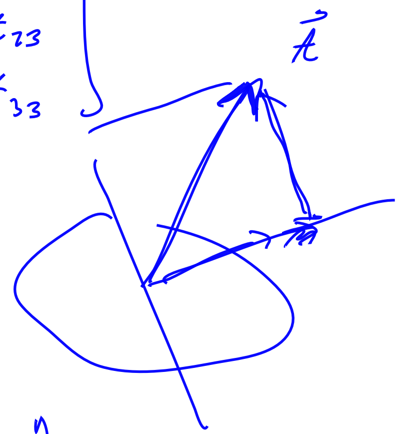
$$\underline{\underline{t}} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33} \end{bmatrix}$$

Predavanje 21. april 2021

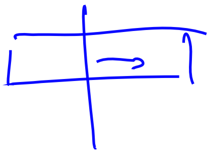
$$\vec{t} = \underline{\underline{t}}\,\vec{n}$$

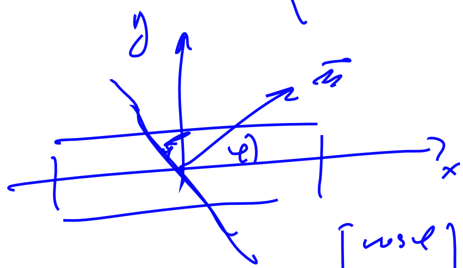


Osnovna napetostna stanja

- Enoosno napetostno stanje; izrčun normalne in strižne napetosti.

$$\underline{\underline{t}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$





$$\vec{n} = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix}$$

$$\vec{t} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma\cos\varphi \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{t} = \sigma\cos\varphi\,\vec{i}$$

$$\vec{t} = \vec{t}_n + \vec{t}_s$$

$$t_n = \vec{t}\cdot\vec{n} = \sigma\cos^2\varphi \qquad t_s^2 = |\vec{t}|^2 - t_n^2$$

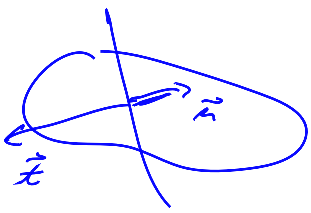
- Hidrostatično napetostno stanje $\underline{\underline{t}} = -p\,\underline{\underline{I}}$; v vsaki smeri je normalna napetost enaka $-p$, strižna napetost je enaka nič.

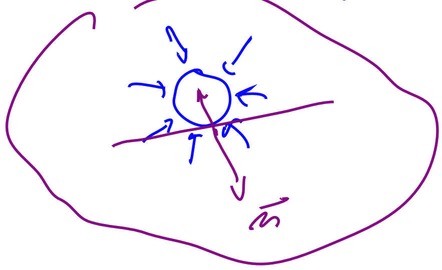
$$= \sigma^2\cos^2\varphi - \sigma^2\cos^4\varphi = \sigma^2\cos^2\varphi\sin^2\varphi = \frac{1}{4}\sigma^2\sin^2 2\varphi$$

$$\cos\varphi\sin\varphi = \frac{1}{2}\sin 2\varphi$$

$$t_s = \frac{1}{2}|\sigma\sin 2\varphi|$$

$$\underline{\underline{t}} = -p\begin{bmatrix} 1 & 0 & \\ 0 & 1 & \\ & & 1 \end{bmatrix} \qquad \vec{t} = \underline{\underline{t}}\,\vec{n} = -p\,\underline{\underline{I}}\,\vec{n} = -p\,\vec{n}$$





$\underline{\underline{t}}$ je krogelni tenzor

- Strižno napetostno stanje, obstaja KS v katerem je vsota diagonalnih elementov napetostnega tenzorja enaka nič. Izračun normalne in strižne napetosti.

$$\underline{\underline{t}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & -\sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Vsota diagonalnih elementov matrike je sled matrike.

Sled tenzorja je neodvisna od izbire KS.

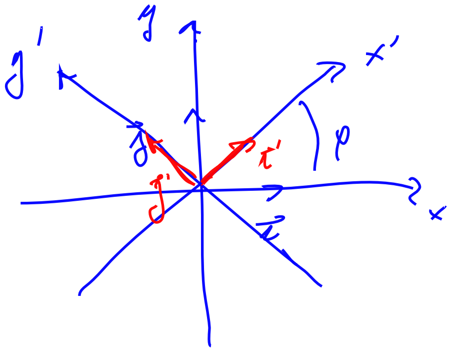
$$\underline{\underline{t}} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{t} = \underline{\underline{t}}\,\vec{n} = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \tau n_2 \\ \tau n_1 \\ 0 \end{bmatrix}$$

$$t_n = \vec{t}\cdot\vec{n} = \tau\begin{bmatrix} n_2 \\ n_1 \\ 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \tau(n_2 n_1 + n_1 n_2) = 2\tau n_1 n_2$$

$$t_s^2 = |\vec{t}|^2 - t_n^2 = \tau^2 n_2^2 + \tau^2 n_1^2 - 4\tau^2 n_1^2 n_2^2 = \tau^2(n_1^2 + n_2^2 - 4n_1^2 n_2^2)$$

- Ravninsko napetostno stanje.

$$\underline{\underline{t}} = \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{12} & t_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



Komponentni zapis tenzorja napetosti.

$$\vec{i}' = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j}$$

$$\vec{j}' = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}$$

$$|\vec{i}'| = 1,\quad |\vec{j}'| = 1;\quad \vec{i}'\cdot\vec{j}' = 0$$

$$\underline{\underline{t}} = \begin{bmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{bmatrix}$$

$$\underline{\underline{t}}\cdot\vec{i} = \begin{bmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} t_{11} \\ t_{12} \end{bmatrix}$$

$$\vec{i}\cdot\underline{\underline{t}}\,\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} t_{11} \\ t_{12} \end{bmatrix} = t_{11}$$

$$t_{12} = \vec{i}\cdot\underline{\underline{t}}\,\vec{j}\,;\qquad t_{22} = \vec{j}\cdot\underline{\underline{t}}\,\vec{j}$$

$$t'_{11} = \vec{i}'\cdot\underline{\underline{t}}\,\vec{i}' = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}\cdot\begin{bmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{bmatrix}\begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} =$$

~~Ravnovesna enačba~~

$$= \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix}\cdot\begin{bmatrix} t_{11}\cos\varphi + t_{12}\sin\varphi \\ t_{12}\cos\varphi + t_{22}\sin\varphi \end{bmatrix} =$$

$$= t_{11}\cos^2\varphi + t_{12}\cos\varphi\sin\varphi + t_{12}\sin\varphi\cos\varphi + t_{22}\sin^2\varphi =$$

$$\cos^2\varphi = \frac{1}{2}(1+\cos 2\varphi)$$

$$\sin^2\varphi = \frac{1}{2}(1-\cos 2\varphi)$$

$$\boxed{t'_{11} = \frac{1}{2}(t_{11}+t_{22}) + \frac{1}{2}(t_{11}-t_{22})\cos 2\varphi + t_{12}\sin 2\varphi}$$

$$t'_{22} = \vec{j}'\cdot\underline{\underline{t}}\,\vec{j}' = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}\cdot\begin{bmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{bmatrix}\begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix} =$$

$$= \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}\begin{bmatrix} -t_{11}\sin\varphi + t_{12}\cos\varphi \\ -t_{12}\sin\varphi + t_{22}\cos\varphi \end{bmatrix} =$$

$$= t_{11}\sin^2\varphi - t_{12}\sin\varphi\cos\varphi - t_{12}\cos\varphi\sin\varphi + t_{22}\cos^2\varphi =$$

$$t'_{22} = \frac{1}{2}(t_{11}+t_{22}) + \underbrace{\frac{1}{2}(-t_{11}+t_{22})\cos 2\varphi}_{-\frac{1}{2}(t_{11}-t_{22})\cos 2\varphi} - t_{12}\sin 2\varphi$$

Ravninsko napetostno stanje

Odvisnost komponent napetostnega tenzorja od postavitve koordinatnega sistema.

$$t'_{12} = \vec{i}' \cdot \underline{\underline{t}} \, \vec{j}' = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \cdot \begin{bmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{bmatrix} \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \begin{bmatrix} -t_{11}\sin\varphi + t_{12}\cos\varphi \\ -t_{12}\sin\varphi + t_{22}\cos\varphi \end{bmatrix} =$$

$$= -t_{11}\cos\varphi\sin\varphi + t_{12}\cos^2\varphi - t_{12}\sin^2\varphi + t_{22}\sin\varphi\cos\varphi =$$

$$t'_{12} = -\frac{1}{2}(t_{11} - t_{22})\sin 2\varphi + t_{12}\cos 2\varphi$$

Ravnovesna enačba

$$\frac{d}{dx}(A\sigma) + p(x) = 0$$

$$\operatorname{div} \underline{\underline{t}} + \vec{f} = \vec{0}$$

$\vec{f}$ volumenska gostota sil

$\rho\vec{g}$ je volumenska gostota sile teže.

$$\operatorname{div} \underline{\underline{t}} = \begin{bmatrix} \frac{\partial t_{11}}{\partial x} + \frac{\partial t_{12}}{\partial y} + \frac{\partial t_{13}}{\partial z} \\ \frac{\partial t_{12}}{\partial x} + \frac{\partial t_{22}}{\partial y} + \frac{\partial t_{23}}{\partial z} \\ \frac{\partial t_{13}}{\partial x} + \frac{\partial t_{23}}{\partial y} + \frac{\partial t_{33}}{\partial z} \end{bmatrix}$$

$$\frac{\partial}{\partial x}(x^2 y^3) = 2xy^3$$

$$\frac{\partial}{\partial y}(x^2 y^3) = 3x^2 y^2$$

$$\underline{\underline{t}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\operatorname{div} \underline{\underline{t}} = \begin{bmatrix} \frac{\partial \sigma}{\partial x} \\ 0 \\ 0 \end{bmatrix};$$

$$\frac{\partial \sigma}{\partial x}\vec{i} + \vec{f} = \vec{0}$$

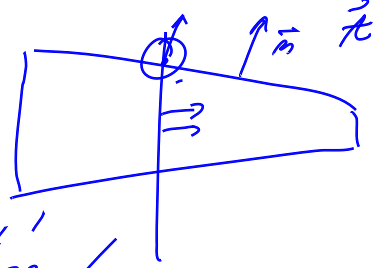
$$\vec{f} = f\vec{i};$$

$$\frac{\partial\sigma}{\partial x} + f = 0 \qquad f = \rho g \qquad \frac{d}{dx}(A\sigma) + p = 0$$

$$\frac{d}{dx}(A\sigma) + fA = 0 \qquad p = f \cdot A$$

V koordinatnem sistemu z osema $\vec{\imath}\,' = \cos\varphi\vec{\imath} + \sin\varphi\vec{\jmath}$ in $\vec{\jmath}\,' = -\sin\varphi\vec{\imath} + \cos\varphi\vec{\jmath}$ je

$$t'_{11} = \frac{1}{2}(t_{11} + t_{22}) + \frac{1}{2}(t_{11} - t_{22})\cos 2\varphi + t_{12}\sin 2\varphi,$$
$$t'_{22} = \frac{1}{2}(t_{11} + t_{22}) - \frac{1}{2}(t_{11} - t_{22})\cos 2\varphi - t_{12}\sin 2\varphi,$$
$$t'_{12} = -\frac{1}{2}(t_{11} - t_{22})\sin 2\varphi + t_{12}\cos 2\varphi.$$



$$\vec{t} = \underline{t}\vec{n} = 0$$

$$\mathrm{Sl}\,\underline{t} = t_{11} + t_{22} = t'_{11} + t'_{22} \checkmark$$

$$t'_{11} = a + b \qquad t'_{22} = a - b$$

$$t_{11} t_{22} - t_{12}^2 = t'_{11} t'_{22} - t'^{2}_{12}$$

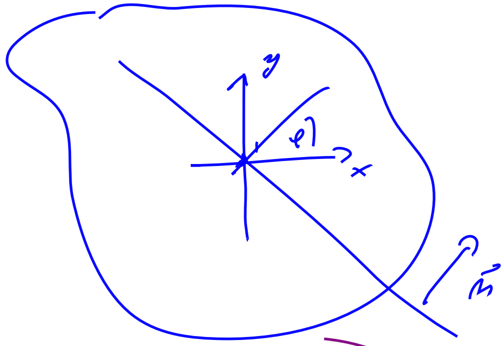
$$t'_{11} t'_{22} - t'^{2}_{12} = \left(\frac{1}{2}(t_{11} + t_{22})\right)^2 - \left(\frac{1}{2}(t_{11} - t_{22})\cos 2\varphi + t_{12}\sin 2\varphi\right)^2 - \left(-\frac{1}{2}(t_{11} - t_{22})\sin 2\varphi + t_{12}\cos 2\varphi\right)^2 =$$

$$= \left(\frac{1}{2}(t_{11} + t_{22})\right)^2 - \left(\left(\frac{1}{2}(t_{11} - t_{22})\right)^2 + t_{12}^2\right) = \frac{1}{4}\left(t_{11}^2 + t_{22}^2 + 2t_{11}t_{22}\right) - \frac{1}{4}\left(t_{11}^2 + t_{22}^2 - 2t_{11}t_{22}\right) - t_{12}^2 =$$

$$= t_{11} t_{22} - t_{12}^2$$

Invariante napetostnega tenzorja sta:

- sled napetostnega tenzorja (vsota diagonalnih elementov)
- determinanta napetostnega tenzorja



Ekstremalne lastnosti napetostnega tenzorja

Določitev smeri ekstremalne normalne napetosti

$$t_n = \sigma = t'_{11} = \frac{1}{2}(t_{11}+t_{22}) + \frac{1}{2}(t_{11}-t_{22})\cos 2\varphi + t_{12}\sin 2\varphi$$

$$A\cos 2\varphi + B\sin 2\varphi = \qquad A = \frac{1}{2}(t_{11}-t_{22})\,; \quad B = t_{12}$$

$$= C\cos(2\varphi-\delta) = C\cos 2\varphi\cos\delta + C\sin 2\varphi\sin\delta$$

$$\boxed{A = C\cos\delta} \qquad B = C\sin\delta \qquad A^2+B^2 = C^2 \Rightarrow C = \pm\sqrt{A^2+B^2}$$

$$A\neq 0 \quad \frac{B}{A} = \frac{C\sin\delta}{C\cos\delta} = \tan\delta \Rightarrow \delta = \arctan\frac{B}{A} + k\pi\,; \quad \delta\in(-\pi/2, \pi/2)$$

Izberimo $C>0$; $A>0$; ✓

$A<0$; $\cos(\delta+\pi) = -\cos\delta$

$\delta\in(\pi/2, 3\pi/2)$ ✓

$$t'_{11} = \frac{1}{2}(t_{11}+t_{22}) + \sqrt{A^2+B^2}\cos(2\varphi-\delta)$$

$\delta\in(-\pi/2, \pi/2)$ $A>0$

$\delta\in(\pi/2, 3\pi/2)$ $A<0$

$$\sigma_{1,2} = \frac{1}{2}(t_{11}+t_{22}) \pm \sqrt{\left(\frac{1}{2}(t_{11}-t_{22})\right)^2 + t_{12}^2} = \frac{1}{2}\left(t_{11}+t_{22} \pm \sqrt{(t_{11}-t_{22})^2 + 4t_{12}^2}\right)$$

$2\varphi-\delta = 0$ max ; $2\varphi-\delta = \pi$ min.

$t_{11} > t_{22}$ max: $2\varphi = \arctan\dfrac{t_{12}}{\frac{1}{2}(t_{11}-t_{22})} \Rightarrow \varphi_{max} = \dfrac{1}{2}\arctan\dfrac{2t_{12}}{t_{11}-t_{22}}$

min $\varphi_{min} = \dfrac{1}{2}\arctan\dfrac{2t_{12}}{t_{11}-t_{22}} + \dfrac{\pi}{2}$

$t_{11} < t_{22}$ max: $2\varphi - \arctan\dfrac{2t_{12}}{t_{11}-t_{22}} + \pi = 0 \Rightarrow \varphi = \dfrac{1}{2}\arctan\dfrac{2t_{12}}{t_{11}-t_{22}} - \dfrac{\pi}{2}$

min: $\varphi = \frac{1}{2}\,\mathrm{arctg}\,\frac{2t_{12}}{t_{11}-t_{22}}$

$$t_{-\vec{n}} = (-\vec{n})\cdot \underline{\underline{t}}(-\vec{n}) = t_n$$

Največja normalna napetost je

$$\sigma_{\max} = \frac{1}{2}\left(t_x + t_y + \sqrt{(t_x - t_y)^2 + 4t_{xy}^2}\right) = \frac{1}{2}\left(\mathrm{sl}\,\underline{\underline{t}} + \sqrt{(\mathrm{sl}\,\underline{\underline{t}})^2 - 4\det\underline{\underline{t}}}\right),$$

najmanjša pa

$$\sigma_{\min} = \frac{1}{2}\left(t_x + t_y - \sqrt{(t_x - t_y)^2 + 4t_{xy}^2}\right) = \frac{1}{2}\left(\mathrm{sl}\,\underline{\underline{t}} - \sqrt{(\mathrm{sl}\,\underline{\underline{t}})^2 - 4\det\underline{\underline{t}}}\right).$$

$t_x = t_{11}$

$t_y = t_{22}$

$t_{xy} = t_{12}$

$$(\mathrm{sl}\,\underline{\underline{t}})^2 - 4\det\underline{\underline{t}} = (t_{11}+t_{22})^2 - 4(t_{11}t_{22} - t_{12}^2) =$$

$$= \underbrace{t_{11}^2 + t_{22}^2 + \underbrace{2t_{11}t_{22} - 4t_{11}t_{22}}_{-2t_{11}t_{22}}}_{(t_{11}-t_{22})^2} - 4t_{12}^2 = (t_{11}-t_{22})^2 + 4t_{12}^2$$

Smeri, v katerih ima napetostni tenzor ekstremalno normalno napetost pravimo glavne smeri.

Glavni smeri sta med seboj pravokotni.

Smeri ekstremalne normalne napetosti sta

$$\varphi_\sigma^1 = \frac{1}{2}\arctan\frac{2t_{xy}}{t_x - t_y}, \quad \varphi_\sigma^2 = \varphi_\sigma^1 + \frac{\pi}{2}.$$

$$\varphi_{\max} = \frac{1}{2}\arctan\frac{2t_{12}}{t_{11} - t_{22}} + \begin{cases} 0 & t_{11} \geq t_{22} \\ \pi/2 & t_{11} < t_{22} \end{cases},$$

minimalna pa je

$$\varphi_{\min} = \varphi_{\max} + \frac{\pi}{2}.$$

Smeri največje in najmanjše normalne napetosti oklepata pravi kot.

Ekstremalnim normalnim napetostim pravimo tudi glavne napetosti, njunima smerema pa glavne smeri.

$$k_n = \operatorname{tg} \delta_n = \frac{2t_{12}}{t_{11}-t_{22}} \qquad k_n \cdot k_s = -1$$

Določitev smeri ekstremalne strižne napetosti

$$t_{12}' = -\frac{1}{2}(t_{11}-t_{22})\sin 2\varphi + \underline{t_{12}}\cos 2\varphi = C\cos(2\varphi-\delta)$$

$$\underline{C = \sqrt{\frac{1}{4}(t_{11}-t_{22})^2 + t_{12}^2}} \qquad \| \qquad \underline{C}\cos 2\varphi\,\underline{\cos\delta} + C\sin 2\varphi \sin\delta$$

$$k_s = \operatorname{tg}\delta = \frac{-\frac{1}{2}(t_{11}-t_{22})}{t_{12}} = -\frac{t_{11}-t_{22}}{2t_{12}} \qquad \left[\begin{array}{l} t_{12} = C\cos\delta \\ -\frac{1}{2}(t_{11}-t_{22}) = C\sin\delta \end{array}\right.$$

$$\sigma_{max} = \frac{1}{2}(t_{11}+t_{22}) + \sqrt{\frac{1}{4}(t_{11}-t_{22})^2 + t_{12}^2}$$

$$\sigma_{min} = \frac{1}{2}(t_{11}+t_{22}) - \sqrt{\frac{1}{4}(t_{11}-t_{22})^2 + t_{12}^2}$$

$$\frac{1}{2}(\sigma_{max} - \sigma_{min}) = \tau_{max}$$

Ekstremalna strižna napetost je

$$\tau_{\text{ext}} = \pm\frac{1}{2}(\sigma_{\max} - \sigma_{\min}).$$

V koordinatnem sistemu z osema v smereh ekstremalnih normalnih napetosti je strižna kompo-

$$\varphi_{ext} = \underline{-\frac{1}{2}\operatorname{arctg}}\frac{t_{11}-t_{22}}{2t_{12}};\ \left(+\frac{\pi}{2}\right)$$

nenta pripadajoče matrike napetostnega tenzorja enaka nič.

Pripadajoča smer ekstremalne stržne napetosti oklepa kot $\pi/4$ s smerjo ekstremalne normalne napetosti.

v ext. smeri

norm. nap.

$$t_{12}' = -\frac{1}{2}(t_{11}-t_{22})\sin 2\varphi + t_{12}\cos 2\varphi =$$

$$= \cos 2\varphi\left(-\frac{1}{2}(t_{11}-t_{22})\operatorname{tg} 2\varphi + t_{12}\right) \qquad \operatorname{tg} 2\varphi = \frac{2t_{12}}{t_{11}-t_{22}}$$

$$= \cos 2\varphi \left(-\frac{1}{2} \frac{\cancel{(t_{11}-t_{22})}}{\cancel{t_{11}-t_{22}}} \cdot 2t_{12} + t_{12} \right) = 0$$

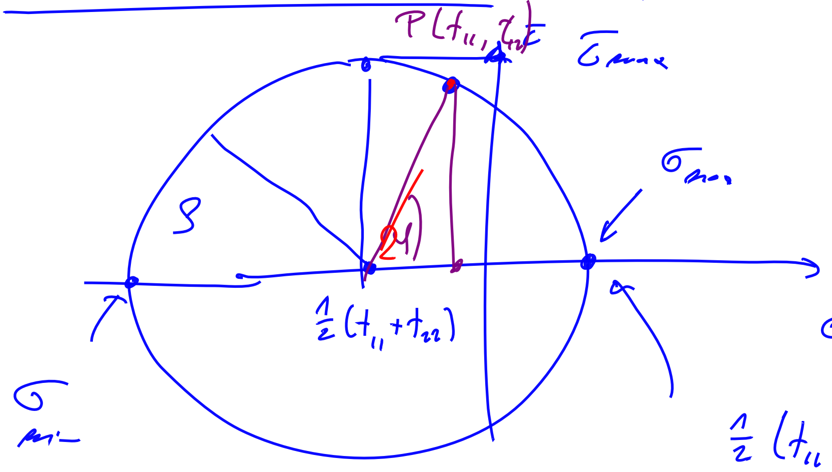
$$t = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

Mohrova kružnica

$$\sigma = t'_{11} = \frac{1}{2}(t_{11}+t_{22}) + \frac{1}{2}(t_{11}-t_{22})\cos 2\varphi + t_{12} \sin 2\varphi$$

$$\tau = t'_{12} = -\frac{1}{2}(t_{11}-t_{22}) \sin 2\varphi + t_{12} \cos 2\varphi$$

$$\left(\sigma - \frac{1}{2}(t_{11}+t_{22})\right)^2 + \tau^2 = \left(\frac{1}{2}(t_{11}+t_{22})\right)^2 + t_{12}^2 = \rho^2$$

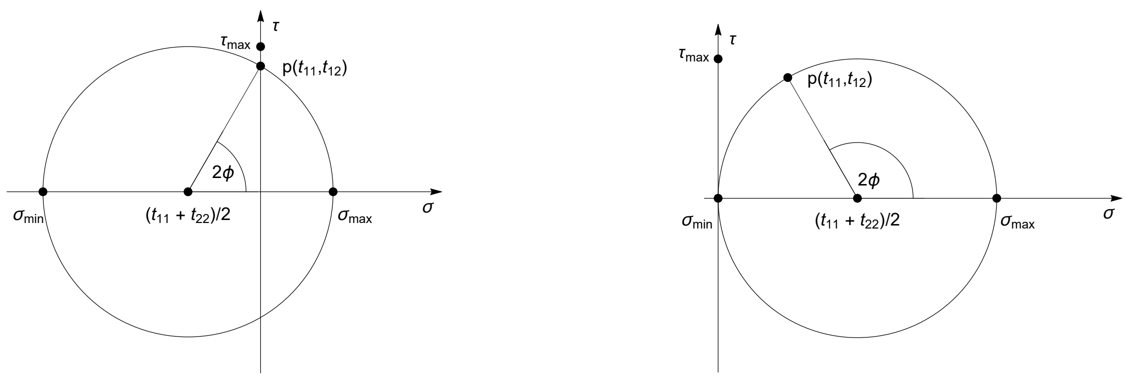


$$(x-a)^2 + y^2 = r^2$$

$$\rho = \sqrt{\frac{1}{2}(t_{11}+t_{22})^2 + t_{12}^2}$$

$$\frac{1}{2}(t_{11}+t_{22}) + \sqrt{} = \sigma_{max}$$

$$\operatorname{tg} 2\varphi = \frac{t_{12}}{t_{22} - \frac{1}{2}(t_{11}-t_{22})} = \frac{t_{12}}{\frac{1}{2}(t_{11}-t_{22})}$$



Slika 1: Mohrova kružnica. Levo $t_{11} > t_{22}$, desno $t_{11} < t_{22}$.

Mohrove krožnice osnovnih napetostnih stanj