

Predavanje 5. maj 2021

$$\vec{r} = \vec{R} + \vec{U}(\vec{R})$$

$$\vec{U}(\vec{R}) = \vec{u}(\vec{r})$$

$$\vec{U}(\vec{R}) - \vec{U}(\vec{R}_0) = \vec{U}(\vec{R}) - \vec{U}(\vec{R} + \vec{U}(\vec{R})) = -\text{grad} \vec{U} \cdot \vec{U}(\vec{R})$$

$$\vec{U}(\vec{R}) \approx \vec{U}(\vec{r}) = \vec{u}(\vec{r})$$

Majhna deformacija $\vec{U}(\vec{R}) \approx \vec{U}(\vec{r}) = \vec{u}(\vec{r})$.

$$\left. \begin{aligned} \varepsilon_2(\vec{n}) &= 2\vec{n} \cdot \underline{\underline{\varepsilon}} \vec{n} \\ \varepsilon_2 &\doteq 2\varepsilon_1 \end{aligned} \right\} \Rightarrow \varepsilon_1 = \vec{n} \cdot \underline{\underline{\varepsilon}} \vec{n} = \underline{\underline{m}} \cdot \underline{\underline{\varepsilon}} \vec{n}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{2} (\text{grad} \vec{U} + (\text{grad} \vec{U})^T) = \frac{1}{2} (\text{grad} \vec{u} + (\text{grad} \vec{u})^T)$$

Osnovni načini deformacij:

- enoosna;
- enakomerni razteg ali skrčitev, $\underline{\underline{\varepsilon}} = \varepsilon \underline{\underline{I}}$;
- strižna deformacija, $\text{sl} \underline{\underline{\varepsilon}} = 0$;
- ravninska deformacija; $u_1 = u_1(X, Y), u_2 = u_2(X, Y), u_3 = 0$.

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & \varepsilon_{12} & \varepsilon_{21} \end{bmatrix}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$b \rightarrow \sqrt{b^2 + (a\gamma)^2} = b\sqrt{1 + \left(\frac{a\gamma}{b}\right)^2} = b\left(1 + \frac{1}{2}\left(\frac{a\gamma}{b}\right)^2\right)$$

$$\sqrt{1+x} \doteq 1 + \frac{1}{2}x$$

$$\frac{a\gamma}{b} \ll 1$$

$$\underline{f(x+h) \doteq f(x) + f'(x) \cdot h}$$

$$x = r; \quad h = x$$

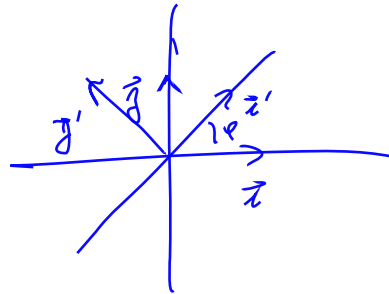
$$\epsilon_n = \sigma = \vec{n} \cdot \underline{\underline{\epsilon}} \vec{n}$$

$$\epsilon(\vec{n}) = \vec{n} \cdot \underline{\underline{\epsilon}} \vec{n}$$

$$\vec{n} = \vec{e}_1 ; \quad \epsilon(\vec{e}_1) = \epsilon_{11}$$

Ravninska deformacija

Odvisnost komponente deformacijskega tenzorja od koordinatnega sistema, analogija z napetostnim tenzorjem.



$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_x & \frac{1}{2} \gamma_{xy} \\ \frac{1}{2} \gamma_{xy} & \epsilon_y \end{bmatrix}$$

$$\epsilon_{12} = \frac{1}{2} \gamma_{xy}$$

$$\epsilon'_{12} = \frac{1}{2} \gamma'_{xy}$$

V koordinatnem sistemu z osema $\vec{i}' = \cos \varphi \vec{i} + \sin \varphi \vec{j}$ in $\vec{j}' = -\sin \varphi \vec{i} + \cos \varphi \vec{j}$ je

$$\begin{aligned} \epsilon'_{11} &= \frac{1}{2}(\epsilon_{11} + \epsilon_{22}) + \frac{1}{2}(\epsilon_{11} - \epsilon_{22}) \cos 2\varphi + \epsilon_{12} \sin 2\varphi, \\ \epsilon'_{22} &= \frac{1}{2}(\epsilon_{11} + \epsilon_{22}) - \frac{1}{2}(\epsilon_{11} - \epsilon_{22}) \cos 2\varphi - \epsilon_{12} \sin 2\varphi, \\ \epsilon'_{12} &= -\frac{1}{2}(\epsilon_{11} - \epsilon_{22}) \sin 2\varphi + \epsilon_{12} \cos 2\varphi. \end{aligned}$$

$$\gamma'_{xy} = -(\epsilon_x - \epsilon_y) \sin 2\varphi + \gamma_{xy} \cos 2\varphi$$

Osna deformacija v smeri $\vec{n} = \cos \varphi \vec{i} + \sin \varphi \vec{j}$ je

$$\epsilon_1(\vec{n}) = \vec{n} \cdot \underline{\underline{\epsilon}} \vec{n} = \epsilon_1(\varphi) = \frac{1}{2}(\epsilon_x + \epsilon_y) + \frac{1}{2}(\epsilon_x - \epsilon_y) \cos 2\varphi + \frac{1}{2} \gamma_{xy} \sin 2\varphi.$$

Ekstremalne lastnosti deformacijskega tenzorja

Smeri ekstremalne osne deformacije sta

$$\varphi_1^a = \frac{1}{2} \arctan \frac{2\epsilon_{xy}}{\epsilon_x - \epsilon_y}, \quad \varphi_2^a = \varphi_1^a + \frac{\pi}{2}.$$

$$\begin{aligned} \epsilon_{ext} &= \frac{1}{2} \left(\epsilon_{11} + \epsilon_{22} \pm \sqrt{\left(\frac{1}{2}(\epsilon_{11} - \epsilon_{22})\right)^2 + \epsilon_{12}^2} \right) = \\ &= \frac{1}{2} \left(\epsilon_{11} + \epsilon_{22} \pm \sqrt{(\epsilon_{11} - \epsilon_{22})^2 + 4\epsilon_{12}^2} \right) \end{aligned} \quad \epsilon_{12} = \frac{1}{2} \gamma_{xy}$$

Ekstremalna osna deformacija je

$$\epsilon_1^{\max, \min} = \frac{1}{2} \left(\epsilon_x + \epsilon_y \pm \sqrt{(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2} \right).$$

$$\vec{m} \perp \vec{n}$$

$$\vec{m} \cdot \underline{\underline{2\vec{n}}} = \vec{m} \cdot \underline{\underline{2\vec{m}}}$$

Mera strižne deformacije v ravnini med seboj pravokotnih enotskih vektorjev \vec{m} in \vec{n} je $\gamma(\vec{m}, \vec{n}) = 2\vec{m} \cdot \underline{\underline{\epsilon}}\vec{n}$.

Smeri ekstremalne osne deformacije oklepajo s smerema ekstremalne strižne deformacije kot $\pi/4$.

Ekstremalna strižna deformacija je

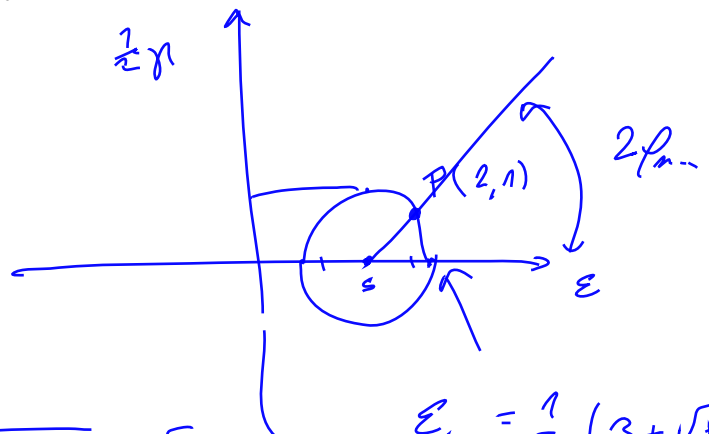
$$\gamma_{\text{ext}} = \pm(\epsilon_1^{\text{max}} - \epsilon_1^{\text{min}}).$$

V koordinatnem sistemu z osema v smereh ekstremalnih osnih deformacij je strižna komponenta pripadajočega deformacijskega tenzorja enaka nič.

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$S = \frac{1}{2} 3 \cdot \epsilon_0$$

$$S = \sqrt{\left(\frac{1}{2}(1)\right)^2 + 1^2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$



$$\epsilon_{\text{max}} = \frac{1}{2} (3 + \sqrt{5})$$

$$\epsilon_{\text{min}} = \frac{1}{2} (3 - \sqrt{5})$$

$$\frac{1}{2} \gamma_{\text{max}} = \frac{\sqrt{5}}{2}$$

$$\underline{\sigma} = \underline{E} \underline{\epsilon}$$

$$\underline{\epsilon} = \frac{1}{E} \underline{\sigma}$$

$$t_{ij} = \sum_{k,l=1}^3 C_{ijkl} \epsilon_{kl}$$

$$t_{ij} = t_{ji} \quad C_{ijkl} \quad \epsilon_{kl} = \epsilon_{lk}$$

Posplošeni Hookov zakon

so komponente elastičnega tenzorja

$$C_{ijkl} = C_{jikl} = C_{jilk}$$

$$\text{Vilja } \left\{ \begin{array}{l} C_{ijkl} = C_{klij} \\ C_{ijkl} = C_{klij} \end{array} \right\}$$

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Linearna zveza med napetostjo in deformacijo, elastični tenzor, podajnostni tenzor.

Zapis zveze med \underline{t} in $\underline{\epsilon}$, Voigtov zapis z elastično matriko reda 6×6 .

$$\epsilon_{ij} = \sum_{k,l=1}^3 S_{ijkl} t_{kl}$$

S_{ijkl} so komponente podajnostnega tenzorja

$$\underline{S} = \underline{E}^{-1}$$

$$\underline{t} \rightarrow \begin{bmatrix} t_{11} \\ t_{22} \\ t_{33} \\ \epsilon_{23} \\ \epsilon_{13} \\ t_{12} \end{bmatrix} \quad \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ & t_{22} & t_{23} \\ & & t_{33} \end{bmatrix}$$

$$\underline{\epsilon} \rightarrow \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{bmatrix}$$

Voigtov zapis

$$\begin{bmatrix} t_{11} \\ t_{22} \\ t_{33} \\ t_{23} \\ t_{13} \\ t_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & - & - & - & - \\ & & & C_{55} & C_{56} & \\ & & & & C_{66} & \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix}$$

Simetrije elastičnega tenzorja

1. anizotropija (21);
2. monoklinična (13);
3. ortotropična (9);
4. kubična (3);
5. tranzverzalna izotropija (5); $C_{66} = \frac{1}{2}(C_{11} - C_{12})$ oziroma $S_{66} = 2(S_{11} - S_{12})$.
6. izotropija (2);

$$\begin{array}{c|c} \vec{e}_1, \vec{e}_2, \vec{e}_3 & \vec{e}'_1, \vec{e}'_2, \vec{e}'_3 \\ \hline & \vec{e}'_1 = \vec{e}_1; \vec{e}'_2 = \vec{e}_2; \vec{e}'_3 = -\vec{e}_3 \end{array}$$

$$\boxed{C'_{ij} = C_{ij}}$$

$$\left[\begin{array}{l} \tau'_{ij} = \vec{e}'_i \cdot \underline{\underline{\tau}} \vec{e}'_j \\ \tau'_{11} = \tau_{11}; \quad \tau'_{22} = \tau_{22}, \quad \tau'_{33} = \vec{e}'_3 \cdot \underline{\underline{\tau}} \vec{e}'_3 = (-\vec{e}_3) \cdot \underline{\underline{\tau}} (-\vec{e}_3) = \tau_{33} \\ \tau'_{12} = \tau_{12}; \quad \tau'_{13} = \vec{e}'_1 \cdot \underline{\underline{\tau}} \vec{e}'_3 = \vec{e}_1 \cdot \underline{\underline{\tau}} (-\vec{e}_3) = -\tau_{13}; \quad \tau'_{23} = \tau_{23} \\ \varepsilon'_{ij} = \vec{e}'_i \cdot \underline{\underline{\varepsilon}} \vec{e}'_j \Rightarrow \varepsilon'_{13} = -\varepsilon_{13}; \quad \varepsilon'_{23} = -\varepsilon_{23} \end{array} \right.$$

$$\Leftrightarrow \tau_{11} = C_{11} \varepsilon_{11} + C_{12} \varepsilon_{22} + C_{13} \varepsilon_{33} + C_{14} 2\varepsilon_{23} + C_{15} 2\varepsilon_{13} + C_{16} 2\varepsilon_{12}$$

$$\tau'_{11} = C'_{11} \varepsilon'_{11} + C'_{12} \varepsilon'_{22} + C'_{13} \varepsilon'_{33} + C'_{14} 2\varepsilon'_{23} + C'_{15} 2\varepsilon'_{13} + C'_{16} 2\varepsilon'_{12}$$

$$\Leftrightarrow \tau_{11} = C_{11} \varepsilon_{11} + C_{12} \varepsilon_{22} + C_{13} \varepsilon_{33} - C_{14} 2\varepsilon_{23} - C_{15} 2\varepsilon_{13} + C_{16} 2\varepsilon_{12}$$

$$0 = 4C_{14} \varepsilon_{23} + 4C_{15} \varepsilon_{13} \Rightarrow \left. \begin{array}{l} C_{14j} = C_{15} = 0 \\ C_{24j} = C_{25} = 0 \\ C_{34j} = C_{35} = 0 \end{array} \right\}$$

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$$\boxed{C_{64} = C_{65} = 0} \quad \checkmark$$

$$\rightarrow \tau_{23} = C_{41} \epsilon_{11} + C_{42} \epsilon_{22} + C_{43} \epsilon_{33} + \underbrace{C_{44}}_2 2\epsilon_{23} + \underbrace{C_{45}}_2 2\epsilon_{13} + C_{46} 2\epsilon_{12}$$

$$\tau'_{23} = C_{41} \epsilon_{11} + C_{42} \epsilon_{22} + C_{43} \epsilon_{33} + C_{44} 2\epsilon'_{23} + C_{45} 2\epsilon'_{13} + C_{46} 2\epsilon_{12}$$

$$\rightarrow -\tau_{23} = C_{41} \epsilon_{11} + C_{42} \epsilon_{22} + C_{43} \epsilon_{33} - \underbrace{C_{44}}_2 2\epsilon_{23} - \underbrace{C_{45}}_2 2\epsilon_{13} + C_{46} 2\epsilon_{12}$$

$$0 = 2C_{41} \epsilon_{11} + 2C_{42} \epsilon_{22} + 2C_{43} \epsilon_{33} + 4C_{46} \epsilon_{12}$$

$$C_{41} = C_{41} = 0 \quad C_{42} = 0, \quad C_{43} = 0, \quad C_{46} = 0$$

$\begin{matrix} \text{"} \\ C_{24} \end{matrix}, \quad \begin{matrix} \text{"} \\ C_{34} \end{matrix}, \quad \boxed{C_{46} = 0}$

$$\left. \begin{matrix} \tau_{13} = \dots \\ \tau'_{13} = \dots \end{matrix} \right\} \Rightarrow \boxed{C_{56} = 0}$$

$$21 - 8 = \textcircled{13}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}$$

ortotropna simetrija

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

$$6 + 3 = 9$$

kubična simetrija

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix}$$

$$\textcircled{3}$$

transverzalna simetrija
invarianten za rotacije
duž osi \vec{e}_3

$$\left[\begin{matrix} C_{11} & C_{12} & C_{13} \\ & C_{11} & C_{13} \\ & & C_{33} \\ & & & C_{44} & C_{44} \\ & & & & C_{66} \end{matrix} \right]$$

$$\textcircled{5}$$

$$C_{66} = \frac{1}{2}(C_{11} - C_{12})$$

Isotropni material

$S_{66} = 2(S_{11} - S_{12})$

(2)

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{11} & C_{13} \\ C_{13} & C_{13} & C_{11} \end{bmatrix}$$

$C_{44} = \frac{1}{2}(C_{11} - C_{12})$

Podajnostni tenzor, zapis zveze med $\underline{\epsilon}$ in \underline{t} za ortotropičen material

$$\underline{S} = \begin{bmatrix} 1/E_1 & -\nu_{12}/E_2 & -\nu_{13}/E_3 & 0 & 0 & 0 \\ -\nu_{21}/E_1 & 1/E_2 & -\nu_{23}/E_3 & 0 & 0 & 0 \\ -\nu_{31}/E_1 & -\nu_{32}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/\mu_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/\mu_{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/\mu_{12} \end{bmatrix}$$

E_i Youngovi moduli, ν_{ij} Poissonovi količniki, strižni moduli $\mu_{ij} = G_{ij}$.

$S_{11} = \frac{1}{E_1}$ $S_{12} = -\frac{\nu_{12}}{E_2}$ $S_{13} = -\frac{\nu_{13}}{E_3}$ $S_{44} = \frac{1}{\mu_{23}}$

$S_{21} = -\frac{\nu_{21}}{E_1}$

$S_{12} = S_{21} \Rightarrow \frac{\nu_{12}}{E_2} = \frac{\nu_{21}}{E_1}$ $\nu_{21} = \frac{E_1}{E_2} \nu_{12}$

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \underline{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{13} \\ 2\epsilon_{12} \end{bmatrix} = \underline{S} \underline{t} = \begin{bmatrix} \epsilon_{11} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} \epsilon_{11} = \frac{1}{E_1} t_{11} \\ \epsilon_{22} = -\frac{\nu_{21}}{E_1} t_{11} \\ \epsilon_{33} = -\frac{\nu_{31}}{E_1} t_{11} \end{cases}$$

$\Rightarrow \epsilon_{22} = -\frac{\nu_{21}}{E_1} \epsilon_{11}$ $\epsilon_{33} = \frac{\nu_{31}}{E_1} \epsilon_{11}$

$\epsilon_{13} = \epsilon_{23} = \epsilon_{12} = 0$

$\nu_{21} = -\frac{\epsilon_{22}}{\epsilon_{11}} E_1$

$2\epsilon_{12} = \frac{1}{\mu_{12}} t_{12}$

$t_{12} = 2\mu_{12} \epsilon_{12}$

Isotropij: $E = E_1 = E_2 = E_3$; $\nu_{12} = \nu_{13} = \nu_{23} = \nu$
 $\mu_{23} = \mu_{13} = \mu_{12} = \mu = G$

$\frac{1}{G} = 2 \left(\frac{1}{E} + \frac{\nu}{E} \right) = \frac{2(1+\nu)}{E} \Rightarrow G = \frac{E}{2(1+\nu)}$

$\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} \\ & \frac{1}{E} & -\frac{\nu}{E} \\ & & \frac{1}{E} \end{bmatrix} \quad \underline{\epsilon} = \underline{S} \underline{t}$

$$S = \begin{pmatrix} \nu/E & & & \\ & -\nu/E & & \\ & & \nu/E & \\ & & & \dots \dots \nu/E \end{pmatrix}$$

$$\epsilon_{11} = \frac{1}{E} t_{11} - \frac{\nu}{E} t_{11} - \frac{\nu}{E} t_{22} = \frac{1-\nu}{E} t_{11} - \frac{\nu}{E} (t_{11} + t_{22} + t_{33})$$

Hookov zakon za izotropičen material

$$\epsilon_{ij} = \frac{1+\nu}{E} t_{ij} - \frac{\nu}{E} (t_{11} + t_{22} + t_{33}) \delta_{ij}$$

oziroma

$$\underline{\underline{\epsilon}} = \frac{1+\nu}{E} \underline{\underline{t}} - \frac{\nu}{E} (\text{sl } \underline{\underline{t}}) \underline{\underline{I}}$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$i=j=1,2,3$$

$$2\epsilon_{12} = \frac{1}{E} t_{12} \Rightarrow \epsilon_{12} = \frac{1}{2E} t_{12} = \frac{2(1+\nu)}{2E} t_{12} = \frac{1+\nu}{E} t_{12}$$

$$\nu=1, \nu=2$$

Zveza $G = \frac{E}{2(1+\nu)}$

$$\underline{\underline{t}} = 2\mu \underline{\underline{\epsilon}} + \lambda (\text{sl } \underline{\underline{\epsilon}}) \underline{\underline{I}}$$

$$\text{sl } \underline{\underline{t}} = 2\mu \text{sl } \underline{\underline{\epsilon}} + \lambda (\text{sl } \underline{\underline{\epsilon}}) 3 = (2\mu + 3\lambda) \text{sl } \underline{\underline{\epsilon}}$$

$$\underline{\underline{\epsilon}} = \frac{1}{2\mu} (\underline{\underline{t}} - \lambda \text{sl } \underline{\underline{\epsilon}} \underline{\underline{I}}) = \frac{1}{2\mu} \underline{\underline{t}} - \frac{\lambda}{2\mu(2\mu+3\lambda)} \text{sl } \underline{\underline{t}} \underline{\underline{I}}$$

Lamejeva koeficienta $\mu = G, \lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}$

$$\underline{\underline{t}} = 2\mu \underline{\underline{\epsilon}} + \lambda (\text{sl } \underline{\underline{\epsilon}}) \underline{\underline{I}}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)}, \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}$$

λ, μ

Lamejeva koeficienta

$$\frac{1}{2\mu} = \frac{1+\nu}{E} \Rightarrow \mu = \frac{E}{2(1+\nu)}$$

Enakomerna kompresija, kompresijski modul $\kappa = \frac{E}{3(1-2\nu)}$.

Nestisljivi material $\nu = \frac{1}{2}$, vrednosti Poissonovega količnika $\nu \in (-1, \frac{1}{2}]$.