Predavanje 10. marec 2021

Redukcija ravninskega sistema

Redukcija ravninskega sistema $\{(P_1, \vec{F}_1), (P_2, \vec{F}_2)\}$ dveh sil na skupno prijemališče.

1. $\vec{F}_1 \nparallel \vec{F}_2$.

2. $\vec{F}_1 \parallel \vec{F}_2$ in $\vec{F}_1 \cdot \vec{F}_2 > 0$

3. $\vec{F}_1 \parallel \vec{F}_2$, $\vec{F}_1 \cdot \vec{F}_2 < 0$ in $\left|\vec{F}_1\right| \neq \left|\vec{F}_2\right|$.

Poljuben ravninski sistem dveh sil $\{(P_1, \vec{F}_1), (P_2, \vec{F}_2)\}$, ki ni dvojica, moremo reducirati na sistem z eno samo silo $\{(P_0, \vec{F}_1 + \vec{F}_2)\}$, kje je P_0 skupno prijemališče.

Moment dvojice je neodvisen od pola: $\vec{N}(\mathcal{F}, O_1) = \vec{N}(\mathcal{F}, O_2)$ za poljubna pola O_1 in O_2. Pravimo, da je navor prosti vektor.

Ekvivalentnost dvojice sil in navora; konstrukcija dvojice sil za dani navor.

Unija sistema sil $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Velja

$$\vec{R}(\mathcal{F}_1 \cup \mathcal{F}_2) = \vec{R}(\mathcal{F}_1) + \vec{R}(\mathcal{F}_2), \qquad \vec{N}(\mathcal{F}_1 \cup \mathcal{F}_2, O) = \vec{N}(\mathcal{F}_1, O) + \vec{N}(\mathcal{F}_2, O).$$

$\mathcal{F}_1 \quad \mathcal{F}_2$ dvojici

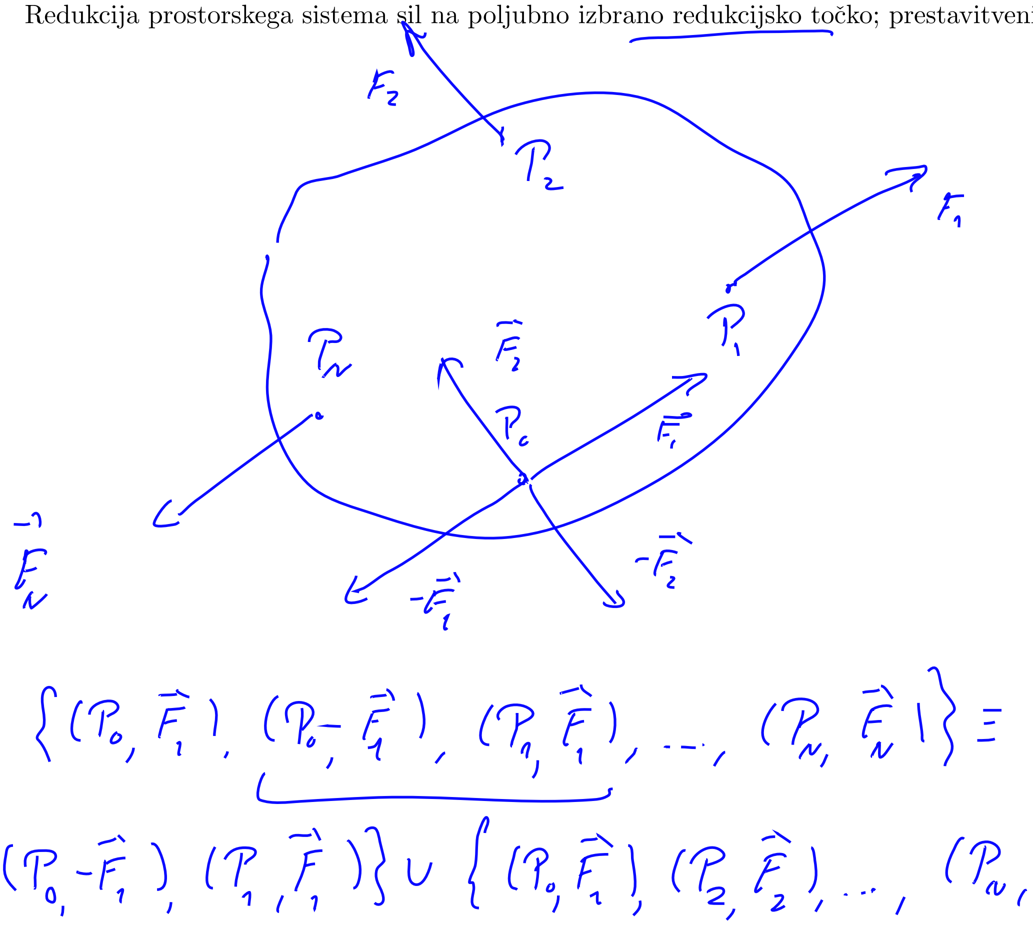
$$\vec{R}(\mathcal{F}_1 \cup \mathcal{F}_2) = \underbrace{\vec{R}(\mathcal{F}_1)}_{=\vec 0} + \underbrace{\vec{R}(\mathcal{F}_2)}_{=\vec 0} = \vec{0} \checkmark$$

Unija dvojice je dvojica ali ravnovesni sistem sil.

$$\vec{N}(\mathcal{F}_1 \cup \mathcal{F}_2, O) = \underbrace{\vec{N}(\mathcal{F}_1, O)}_{\neq \vec 0} + \underbrace{\vec{N}(\mathcal{F}_2, O)}_{\neq \vec 0} = \begin{cases} \neq 0 \\ = \vec{0} \quad (\vec{N}(\mathcal{F}_1, O) = -\vec{N}(\mathcal{F}_2, O)) \end{cases}$$

Redukcija prostorskega sistema sil

Redukcija prostorskega sistema sil na poljubno izbrano redukcijsko točko; prestavitveni moment.



$$\mathcal{F} = \{(P_0, \vec F_1), \underbrace{(P_0, -\vec F_1), (P_1, \vec F_1)}, \dots, (P_N, \vec F_N)\} =$$

$$\{\underbrace{(P_0, -\vec F_1), (P_1, \vec F_1)}_{\mathcal{D}_1}\} \cup \{(P_0, \vec F_1), (P_2, \vec F_2), \dots, (P_N, \vec F_N)\} =$$

$$= \mathcal{D}_1 \cup \{(P_0, \vec F_1), (P_0, \vec F_2), \underbrace{(P_0, -\vec F_2), (P_2, \vec F_2)}_{\mathcal{D}_2}, \dots, (P_N, \vec F_N)\} =$$

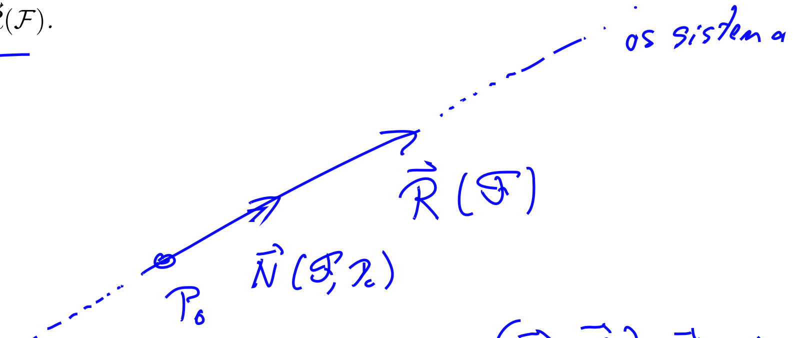
$$= \mathcal{D}_1 \cup \mathcal{D}_2 \cup \{(P_0, \vec F_1 + \vec F_2), (P_3, \vec F_3), \dots, (P_N, \vec F_N)\} =$$

$$= \mathcal{D}_1 \cup \dots \cup \mathcal{D}_N \cup \{(P_0, \vec F_1 + \vec F_2 + \dots + \vec F_N)\} = \mathcal{D} \cup \{(P_0, \vec R(\mathcal{F}))\}$$

$\mathcal{D}$ je ravnovesni sistem $\Rightarrow$ $\mathcal{F} \equiv \{(P_0, \vec{R}(\mathcal{F}))\}$
Reducijska točka je skupno prijemališče.
$\mathcal{D}$ je dvojica $\qquad \mathcal{F} \equiv \mathcal{D} \cup \{(P_0, \vec{R}(\mathcal{F}))\}$ $\qquad \mathcal{D}$ je premestitveni navor.

Definicija 1. *Sistem sil $\mathcal{F}$ je dinama, če je $\vec{R}(\mathcal{F}) \neq \vec{0}$ in obstaja taka točka P_0, da velja*

1. $\vec{N}(\mathcal{F}, P_0) \neq \vec{0}$;
2. $\vec{N}(\mathcal{F}, P_0) \parallel \vec{R}(\mathcal{F})$.



$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

Definicija 2. *Os sistema sil $\mathcal{F}$ je premica v smeri $\vec{R}(\mathcal{F})$, ki gre skozi točko P_0 v kateri je $\vec{N}(\mathcal{F}, P_0) \parallel \vec{R}(\mathcal{F})$.*

$$\vec{N}(\mathcal{F}, P_0) = \vec{P_0O} \times \vec{R}(\mathcal{F}) + \vec{N}(\mathcal{F}, O) \Big| \times \vec{R}(\mathcal{F})$$

$$\vec{0} = \vec{N}(P_0) \times \vec{R} = (\vec{P_0O} \times \vec{R}) \times \vec{R} + \vec{N}(O) \times \vec{R}$$

$$\vec{0} = (\vec{P_0O} \cdot \vec{R})\vec{R} - (\vec{R} \cdot \vec{R})\vec{P_0O} + \vec{N}(O) \times \vec{R}$$

$$\rightarrow \vec{P_0O} = \frac{1}{|\vec{R}|^2}\left((\vec{P_0O} \cdot \vec{R})\vec{R} + \vec{N}(O) \times \vec{R}\right); \qquad \vec{R}(\mathcal{F}) \neq 0$$

$$\vec{OP_0} = -\vec{P_0O}$$

Analitična določitev osi sistema.
Krajevni vektor od poljubnega pola O do točke P_0 na osi sistema je

$$\boxed{\vec{OP_0} = \frac{\vec{R}(\mathcal{F}) \times \vec{N}(\mathcal{F}, O)}{\left|\vec{R}(\mathcal{F})\right|^2}.}$$

$\mathcal{F}$; $\vec{R}(\mathcal{F})$; $\vec{N}(\mathcal{F}, O)$ $\qquad$ $I(\mathcal{F}) = \vec{R}(\mathcal{F}) \cdot \vec{N}(\mathcal{F}, O)$

$(\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Invarianta sistema sil

Invarianta sistema sil $I(\mathcal{F}) = \vec{R}(\mathcal{F}) \cdot \vec{N}(\mathcal{F}, O)$.

$\vec{N}(\mathcal{F}, P_0) = \vec{P_0 O} \times \vec{R}(\mathcal{F}) + \vec{N}(\mathcal{F}, O) \quad / \cdot \vec{R}(\mathcal{F})$

$\vec{N}(\mathcal{F}, P_0) \cdot \vec{R}(\mathcal{F}) = \underbrace{(\vec{P_0 O} \times \vec{R}(\mathcal{F})) \cdot \vec{R}(\mathcal{F})}_{= \vec{0}} + \vec{N}(\mathcal{F}, O) \cdot \vec{R}(\mathcal{F})$

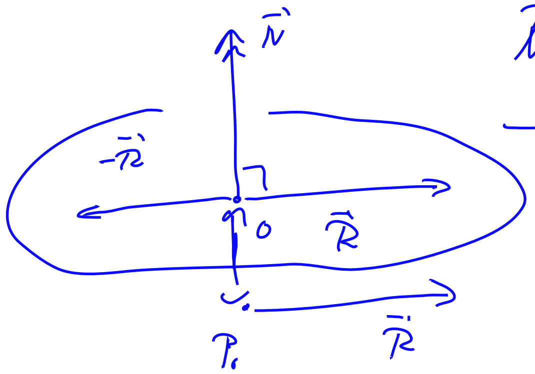
$\vec{a} \cdot \vec{b} = 0$

1. $\vec{a} = \vec{b} = \vec{0}$
2. $\vec{a} \neq \vec{0}$, $\vec{b} = \vec{0}$

Invarianta sistema sil je neodvisna od pola.

Redukcija sistema sil:

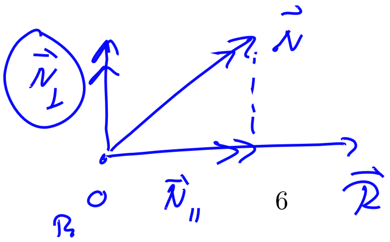
- $I(\mathcal{F}) = 0$
  1. $\vec{R}(\mathcal{F}) = \vec{0}$ in $\vec{N}(\mathcal{F}, O) = \vec{0}$: *ravnovesni sistem sil*;
  2. $\vec{R}(\mathcal{F}) \neq \vec{0}$ in $\vec{N}(\mathcal{F}, O) = \vec{0}$: *sistem sil s skupnim prijemališčem v O*;
  3. $\vec{R}(\mathcal{F}) = \vec{0}$ in $\vec{N}(\mathcal{F}, O) \neq \vec{0}$: *dvojica sil*;
  4. $\vec{R}(\mathcal{F}) \neq \vec{0}$, $\vec{N}(\mathcal{F}, O) \neq \vec{0}$ in $\vec{R}(\mathcal{F}) \perp \vec{N}(\mathcal{F}, O)$: *sistem sil ima skupno prijemališče* na osi sistema;
- $I(\mathcal{F}) \neq 0$: sistem sil nima skupnega prijemališča. Sistem sil je *dinama*.



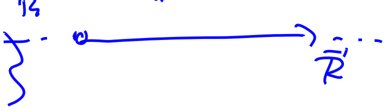
$\vec{N}(\mathcal{F}, O) \equiv \{(O, -\vec{R}), (P_0, \vec{R})\}$

$\mathcal{F} = \vec{N}(\mathcal{F}, O) \cup \{(O, \vec{R})\} = \{(O, -\vec{R}), (P_0, \vec{R}), (O, \vec{R})\} = \{(P_0, \vec{R})\}$

$I(\mathcal{F}) \neq 0$



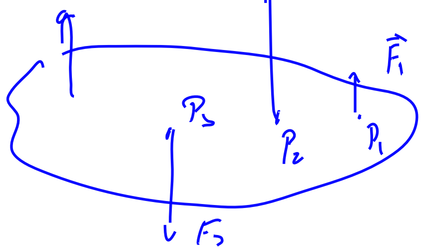
$\vec{N}_\perp \cup \mathcal{F} \equiv \{(P_0, \vec{R})\}$



$\mathcal{F} \equiv \vec{N}_\parallel \cup \{(P_0, \vec{R})\}$

$\vec{N}(\mathcal{F}, P_0) = \vec{N}_\parallel \parallel \vec{R}$ dinam-

$\vec{F}_3$



Vsa prijemališča so v isti ravnini, kjer leži O.

Primer: sistem sil z vzporednimi silami $\vec{F}_i = m_i \vec{F}_0$. Če je $m = \sum_{i=1}^{N} m_i \neq 0$ je ta sistem sil ekvipolenten rezultanti $m\vec{F}_0$, ki ima prijemališče v masnem središču.

$\vec F_i = m_i \vec F_0$ $\quad \mathcal{F} = \{(P_1, m_1 \vec F_0), \ldots, (P_N, m_N \vec F_0)\}$

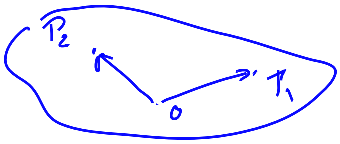
$\vec R(\mathcal F) = \sum_{i=1}^N \vec F_i = \left(\sum_{i=1}^N m_i\right) \vec F_0 = m \vec F_0 \qquad m = \sum_{i=1}^N \vec F_i$

$m = 0 \Rightarrow \vec R(\mathcal F) = \vec 0 \Rightarrow I(\mathcal F) = 0.$

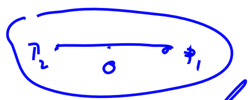
$\vec N(\mathcal F, O) = \sum_{i=1}^N \vec{OP_i} \times \vec F_i = \sum_{i=1}^N \vec{OP_i} \times m_i \vec F_0 = \left(\sum_{i=1}^N m_i \vec{OP_i}\right) \times \vec F_0 \quad ; \quad \vec F_0 \neq \vec 0$

$\vec N(\mathcal F, O) = \vec 0 \qquad$ 1) $\sum_{i=1}^N m_i \vec{OP_i} = \vec 0 \qquad$ 2) $\sum_{i=1}^N m_i \vec{OP_i} \parallel \vec F_0$

ni možna



$m_1 = m_2 = 0$



V posebnem primeru je sistem sil lahko ravnovesen.

$m \neq 0$; $\quad \vec R(\mathcal F) = m \vec F_0 \neq \vec 0. \qquad \vec N(\mathcal F, O) = \sum_{i=1}^N m_i \vec{OP_i} \times \vec F_0$

$I(\mathcal F) = \vec R(\mathcal F) \cdot \vec N(\mathcal F, O) = m \vec F_0 \cdot \left(\left(\sum m_i \vec{OP_i}\right) \times \vec F_0\right) = \vec 0$

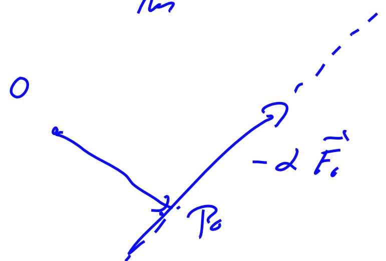
(Poljubni ravninski sistem sil lahko reduciramo na dve sili, ki imata prijemališči v poljubno izbranih točkah.)

$I(\mathcal F) = 0 \Rightarrow \mathcal F$ ima skupno prijemališče. Poiščimo ga.

$$\vec{OP_0} = \frac{\vec R(\mathcal F) \times \vec N(\mathcal F, O)}{|\vec R(\mathcal F)|^2} = \frac{m \vec F_0 \times \left(\left(\sum m_i \vec{OP_i}\right) \times \vec F_0\right)}{m^2 |\vec F_0|^2} =$$

$$= \frac{1}{m |\vec F_0|^2} \left((\vec F_0 \cdot \vec F_0)\left(\sum m_i \vec{OP_i}\right) - \left(\left(\sum m_i \vec{OP_i}\right) \cdot \vec F_0\right) \vec F_0 \right) =$$

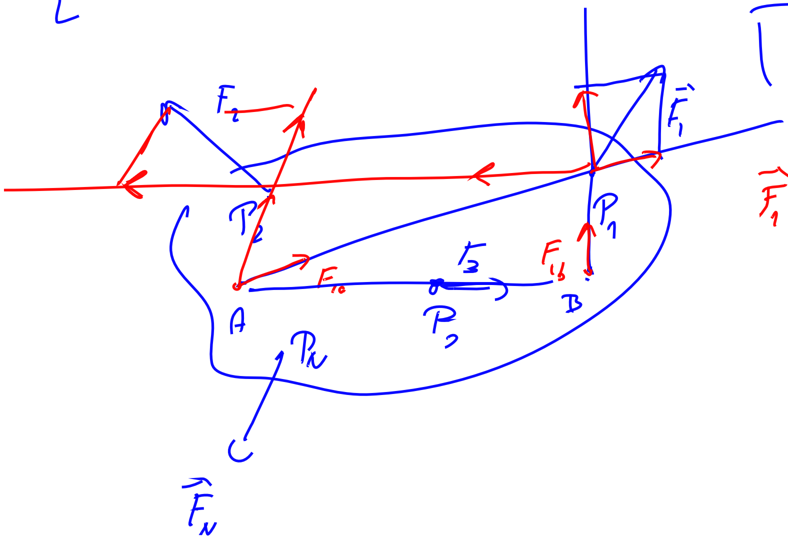
$$= \frac{1}{m} \sum m_i \vec{OP_i} - \alpha \vec F_0 \qquad \alpha = \frac{\left(\sum m_i \vec{OP_i}\right) \cdot \vec F_0}{m |\vec F_0|^2}$$



$$\boxed{\vec{OP_0} = \frac{1}{m} \sum m_i \vec{OP_i}}$$

$\vec{F}_i = m_i \cdot \vec{g}$; $m_i > 0$; $\Rightarrow P_0 = P_*$

Poljubni prostorski sistem sil lahko reduciramo na tri sile, ki imajo prijemališča v poljubno izbranih treh nekolinearnih točkah.



$\boxed{\mathcal{S} \equiv \{(A, \vec{Q}_1), (B, \vec{Q}_2)\}}$

$\vec{F}_1 = \vec{F}_{10} + \vec{F}_{11}$

A, B, C