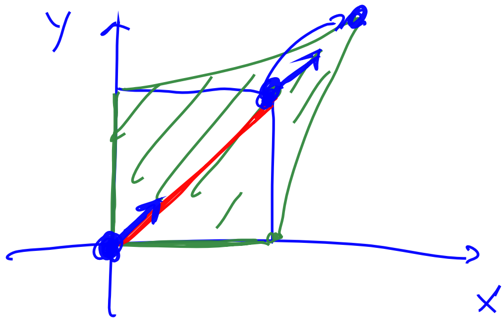
Vaje 6. maj 2021

1. Podana je funkcija pomika $\vec{U} = \alpha(XY^2\vec{\imath} + X^2Y\vec{\jmath})$.
   (a) Skiciraj, kam se preslika kvadrat $[0, a] \times [0, a]$.
   (b) Izračunaj infinitezimalni deformacijski tenzor.
   (c) Izračunaj deformacijski tenzor in ugotovi kdaj je deformacija majhna.



$\vec{r} = \vec{R} + \vec{U}$

$\vec{U}\big|_{X=0} = \vec{0}$

$\vec{U}\big|_{Y=0} = \vec{0}$

$X = a;\quad \vec{U} = \alpha(aY^2\vec{\imath} + a^2Y\vec{\jmath})$

$\alpha = a = 1$

$X = Y = a;\quad \vec{U} = \alpha(a^3\vec{\imath} + a^3\vec{\jmath})$ $\quad Y = a;\quad \vec{U} = \alpha(Xa^2\vec{\imath} + X^2a\vec{\jmath})$

$\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\operatorname{grad}\vec{U} + (\operatorname{grad}\vec{U})^T\right)$ $\qquad U_1 = \alpha XY^2;\quad U_2 = \alpha X^2Y$

$$\operatorname{grad}\vec{U} = \begin{bmatrix} \frac{\partial U_1}{\partial X} & \frac{\partial U_1}{\partial Y} \\ \frac{\partial U_2}{\partial X} & \frac{\partial U_2}{\partial Y} \end{bmatrix} = \alpha\begin{bmatrix} Y^2 & 2XY \\ 2XY & X^2 \end{bmatrix} \qquad \underline{\underline{\varepsilon}} = \alpha\begin{bmatrix} Y^2 & 2XY \\ 2XY & X^2 \end{bmatrix}$$

$$\underline{\underline{E}} = \underline{\underline{\varepsilon}} + \frac{1}{2}(\operatorname{grad}\vec{U})^T\operatorname{grad}\vec{U}$$

$$(\operatorname{grad}\vec{U})^T\operatorname{grad}\vec{U} = \alpha^2\begin{bmatrix} Y^2 & 2XY \\ 2XY & X^2 \end{bmatrix}\begin{bmatrix} Y^2 & 2XY \\ 2XY & X^2 \end{bmatrix} = \alpha^2\begin{bmatrix} Y^4 + 4X^2Y^2 & 2XY^3 + 2X^3Y \\ 2XY^3 + 2X^3Y & X^4 + 4X^2Y^2 \end{bmatrix}$$

$$\underline{\underline{E}} = \begin{bmatrix} \alpha Y^2 + \alpha^2(Y^4 + 4X^2Y^2) & \alpha 2XY + \alpha^2(2XY^3 + 2X^3Y) \\ & \alpha X^2 + \alpha^2(X^4 + 4X^2Y^2) \end{bmatrix}$$

$$\vec{n} = \frac{1}{\sqrt{2}}(\vec{\imath} + \vec{\jmath}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

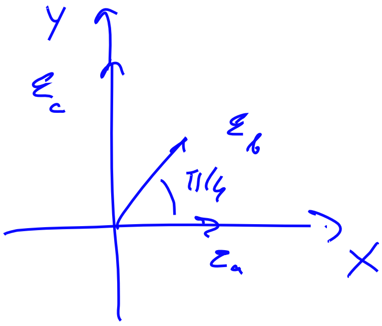
$\varepsilon = \vec{n} \cdot \underline{\underline{\varepsilon}}\,\vec{n} =$

V izhodišču $X = Y = 0 \Rightarrow \underline{\underline{\varepsilon}} = 0 \Rightarrow \varepsilon = 0$

$x = y = a$; $\varepsilon = \alpha \begin{bmatrix} a^2 & 2a^2 \\ 2a^2 & a^2 \end{bmatrix}$

$\varepsilon = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \alpha a^2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \alpha a^2 \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \alpha a^2 6$

2. V treh smereh, ki oklepajo medsebojni kot $\pi/4$ so znane osne deformacije $\epsilon_{\mathrm{a}} = \epsilon_0$, $\epsilon_{\mathrm{b}} = -\frac{3}{2}\epsilon_0$ in $\epsilon_{\mathrm{c}} = 2\epsilon_0$, kjer je $\epsilon_0 = 10^{-3}$.
   (a) Določi deformacijski tenzor.
   (b) Skiciraj Mohrovo krožnico in določi ekstremalne vrednosti in smeri deformacije.



$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_a & x \\ x & \varepsilon_c \end{bmatrix}$

$\varepsilon(\vec{n}) = \frac{1}{2}(\varepsilon_{11} + \varepsilon_{22}) + \frac{1}{2}(\varepsilon_{11} - \varepsilon_{22})\cos 2\varphi + \varepsilon_{12} \sin 2\varphi$

$\vec{n} = \cos\varphi \, \vec{i} + \sin\varphi \, \vec{j}$

ε_{11}' ; $\vec{i}' = \vec{n}$

$\varphi = \frac{\pi}{4}$

$\varepsilon_b = \frac{1}{2}(\varepsilon_a + \varepsilon_c) + \frac{1}{2}(\varepsilon_a - \varepsilon_c)\underbrace{\cos\frac{\pi}{2}}_{=0} + \underbrace{\varepsilon_{12}}_{=x} \underbrace{\sin\frac{\pi}{2}}_{=1}$

$\varepsilon_b = \frac{1}{2}(\varepsilon_a + \varepsilon_c) + x \Rightarrow x = \varepsilon_b - \frac{1}{2}(\varepsilon_a + \varepsilon_c) =$

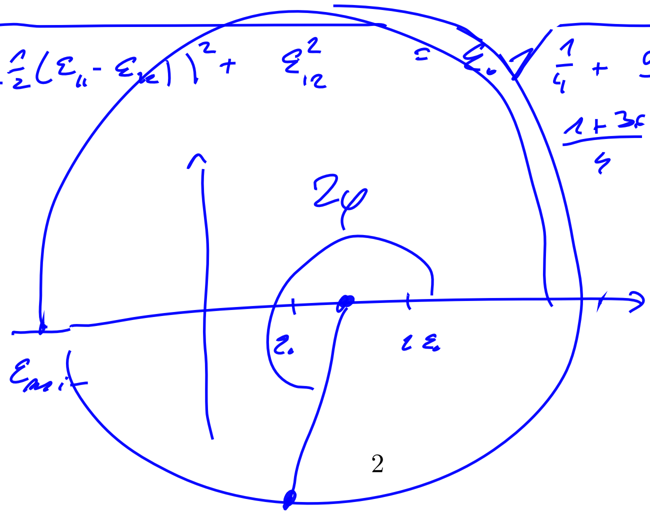
$x = -\frac{3}{2}\varepsilon_0 - \frac{1}{2}(\varepsilon_0 + 2\varepsilon_0) = -3\varepsilon_0$

$\underline{\underline{\varepsilon}} = \varepsilon_0 \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix}$ $\qquad s = \frac{1}{2}(\varepsilon_{11} + \varepsilon_{22}) = \frac{3}{2}\varepsilon_0$

$r = \sqrt{\left(\frac{1}{2}(\varepsilon_{11} - \varepsilon_{22})\right)^2 + \varepsilon_{12}^2} = \varepsilon_0 \sqrt{\frac{1}{4} + 9} = \frac{1}{2}\varepsilon_0\sqrt{37}$

$\frac{1+36}{4}$

$\varepsilon_{min} = s - r$



$\varepsilon_{22} > \varepsilon_{11}$

$P(1, -3)$

$\varphi_{max} = \frac{1}{2}\operatorname{arctg}\frac{2\varepsilon_{12}}{\varepsilon_{11} - \varepsilon_{22}} \left(+\frac{\pi}{2}\right) =$

$= \frac{1}{2} \operatorname{arccotg} 6 + \frac{\pi}{2} \approx \underline{\underline{130^\circ}}$

3. Za ravninsko deformacijsko stanje sta podani glavni osni deformaciji $\epsilon_1 = 2\epsilon_0$ in $\epsilon_2 = -\epsilon_0$, kjer je ϵ_0 majhno pozitivno število. Določi osi x' in y' pri katerih je $e'_{11} = 0$ in $e'_{12} > 0$.

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} 2\varepsilon_0 & 0 \\ 0 & -\varepsilon_0 \end{bmatrix}$$

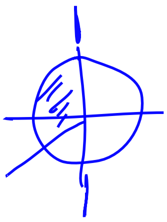
$$\varepsilon'_{11} = \frac{1}{2}(\varepsilon_{11} + \varepsilon_{22}) + \frac{1}{2}(\varepsilon_{11} - \varepsilon_{22})\cos 2\varphi + \underline{\varepsilon_{12}}\sin 2\varphi$$

$$0 = \varepsilon'_{11} = \frac{1}{2}\varepsilon_0 + \frac{1}{2}3\varepsilon_0\cos 2\varphi \Rightarrow \underline{\cos 2\varphi = -\frac{1}{3}}$$

$$\varphi = \underline{\pm\frac{1}{2}\arccos}\left(-\frac{1}{3}\right)$$

$\underline{\varepsilon'_{12} > 0}$

$$0 < \varepsilon'_{12} = -\frac{1}{2}(\varepsilon_{11} - \varepsilon_{22})\sin 2\varphi + \varepsilon_{12}\cos 2\varphi$$



$$0 < -\frac{3}{2}\varepsilon_0\sin 2\varphi \Rightarrow \underline{\sin 2\varphi < 0}$$

$$\left.\begin{aligned} 2\varphi &\in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \\ 2\varphi &\in (\pi, 2\pi) \end{aligned}\right\} \Rightarrow 2\varphi \in \left(\pi, \frac{3\pi}{2}\right) \Rightarrow \varphi \in \underline{\left(\frac{\pi}{2}, \frac{3\pi}{4}\right)}$$

$$\varphi = \frac{1}{2}\arccos\left(-\frac{1}{3}\right) \Rightarrow \cos 2\varphi = -\frac{1}{3}$$

$$\vec{i}' = \cos\varphi\,\vec{i} + \sin\varphi\,\vec{j} \qquad \vec{j}' = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j}$$

$$\cos^2\varphi = \frac{1}{2}(1 + \cos 2\varphi) = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{3} \qquad \sin^2\varphi = \frac{1}{2}(1 - \cos 2\varphi) = \frac{2}{3}$$

$$\cos\varphi = -\frac{1}{\sqrt{3}} \ ; \quad \sin\varphi = \sqrt{\frac{2}{3}}$$

$$\vec{i}' = \frac{1}{\sqrt{3}}\left(-\vec{i} + \sqrt{2}\,\vec{j}\right) \qquad \vec{j}' = \frac{1}{\sqrt{3}}\left(-\sqrt{2}\,\vec{i} - \vec{j}\right)$$

$\varepsilon'_{22} = ?$ $\qquad$ $\varepsilon'_{12} = ?$

4. Izotropični material je v homogenem napetostnem stanju. V smeri vektorja $\vec{\imath}$ je normalna napetost enaka σ_0, v smeri vektorja $\vec{\jmath}$, ki je pravokoten na $\vec{\imath}$ pa je normalna napetost enaka $2\sigma_0$. Določi Yongov modul in Poissonov količnik, če je matrika pripadajočega deformacijskega tenzorja v bazi $\vec{\imath}$, $\vec{\jmath}$ enaka

$$\mathbf{e} = \epsilon_0 \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Določi tudi napetostni tenzor.

$$\longrightarrow \quad \underline{\underline{t}} = 2\mu \, \underline{\underline{e}} + \lambda \, \mathrm{sl}\, \underline{\underline{e}} \, \underline{\underline{I}} =$$

$$= 2\mu \varepsilon_0 \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \lambda \cdot 4\varepsilon_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \varepsilon_0 \begin{bmatrix} 2\mu + 4\lambda & 2\mu & 0 \\ 2\mu & 6\mu + 4\lambda & 0 \\ 0 & 0 & 4\lambda \end{bmatrix}$$

$$\sigma_0 = \vec{\imath} \cdot \underline{\underline{t}} \, \vec{\imath} = \varepsilon_0 (2\mu + 4\lambda)$$

$$2\sigma_0 = \vec{\jmath} \, \underline{\underline{t}} \, \vec{\jmath} = \varepsilon_0 (6\mu + 4\lambda)$$

$$\sigma_0 = 4\mu\varepsilon_0 \Rightarrow \mu = \frac{1}{4} \frac{\sigma_0}{\varepsilon_0}$$

$$\sigma_0 = \varepsilon_0 \left(2 \cdot \frac{1}{4} \frac{\sigma_0}{\varepsilon_0} + 4\lambda \right) \Rightarrow \sigma_0 - \frac{1}{2}\sigma_0 = 4\lambda\varepsilon_0 \Rightarrow \lambda = \frac{1}{8} \frac{\sigma_0}{\varepsilon_0}$$

$$\longrightarrow \quad \mu = G = \frac{E}{2(1+\nu)} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\frac{\mu}{\lambda} = \frac{1\,(1-2\nu)}{2\,\nu} \; ; \quad \frac{\sigma_0 \cdot 8\varepsilon_0}{4\varepsilon_0 \, \sigma_0} = 2 = \frac{1-2\nu}{2\nu}$$

$$4\nu = 1 - 2\nu$$

$$\nu = \frac{1}{6}$$

$$E = 2(1+\nu)\mu = 2\left(1 + \frac{1}{6}\right) \cdot \frac{1}{4} \frac{\sigma_0}{\varepsilon_0} = \frac{2 \cdot 7 \, \sigma_0}{4 \cdot 6 \, \varepsilon_0} = \frac{7}{12} \frac{\sigma_0}{\varepsilon_0}$$

.

5. Izotropični material je v homogenem napetostnem stanju. V smeri vektorja $\vec{\imath}$ je osna deformacija enaka ϵ_0, v smeri vektorja $\vec{\jmath}$, ki je pravokoten na $\vec{\imath}$ pa je osna deformacija enaka $-\frac{2}{3}\epsilon_0$. Določi Yongov modul in Poissonov količnik, če je matrika napetostnega tenzorja v bazi $\vec{\imath}$, $\vec{\jmath}$ enaka

$$\mathbf{t} = \sigma_0 \begin{bmatrix} 2 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Določi tudi deformacijski tenzor.

$$\underline{\underline{\varepsilon}} = \frac{1+\nu}{E}\underline{\underline{t}} - \frac{\nu}{E}\,\mathrm{Sl}\,\underline{\underline{t}}\,\underline{\underline{I}} = \qquad \mathrm{Sl}\,\underline{\underline{t}} = \sigma_0$$

$$= \frac{1+\nu}{E}\sigma_0 \begin{bmatrix} 2 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{\nu}{E}\sigma_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \frac{\sigma_0}{E} \begin{bmatrix} 2(1+\nu)-\nu & -(1+\nu) & 0 \\ -(1+\nu) & -(1+\nu)-\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix} =$$

$$\frac{\sigma_0}{E} \begin{bmatrix} 2+\nu & -1-\nu & 0 \\ -1-\nu & -1-2\nu & 0 \\ 0 & 0 & -\nu \end{bmatrix}$$

$$\vec{\imath}\cdot\underline{\underline{\varepsilon}}\,\vec{\imath} = \varepsilon_0 \Rightarrow \frac{(2+\nu)\sigma_0}{E} = \varepsilon_0$$

$$\vec{\jmath}\cdot\underline{\underline{\varepsilon}}\,\vec{\jmath} = -\frac{2}{3}\varepsilon_0 \Rightarrow +\frac{(1+2\nu)\sigma_0}{E} = +\frac{2}{3}\varepsilon_0$$

$$\frac{2+\nu}{1+2\nu} = \frac{3}{2} \Rightarrow 4+2\nu = 3+6\nu \Rightarrow 1 = 4\nu \Rightarrow \underline{\underline{\nu = \frac{1}{4}}}$$

$$E = \frac{(2+\nu)\sigma_0}{\varepsilon_0} = \frac{\sigma_0}{\varepsilon_0}\left(2+\frac{1}{4}\right) = \underline{\underline{\frac{9\sigma_0}{4\varepsilon_0}}}$$

$$\underline{\underline{\varepsilon}} = \frac{\sigma_0}{9\sigma_0}\cdot 4\varepsilon_0 \begin{bmatrix} 2+\frac{1}{4} & -1-\frac{1}{4} & 0 \\ -1-\frac{1}{4} & -1-\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} = \frac{4\varepsilon_0}{9} \begin{bmatrix} \frac{9}{4} & -\frac{5}{4} & 0 \\ -\frac{5}{4} & -\frac{3}{2} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}$$

$$\underline{\underline{\varepsilon}} = \frac{1}{9}\varepsilon_0 \begin{bmatrix} 9 & -5 & 0 \\ -5 & -6 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$