim_{\Delta t\to 0}\beta=\beta$$

$$(\alpha+\beta t)^{\circ}=\beta$$

$$f(t)=\alpha+\beta t+\gamma t^2$$

$$\dot{f}=(\alpha+\beta t+\gamma t^2)^{\circ}=\dot{\alpha}+(\beta t)^{\circ}+(\gamma t^2)^{\circ}=0+\beta+\gamma(t^2)^{\circ}$$

$$(t^2)^{\circ}=(t\cdot t)^{\circ}=\dot{t}\cdot t+t\dot{t}=t+t=2t$$

$$\dot{t}=1$$

$$(\alpha+\beta t+\gamma t^2)^{\circ}=\beta+2\gamma t$$

$$(t^n)^{\circ}=nt^{n-1}$$

$$p_n(t)=\alpha_0+\alpha_1 t+\alpha_2 t^2+\dots+\alpha_n t^n \quad\Big|\quad \dot{p}_n(t)=\alpha_1+2\alpha_2 t+3\alpha_3 t^2+\dots+n\alpha_n t^{n-1}$$

$f(t) = \sin t$

$$\lim_{\Delta t \to 0} \frac{\sin(t+\Delta t) - \sin t}{\Delta t} = \lim_{\Delta t \to 0} \frac{\sin t \cos \Delta t + \sin \Delta t \cos t - \sin t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cos t \sin \Delta t + \sin t (\underbrace{\cos \Delta t - 1}_{})}{\Delta t} = \cos t$$

1

$$\lim_{\Delta t \to 0} \frac{\cos t \sin \Delta t}{\Delta t} = \cos t \lim_{\Delta t \to 0} \frac{\sin \Delta t}{\Delta t} = \cos t$$

1

$$\lim_{\Delta t \to 0} \frac{\sin t (\cos \Delta t - 1)}{\Delta t} = \sin t \lim_{\Delta t \to 0} \frac{-2\sin^2 \frac{\Delta t}{2}}{\Delta t}$$

1 $\quad \to 0$

$$\cos \Delta t = \cos^2 \frac{\Delta t}{2} - \sin^2 \frac{\Delta t}{2}$$

$$1 = \cos^2 \frac{\Delta t}{2} + \sin^2 \frac{\Delta t}{2}$$

$$= -\sin t \lim_{\Delta t \to 0} \frac{\sin \frac{\Delta t}{2}}{\frac{\Delta t}{2}} \cdot \sin \frac{\Delta t}{2} = 0$$

$$(\sin t)' = \cos t$$

$$(\cos t)' = -\sin t$$

✓

$h(t) = f(g(t))$

$$t \mapsto g(t) \longmapsto f(g(t))$$

$h(t)$

$$h'(t) = \frac{d}{dt}\left(f(g(t))\right) = \underline{f'(g(t)) \cdot g'(t)}$$

$$\sin\left(t + \frac{\pi}{2}\right) = \sin t \underbrace{\cos\frac{\pi}{2}}_{0} + \underbrace{\sin\frac{\pi}{2}}_{=1} \cos t = \cos t$$

$$\underline{(\cos t)'} = \left(\sin\left(t+\frac{\pi}{2}\right)\right)' = \cos\left(t+\frac{\pi}{2}\right) \cdot 1 = \cos t \underbrace{\cos\frac{\pi}{2}}_{=0} - \sin t \underbrace{\sin\frac{\pi}{2}}_{=1}$$

$$t \mapsto t + \frac{\pi}{2} \to \sin\left(t + \frac{\pi}{2}\right)$$

$$= \underline{-\sin t}$$

$$t \mapsto \omega t \to \sin \omega t$$

$(\sin(\omega t))^\circ = (\cos\omega t)\,\omega = \omega\cos\omega t$

$(\cos(\omega t))^\circ = -\omega\sin\omega t$

Pojem pospeševanja, zaviranja:

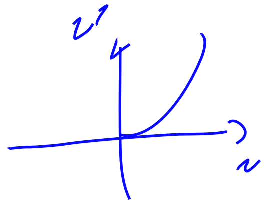
- točka pospešuje, če je $dv/dt > 0 \iff \vec{v}\cdot\vec{a} > 0$;
- točka zavira, če je $dv/dt < 0 \iff \vec{v}\cdot\vec{a} < 0$.

pospešuje $\iff$ hitrost narašča $\quad v(t+\Delta t) \geq v(t)$

$\Delta t > 0$

$\Delta v > 0$

funkcija (strogo) narašča $\iff$ njen odvod je pozitiven



pospešuje $\iff \dfrac{dv}{dt} \geq 0$

v narašča $\iff v^2$ narašča

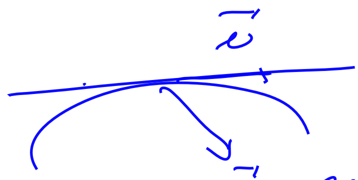
$v^2 = |\vec{v}|^2 = \vec{v}\cdot\vec{v}$ $\qquad \dfrac{dv^2}{dt} \geq 0$

$$\frac{dv^2}{dt} = \frac{d}{dt}(\vec{v}\cdot\vec{v}) = \dot{\vec{v}}\cdot\vec{v} + \vec{v}\cdot\dot{\vec{v}} = \vec{a}\cdot\vec{v} + \vec{v}\cdot\vec{a} = 2\vec{v}\cdot\vec{a}$$

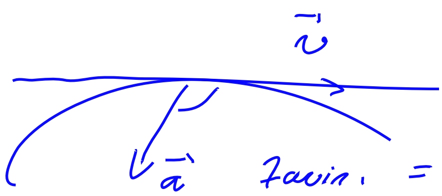
Osnovni primeri gibanja

- premočrtno gibanje: tir leži na premici $\iff \vec{a} \parallel \vec{v}$;

$\dfrac{dv}{dt} \geq 0 \iff \vec{v}\cdot\vec{a} \geq 0$



pospeš.



zavir. $= \dfrac{dv}{dt} \leq 0$

- enakomerno gibanje: $\iff \vec{a} = \vec{0} \iff \vec{v} = \vec{v}_0$, gibanje je premočrtno; Za začetne pogoje $x(t=t_0) = x_0$ je $x = v_0(t-t_0) + x_0$.

$\vec{a} = \vec{0}$ $\qquad \vec{a} = \dot{\vec{v}} \Rightarrow \vec{v} = \text{konst.}$ $\qquad \vec{a} \parallel \vec{v}$

- enakomerno pospešeno $\iff$ $\vec{a} = \vec{a}_0$, gibanje v splošnem ni premočrtno.

$\vec{a} =$ konst. $= \vec{a}_0 \iff \vec{v} = \vec{a}_0 t + \vec{v}_0$

$\dot{\vec{v}} = \vec{a}_0$

$\vec{a} \parallel \vec{v}$

Je premočrtno, samo če je $\vec{v}_0 \parallel \vec{a}_0$

$\vec{v}(t=0) = \vec{0}$

$\vec{v}_0 = 0$

Rešitev za premočrtno gibanje pri pogoju $x(t = t_0) = x_0$ in $\dot{x}(t = t_0) = v_0$ je

$$x = \frac{1}{2}a_0(t - t_0)^2 + v_0(t - t_0) + x_0.$$