Predavanje 12. maj 2021

Lamejeva koeficienta $\mu = G$, $\lambda = \frac{\nu E}{(1-2\nu)(1+\nu)}$,

$$\underline{\underline{t}} = 2\mu\underline{\underline{\epsilon}} + \lambda(\mathrm{sl}\,\underline{\underline{\epsilon}})\underline{\underline{I}}.$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)}, \qquad E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}.$$

$$\underline{\underline{\epsilon}} = \frac{1+\nu}{E}\underline{\underline{t}} - \frac{\nu}{E}(\mathrm{sl}\,\underline{\underline{t}})\underline{\underline{I}}$$

$$\underline{\underline{t}} = -p\underline{\underline{I}} = -p\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix} \quad ; \quad p>0$$

$$\mathrm{sl}\,\underline{\underline{t}} = -3p$$

$$\underline{\underline{\epsilon}} = \frac{1+\nu}{E}(-p\underline{\underline{I}}) - \frac{\nu}{E}(-3p)\underline{\underline{I}} = -p\left(\frac{1+\nu}{E} - \frac{3\nu}{E}\right)\underline{\underline{I}} = -p\,\frac{1-2\nu}{E}\underline{\underline{I}}$$

$\mathrm{sl}\,\underline{\underline{\epsilon}} \le 0$ $\qquad \mathrm{sl}\,\underline{\underline{\epsilon}} = -p\,\frac{1-2\nu}{E}\cdot 3 \le 0$

$E>0,\ p>0 \Rightarrow 1-2\nu \ge 0 \Rightarrow \nu \le \frac{1}{2}$

$\nu = \frac{1}{2}$ nestisljiv material $\qquad \boxed{-1 < \nu \le \frac{1}{2}}$

$\lambda > 0$

$$\mathrm{sl}\,\epsilon = -p\,\frac{3(1-2\nu)}{E} = \frac{1-2\nu}{E}\,\mathrm{sl}\,\underline{\underline{t}}$$

$$\varkappa = \frac{E}{3(1-2\nu)}$$

kompresijski modul ; $\boxed{-p = \varkappa\,\mathrm{sl}\,\underline{\underline{\epsilon}}}$

Enakomerna kompresija, kompresijski modul $\kappa = \frac{E}{3(1-2\nu)}$.

Nestisljivi material $\nu = \frac{1}{2}$, vrednosti Poissonovega količnika $\nu \in (-1, \frac{1}{2}]$.

Termoelastičnost

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}_T + \underline{\underline{\varepsilon}}_E$$

Termalni raztezek $\underline{\underline{\epsilon}}_{\mathrm{T}} = \alpha \Delta T \underline{\underline{I}}$.

$$\underline{\underline{\epsilon}} = \underbrace{\frac{1+\nu}{E}\underline{\underline{t}} - \frac{\nu}{E}(\mathrm{sl}\,\underline{\underline{t}})\underline{\underline{I}}}_{\varepsilon_E} + \alpha \Delta T \underline{\underline{I}}.$$

Primer: izračun napetostnega stanja elastičnega vključka v togi matriki.

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}_E + \underline{\underline{\varepsilon}}_T \qquad \underline{\underline{\varepsilon}} = 0$$

$$\underline{\underline{\varepsilon}}_E = -\underline{\underline{\varepsilon}}_T = -\alpha \Delta T \underline{\underline{I}}$$

$$\frac{1+\nu}{E}\underline{\underline{t}} - \frac{\nu}{E}\,\mathrm{sl}\,\underline{\underline{t}}\,\underline{\underline{I}} = -\alpha \Delta T \underline{\underline{I}}$$

$\underline{\underline{t}}$ je sferični tenzor $\underline{\underline{t}} = p\underline{\underline{I}}$

Napetostno stanje je hidrostatično.

$$\mathrm{sl}\left(\frac{1+\nu}{E}\underline{\underline{t}} - \frac{\nu}{E}\,\mathrm{sl}\,\underline{\underline{t}}\,\underline{\underline{I}}\right) = \mathrm{sl}\left(-\alpha \Delta T \underline{\underline{I}}\right)$$

$$\frac{1+\nu}{E}\,\mathrm{sl}\,\underline{\underline{t}} - \frac{\nu}{E}\,\mathrm{sl}\,\underline{\underline{t}} \cdot 3 = -\alpha \Delta T\, 3$$

$$\left(\frac{1+\nu}{E} - 3\frac{\nu}{E}\right)\mathrm{sl}\,\underline{\underline{t}} = -3\alpha \Delta T$$

$$\frac{1-2\nu}{E}\,\mathrm{sl}\,\underline{\underline{t}} = -3\alpha \Delta T \Rightarrow \mathrm{sl}\,\underline{\underline{t}} = -\frac{3E}{1-2\nu}\alpha \Delta T$$

$$\boxed{\underline{t} = -\frac{3E\alpha}{1-2\nu}\Delta T\,\underline{I}}$$

$$\underline{\varepsilon} = \frac{1+\nu}{E}\underline{t} - \frac{\nu}{E}\,\mathrm{sl}\,\underline{t}\,\underline{I}$$

Ravninska napetost $t_{13} = t_{23} = t_{33} = 0$

$$\varepsilon_{11} = \frac{1}{E}t_{11} - \frac{\nu}{E}t_{22}$$

$$\varepsilon_{22} = \frac{1}{E}t_{22} - \frac{\nu}{E}t_{11}$$

$$\varepsilon_{12} = \frac{1+\nu}{E}t_{12}$$

$$\varepsilon_{13} = 0\,; \quad \varepsilon_{23} = 0\,;$$

$$\boxed{\varepsilon_{33} = -\frac{\nu}{E}t_{11} - \frac{\nu}{E}t_{22}}$$

Ravninska deformacija

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$$

$$\varepsilon_{11} = \frac{1}{E} t_{11} - \frac{\nu}{E} t_{22} - \frac{\nu}{E} t_{33}$$

$$\varepsilon_{22} = \frac{1}{E} t_{22} - \frac{\nu}{E} t_{11} - \frac{\nu}{E} t_{33}$$

$$0 = \varepsilon_{33} = \frac{1}{E} t_{33} - \frac{\nu}{E} t_{11} - \frac{\nu}{E} t_{22} \Rightarrow t_{33} = \nu(t_{11} + t_{22})$$

$$\varepsilon_{11} = \frac{1}{E} t_{11} - \frac{\nu}{E} t_{22} - \frac{\nu^2}{E}(t_{11} + t_{22}) = \frac{1-\nu^2}{E} t_{11} - \frac{\nu(1+\nu)}{E} t_{22}$$

$$\varepsilon_{22} = \frac{1-\nu^2}{E} t_{22} - \frac{\nu(1+\nu)}{E} t_{11}$$

$$\frac{1-\nu^2}{E} = \frac{1}{E'} \qquad \frac{\nu(1+\nu)}{E} = \frac{\nu'}{E'}$$

$$\boxed{\begin{aligned} \varepsilon_{11} &= \frac{1}{E'} t_{11} - \frac{\nu'}{E'} t_{22} \\ \varepsilon_{22} &= \frac{1}{E'} t_{22} - \frac{\nu'}{E'} t_{11} \end{aligned}}$$

$$E' = \frac{E}{1-\nu^2}$$

$$\nu' = \frac{\nu(1+\nu)E'}{E} = \frac{\nu(1+\nu)E}{(1-\nu^2)E} = \frac{\nu}{1-\nu}$$

$$1-\nu^2 = (1-\nu)(1+\nu)$$

$$0 = \varepsilon_{23} = \frac{1+\nu}{E} t_{23} \Rightarrow t_{23} = 0 \; ; \quad t_{13} = 0$$

$$\varepsilon_{12} = \frac{1+\nu}{E} t_{23} \qquad \frac{1+\nu}{E} \overset{?}{=} \frac{1+\nu'}{E'} \quad \checkmark$$

Modificirana modula

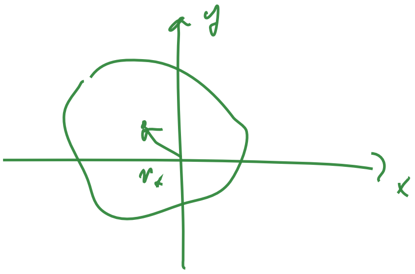
$$\hat{E} = \frac{E}{1-\nu^2}, \qquad \hat{\nu} = \frac{\nu}{1-\nu}.$$

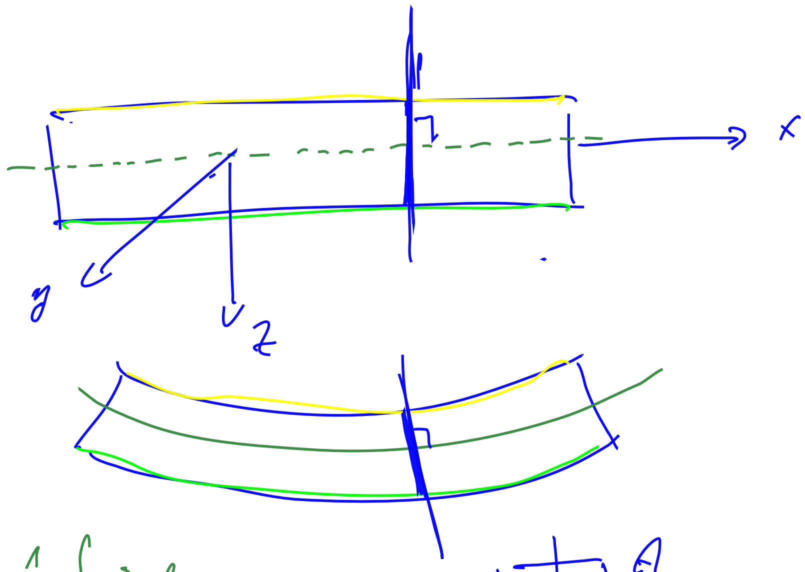
$$\frac{1+\nu'}{E'} = \frac{\left(1+\frac{\nu}{1-\nu}\right)(1-\nu^2)}{E} = \frac{1}{E}\left(\underbrace{1-\nu^2 + \nu(1+\nu)}_{1-\nu^2+\nu+\nu^2}\right) = \frac{1+\nu}{E}$$

Ravninska deformacija je ekvivalentna ravninski napetosti z modificiranima moduloma.

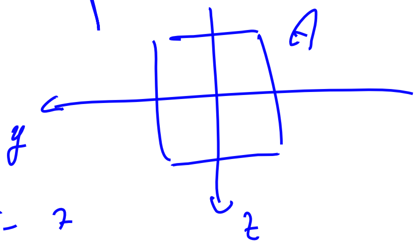
Upogib nosilca

Nevtralna os, centralna os.





$$\vec{r}_* = \frac{1}{A}\int_A \vec{r}\, dA$$



A simetrična glede na osi y in z

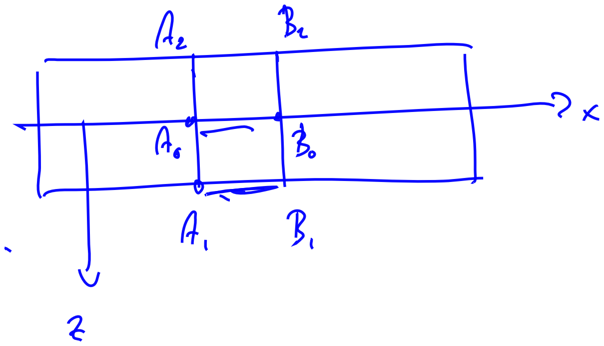
centralna os je os x $\Rightarrow$ $\vec{r}_* = (x, 0, 0)$

$$z_* = \frac{1}{A}\int_A z\, dA = 0$$

Inženirska teorija nosilcev:

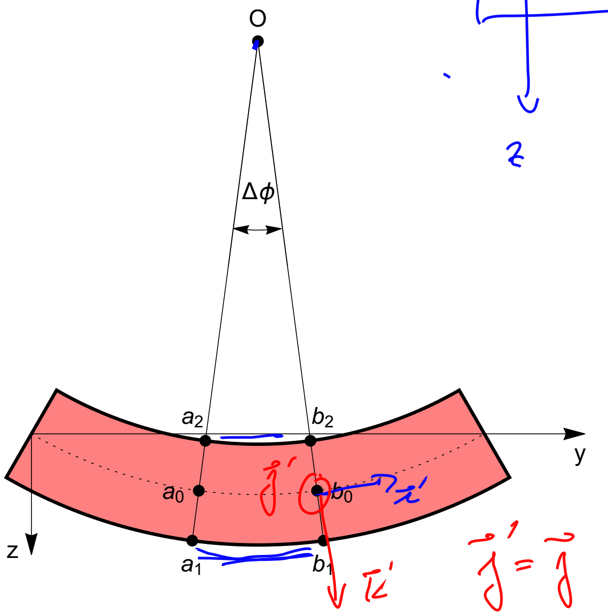
- Dolžina nosilca je bistveno večja od lateralnih dimenzij.
- Nosilec je simetričen glede na ravnini xz in xy.
- Za homogeni nosilec se nevtralna os ujema s centralno osjo.
- Ravnine pravokotne na nevtralno os v referenčni legi se deformirajo v ravnine, ki so pravokotne na deformirano nevtralno os.
- V ravninah, ki so pravokotne na nevtralno ni deformacij.

Deformacija vlaken



$$|A_0A_1| = z$$

$$|A_0B_2| = -z$$



$$R = |Oa_0| = |Ob_0|$$

Slika 1: Upogib nosilca, nevtralna os je označena črtkano.

$$\varepsilon = \frac{|a_1b_1| - |A_1B_1|}{|A_1B_1|} = \frac{|a_1b_1|}{|A_1B_1|} - 1 = \frac{|a_1b_1|}{|a_0b_0|} - 1$$

$$|A_1B_1| = |A_0B_0| = |a_0b_0| \qquad |a_0b_0| = R\Delta\phi$$

$$|Oa_1| = |Ob_1| = R+z$$

$$|a_1b_1| = (R+z)\Delta\phi$$

$$\varepsilon = \frac{(R+z)\Delta\phi}{R\Delta\phi} - 1 = \frac{z}{R}$$

$$\hat{\varepsilon} = \frac{|a_2b_2|}{|a_0b_0|} - 1$$

$$\hat{\varepsilon} = \frac{z}{R}$$

$$|Oa_2| = R + z$$

$$\boxed{\text{Deformacija vlaken } \epsilon = \frac{z}{R}.}$$

$$\varepsilon(x) = \frac{z}{R(x)}$$

$$\sigma = E\varepsilon = \frac{E}{R} z$$

homogeni nosilec

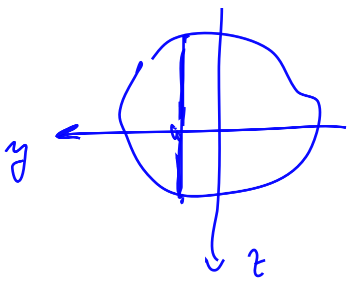
Napetost $\sigma = z\frac{E}{R}$.

$$\vec{t} = \sigma \cdot \vec{i}'$$

$$\vec{0} \overset{?}{=} \int_A \vec{t}\, dA = \int_A \sigma \vec{i}\, dA = \int_A \frac{E}{R} z\, dA = \frac{E}{R} \int_A z\, dA = 0$$

$$\vec{M} = \int_A d\vec{M} = \int_A \vec{r} \times \vec{t}\, dA = \int_A (y\vec{j}' + z\vec{k}') \times \sigma \vec{i}'\, dA =$$

$$\sigma = z\frac{E}{R}$$



$$\vec{r} = y\vec{j}' + z\vec{k}'$$

$$= \frac{E}{R} \int_A (yz\,\vec{k}' + z^2 \vec{j}')\, dA = \left(\frac{E}{R} \int_A yz\, dA\right)\vec{k}' + \frac{E}{R} \int_A z^2 dA\, \vec{j}'$$

$\int_A yz\, dA = 0$ zaradi predpostavljene simetrije prereza

$$\vec{M} = \frac{E}{R} \int_A z^2 dA\, \vec{j}' \qquad \vec{M} = \frac{E}{R} \int_A z^2 dA\, \vec{j}$$

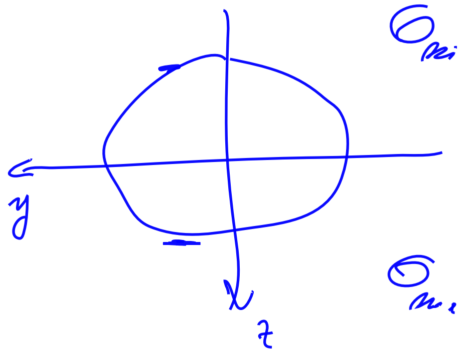
$$\boxed{M = \frac{E}{R} \int_A z^2 dA}$$

Euler Bernoullijeva formula.

$\int_A z^2 dA$ ploskovni moment drugega reda $= I_y = I$

$$M = \frac{EI}{R} \qquad \sigma = \frac{E}{R} z$$

$$\boxed{\sigma = \frac{M}{I} z}$$



$$\sigma_{min} = \frac{M}{I} z_{min}$$

$$\sigma_{max} = \frac{M}{I} z_{max}$$

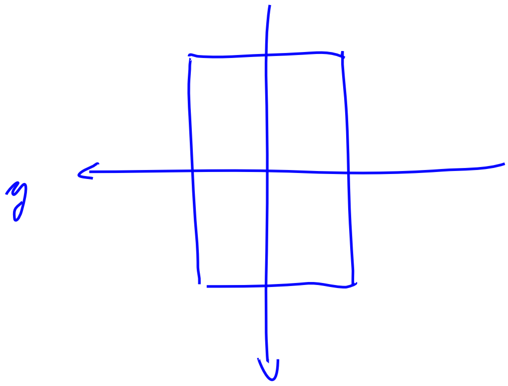
Upogibni moment

Izračun upogibnega momenta.

Euler - Bernoullijeva enačba $M = \frac{EI}{R}$.

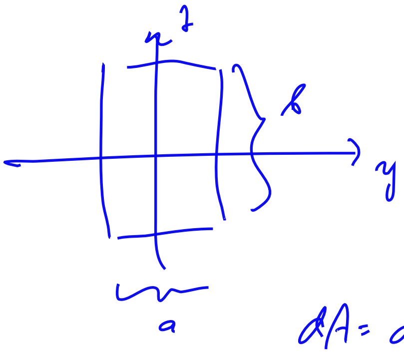
Zveza $\sigma = E\epsilon = \frac{M}{I}z$.

$$\sigma = \frac{M}{I} z$$



Ploskovni moment

Ploskovni moment pravokotnika, $I = ab^3/12$.



$$I = \int_A z^2 dA = \int_{-a/2}^{a/2} dy \int_{-b/2}^{b/2} z^2 dz = 2a \int_0^{b/2} z^2 dz$$

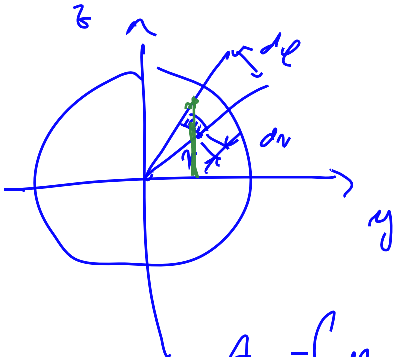
$$dA = dy\, dz$$

$$= 2a\, \frac{1}{3} z^3 \Big|_0^{b/2} = 2a \frac{1}{3} \cdot \frac{b^3}{8}$$

$$\boxed{I = \frac{1}{12} a b^3}$$

$$\boxed{\int_{x_1}^{x_2} f(x)\,dx = F(x_2) - F(x_1) \qquad F'(x) = f(x)}$$

Ploskovni moment kroga $I = \pi R^4/4$.



$$I = \int_A z^2 dA \qquad z = r \sin\varphi$$

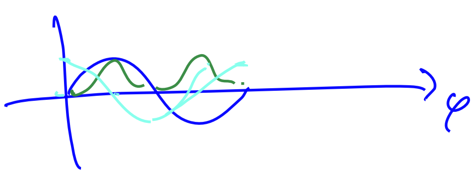
$$dA = r\, d\varphi\, dr$$

$$A = \int_A dA = \int_0^R r\, dr \int_0^{2\pi} d\varphi = \frac{1}{2} r^2 \Big|_0^R \cdot 2\pi = \pi R^2$$

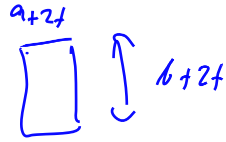
$$I = \int_A r^2 \sin^2\varphi\, dA = \int_0^R r^3 dr \int_0^{2\pi} \sin^2\varphi\, d\varphi = \frac{1}{4} r^4 \Big|_0^R \cdot \pi$$

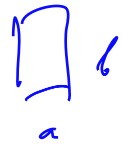
$$J = \int_0^{2\pi} \sin^2\varphi\, d\varphi = \int_0^{2\pi} \cos^2\varphi\, d\varphi$$

$$2J = \int_0^{2\pi} \sin^2\varphi\, d\varphi + \int_0^{2\pi} \cos^2\varphi\, d\varphi = \int_0^{2\pi} d\varphi = 2\pi \Rightarrow J = \pi$$

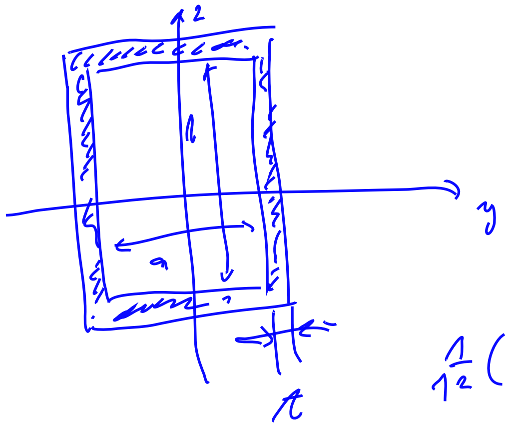


$$\boxed{I = \frac{1}{4} \pi R^4}$$





Ploskovni momenti tankostenskih presekov.



$$A = A_2 - A_1$$

$$I = \int_A z^2 dA = \int_{A_2} z^2 dA - \int_{A_1} z^2 dA =$$

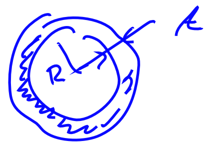
$$\frac{1}{12}(a+2t)(b+2t)^3 - \frac{1}{12} a b^3$$

$$t \ll a; \quad t \ll b$$

$$I = \frac{1}{12}\left[(a+2t)\left(b^3 + 3b^2 2t + 3\cdot b\cdot(2t)^2 + (2t)^3\right) - ab^3\right]$$

$$\underline{ab^3 + 2tb^3 + 3ab^2\cdot 2t + 6b^2t^2}$$

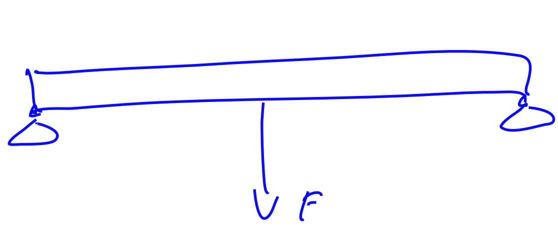
$$\doteq \frac{1}{12}\left[2tb^3 + 6ab^2t\right] = \frac{1}{6}\left(b^3 + 3ab^2\right)t \checkmark$$

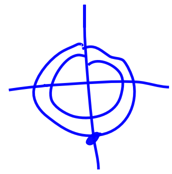


$$I = \int_{A_2} z^2 dA - \int_{A_1} z^2 dA = \frac{1}{4}\pi (R+t)^4 - \frac{1}{4}\pi R^4$$

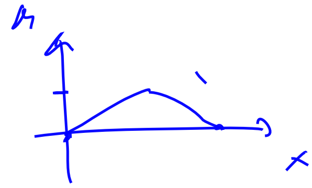
$$t \ll R \;\Big|\; (R+t)^4 \doteq R^4 + 4R^3 t$$

$$\boxed{I = R^3 t \pi}$$





Primer: enostavno podprti nosilec dolžine $l = 2$m s tankostenskim krožnim presekom je točkovno obremenjen na svoji polovici. Določi debelino, da bo osna napetost pod dopustno vrednostjo.



$\sigma \leq \sigma_0$

$$\sigma_{max} = \frac{M_{max}}{I} t_{max} < \sigma_0$$

$M_{max} = \frac{1}{4} lF$

$I = t R^3 \pi$

$M_{max} = \frac{1}{4} lF \; ; \quad t_{max} = R + t$

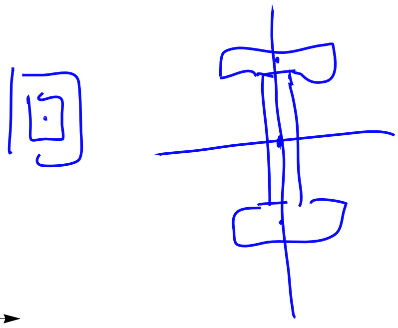
$$\frac{\frac{1}{4} lF \cdot (R+t)}{t R^3 \pi} < \sigma_0 \Rightarrow \frac{1}{4} lF (R+t) < \sigma_0 t R^3 \pi$$

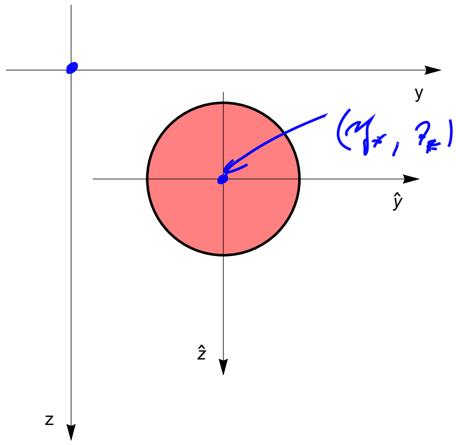
$$\frac{1}{4} lFR < \sigma_0 t R^3 \pi - \frac{1}{4} lF t$$

$$\frac{1}{4} lFR < t \left(\sigma_0 R^3 \pi - \frac{1}{4} lF \right)$$

Predpostavimo $\sigma_0 R^3 \pi > \frac{1}{4} lF$.

Potem mora veljati $$t > \frac{lFR}{4\left(\sigma_0 R^3 \pi - \frac{1}{4} lF\right)}.$$





Slika 2: Paralelna koordinatna sistema.

$$y = y_* + \hat{y} \qquad z = z_* + \hat{z} \qquad dA = dy\,dz = d\hat{y}\,d\hat{z}$$

$$\int_A z^2 dA = \int_A (z_* + \hat{z})^2 dA = \int_A (z_*^2 + 2 z_* \hat{z} + \hat{z}^2) dA =$$

$$= z_*^2 \underbrace{\int_A dA} + 2 z_* \int_A \underline{\hat{z}} dA + \int_A \hat{z}^2 dA =$$

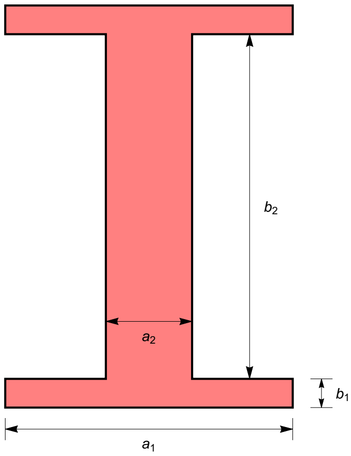
$$= z_*^2 A + \hat{I}$$

$$\boxed{I = \hat{I} + z_*^2 A}$$

$$\int_A \hat{z} dA = \underbrace{\int \hat{z}\, d\hat{z}}_{=0} \int d\hat{y}$$

Izrek o paralelnih oseh, $\hat{I} = z_0^2 A + I$.

Primer: ploskovni moment I nosilca.



Slika 3: I nosilec.