Obstructions for simple embeddings

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Abstract

Suppose that $K \subseteq G$ is a graph embedded in some surface and F is a face of K with singular branches e and f such that $F \cup \partial F$ is homeomorphic to the torus minus an open disk. An embedding extension of K to G is a *simple embedding* if each K-bridge embedded in F is attached to at most one appearance of e and at most one appearance of f on ∂F. Combinatorial structure of minimal obstructions for existence of simple embedding extensions is described. Moreover, a linear time algorithm is presented that either finds a simple embedding, or returns an obstruction for existence of such embeddings.

1 Introduction

Let K be a subgraph of a graph G and suppose that we are given an embedding of K into a (closed) surface Σ. The *embedding extension problem* asks whether it is possible to extend the given embedding of K to an embedding of G, and any such embedding is said to be an *embedding extension* of K to G. An *obstruction* for embedding extensions is a subgraph Ω of $G - E(K)$ such that the embedding of K cannot be extended to $K \cup \Omega$.

Special cases of embedding extension problems have been treated in [5, 6, 7, 8]. In this paper we examine a more involved special case of the embedding extension problem where K is 2-cell embedded in some surface and there is a face F of K such that there are two branches e, f of K appearing twice on ∂F in order e, f, e^-, f^- (where e^-, f^- denote the traversals of the corresponding branches in the reverse direction) and no other part of ∂F is singular (cf. Section 2). We are interested in obstructions for extending the embedding of K such that all bridges of K are embedded in the face F in such a way that each of them is attached to at most one appearance of e and to at most one appearance of f. We refer to such embeddings as *simple embeddings*. Importance of detecting and classifying obstructions for simple embeddings lies in the fact (cf. [9, 10]) that obstructions for general embedding extension problems can be expressed by means of these obstructions and the obstructions described in [6].

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Embedding extension problems can be used, in particular, to construct embeddings of graphs in surfaces, or to find obstructions for such embeddings. Given G, K, and F as above, we are able to find a simple embedding extension in F, or construct a "nice" obstruction for the existence of simple embeddings by means of a linear time algorithm (see Theorem 6.1). It is shown that the obtained obstructions can be transformed into obstructions of bounded size if we allow K to be changed (see Section 7). This result is used as one of the basic steps in the design of linear time algorithms for embeddability of graphs in general surfaces [9, 10].

At the same time, combinatorial structure of minimal obstructions is considered. The structure can be described in terms of "millipedes" which are rather simple but arbitrarily large subgraphs encountered also in other embedding extension problems (cf. [5]–[8]).

Embeddings in orientable surfaces can be described combinatorially by specifying a *rotation system* [3]: for each vertex v of the graph G we have a cyclic permutation π_v of its neighbors, representing their circular order around v on the surface. In order to make a clear presentation of our algorithm, we have decided to use this description only implicitly. Whenever we say that we have an embedding, we mean such a combinatorial description. Whenever used, it is easy to see how one can combine the embeddings of some parts of the graph described this way into the embedding of larger species.

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for basic operations. This model was introduced by Cook and Reckhow [2] and is used also in other similar algorithms, e.g. [4]. More precisely, our model is the *unit-cost* RAM where operations on integers of value $O(n)$ need only constant time (where n is the order of the given graph).

2 Obstructions

Let K be a subgraph of G. A vertex of K of degree in K different from 2 is a *main vertex* of K. For convenience, if a connected component of K is a cycle, then we choose an arbitrary vertex of it and declare it to be a main vertex of K as well. A *branch* of K is any path in K (possibly closed) whose endpoints are main vertices but no internal vertex on this path is a main vertex. A *bridge* of K in G (also called a *K-bridge* in G) is a subgraph of G which is either an edge $xy \in E(G)\backslash E(K)$ (together with its ends) such that $x, y \in V(K)$, or it is a connected component of $G - V(K)$ together with all edges (and their endpoints) joining this component and K. Each edge of a K-bridge B with an endpoint in K is a *foot* of B. The vertices of $B \cap K$ are the *vertices of attachment* of B. If a K-bridge is attached to a single branch of K, it is said to be *local*. The number of branches of K is called the *branch size* of K.

Let $K \subseteq G$ be a subgraph of G with a given 2-cell embedding in some surface. Let F be a face of K. A main vertex x or a branch e of K is *singular* in F if it appears more than once on the facial walk ∂F of F. The face F is *singular* if it contains a singular branch or a singular vertex. If ∂F contains exactly k singular branches (and no other singular parts), then F is said to be *k-singular*. We say that F is *at most 1-singular* if it is either 1-singular, non-singular, or the only singularity is a vertex that appears on ∂F exactly twice.

Suppose that $\Omega \subseteq G - E(K)$. If Ω has the property that there are no embedding

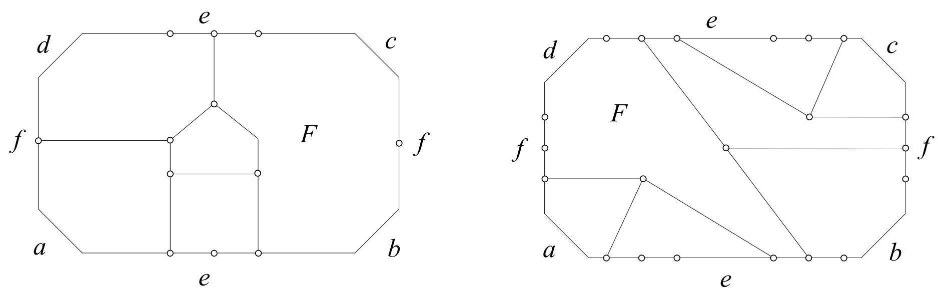

Figure 1: Corner obstructions.

extensions of K to $K \cup \Omega$ with certain properties, then Ω is said to *obstruct* embeddings of G with these properties. We will be interested in obstructions Ω that have no simple embeddings, i.e., Ω obstructs simple embeddings. Such an obstruction will be called a *corner obstruction*. In Figure 1, two examples of corner obstructions for embeddings into a 2-singular face are presented. The first one is a single bridge, while the second consists of three bridges.

To measure the size of Ω we use the number $b(\Omega)$ which is equal to the number of branches of $K \cup \Omega$ that are contained in Ω. We say that Ω is of *bounded size* if $b(\Omega)$ is bounded by certain constant (independent of G).

The next result will enable us to replace every K-bridge in G by a subgraph of bounded size such that the simple embedding extension problem for the new graph is equivalent to the original one.

Lemma 2.1 *([7]) Every K-bridge B in G contains a subgraph $\tilde{B}$ with $b(\tilde{B}) \leq 71$ such that for an arbitrary set of non-local K-bridges $B_1, \ldots, B_k$, every simple embedding of $K \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_k$ can be extended to a simple embedding of $K \cup B_1 \cup \cdots \cup B_k$. Additionally, if a bridge B is attached only to both singular branches of a 2-singular face and B admits a simple embedding, then $b(\tilde{B}) \leq 5$. Moreover, the replacement of all K-bridges B by their subgraphs $\tilde{B}$ can be done in linear time.*

Throughout the paper we will assume that no K-bridge in G is local. By Lemma 2.1 we can replace all bridges B by their subgraphs $\tilde{B}$ of bounded size. From now on we will assume that this replacement has already been made.

If Ω is an obstruction, let $\mathcal{B}(\Omega)$ be the union of those (reduced) K-bridges that contain at least one edge of Ω. By Lemma 2.1, $\mathcal{B}(\Omega)$ has bounded size if and only if the number of bridges in Ω is bounded. Therefore we may work only with obstructions composed of entire K-bridges. Having two such obstructions, Ω_1 and Ω_2, we can *combine* them into a single obstruction by taking their *union*, $\Omega = \Omega_1 \cup \Omega_2$, that obstructs all embedding extensions that are obstructed by either of them. By our assumption, $b(\Omega) \leq b(\Omega_1) + b(\Omega_2)$.

Let F be at most 2-singular face of K with singular branch e. The two appearances of e on ∂F will be distinguished as the *lower* and the *upper* appearance. An *upper-embedding* (respectively, a *lower-embedding*) of a K-bridge B is a simple embedding of B in F such that it is not attached to e at its lower (respectively, upper) appearance.

A *millipede* in G *based* on e is a subgraph M of $G - E(K)$ which can be expressed as $M = B_1^\circ \cup B_2^\circ \cup \cdots \cup B_m^\circ$ $(m \geq 6)$ where:

(M1) Each of B_1° and B_m° is a union of a bounded number of K-bridges in G that has (or is allowed to have) a fixed simple embedding in F.

(M2) $B_2^\circ, \ldots, B_{m-1}^\circ$ are distinct K-bridges.

(M3) For $i = 2, \ldots, m-2$, B_i° and B_{i+1}° have neither simultaneous upper-embeddings nor simultaneous lower-embeddings but they have both simultaneous simple embeddings where B_i° is upper-embedded and B_{i+1}° is lower-embedded, or vice-versa. The bridges $B_1^\circ \cup B_2^\circ \cup B_{m-1}^\circ \cup B_m^\circ$ have unique simple embedding in F. Moreover, if m is even, then, under this embedding, B_2° and B_{m-1}° are embedded in the same way (both upper- or both lower-embedded). If m is odd, then B_2° and B_{m-1}° are embedded in distinct ways.

(M4) For $i \geq 1$ and $i + 2 \leq j \leq m$, no simple embedding of B_i° interferes with any simple embedding of B_j°.

In this paper, we may assume that each of B_1° and B_m° contains at most 23 K-bridges.

By (M1) and (M3) the millipede M obstructs those simple embedding extensions in F for which B_1° and B_m° are embedded in accord to (M1). Millipedes can be arbitrarily long. See Figure 2 for an example. Millipedes which are not of bounded size do not contain obstructions of bounded size since the removal of an arbitrary bridge B_i° from M gives a subgraph which admits simple embeddings (by (M3) and (M4)).

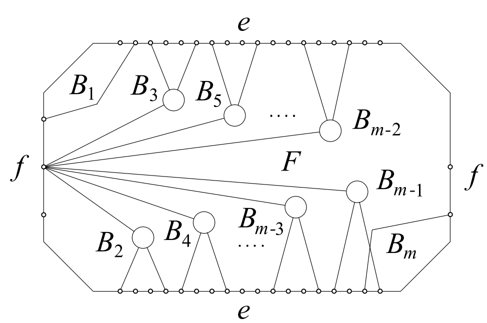

Figure 2: A millipede based on e.

Let M be a millipede. For $i = 2, 3, \ldots, m-1$, denote by l_i and r_i the leftmost and the rightmost attachment of B_i° on e, respectively, where "left" and "right" is with respect to Figure 2. (It follows by (M3) and $m \geq 6$ that B_i° has at least one attachment on e.)

By (M3) and (M4) if follows that $M^\circ := B_2^\circ \cup \cdots \cup B_{m-1}^\circ$ has exactly two (substantially different) simple embeddings. Let $\mathcal{B}_\ell$ be the set of those bridges B_i°, $2 \leq i \leq m-1$, that are not attached to the "right" part of $\partial F - e$. Suppose that $|\mathcal{B}_\ell| \geq 4$. It is easy to see that in each simple embedding of M°, at least one bridge from $\mathcal{B}_\ell$ is lower-embedded and at least one bridge from $\mathcal{B}_\ell$ is upper-embedded. Denote by B_ℓ^{lo} (B_ℓ^{up}) the "leftmost" lower-embedded (upper-embedded, respectively) bridge from $\mathcal{B}_\ell$ (under one of the two simple embeddings of M°). Note that the pair $\{B_\ell^{lo}, B_\ell^{up}\}$ is independent of the choice of the embedding. Similarly we define $B_r^{lo}, B_r^{up} \in \mathcal{B}_\ell$ (with respect to the same embedding of M°). We claim that B_ℓ^{lo} and B_ℓ^{up} have neither simultaneous lower nor simultaneous upper-embedding. Suppose, for example, that they have simultaneous upper-embedding. We may assume that B_ℓ^{up} is embedded to the "left" of B_ℓ^{lo}. By (M3), there exists a bridge $B \subset M^\circ$ whose upper-embedding overlaps with the upper-embedding of B_ℓ^{up}. Then B is embedded in the same way as B_ℓ^{lo} under every simple embedding of M°. Obviously, B is not attached to the "right" part of $\partial F - e$ since in that case it would also overlap with B_ℓ^{lo}. On the other hand, $B \notin \mathcal{B}_\ell$, since it would be attached to e to the "left" of B_ℓ^{lo} which is a contradiction with our choice of B_ℓ^{lo}. We also want that B_r^{lo} and B_r^{up} have neither simultaneous lower nor simultaneous upper-embedding. If this is not the case, suppose that B_r^{up} is attached to e to the right of B_r^{lo}. It is easy to see that B_r^{lo} overlaps only with bridges in $\mathcal{B}_\ell$. Replace B_r^{up} with the "rightmost" bridge of M° that overlaps with B_r^{lo}. Then B_r^{lo} and B_r^{up} have the required property. In the case when B_r^{up} has been changed, we also remove the original B_r^{up} from the set $\mathcal{B}_\ell$.

Let f' be the "rightmost" foot of B_ℓ^{lo} on e. (The rightmost foot is well defined with respect to a simple embedding of B_ℓ^{lo} in F.) Subdivide f' by inserting a new vertex v_ℓ^{lo} of degree 2. Introduce similarly vertices v_ℓ^{up} in B_ℓ^{up}, and v_r^{lo}, v_r^{up} in B_r^{lo}, B_r^{up}, respectively (in the latter two cases with respect to their "leftmost" feet). Then add to M the edges $f_1 = v_\ell^{lo} v_r^{lo}$ and $f_2 = v_\ell^{up} v_r^{up}$. Finally, delete bridges $\mathcal{B}_\ell \setminus \{B_\ell^{lo}, B_\ell^{up}, B_r^{lo}, B_r^{up}\}$ from M.

If $|\mathcal{B}_\ell| < 4$, then we leave M unchanged. We repeat the same procedure with bridges $\mathcal{B}_r$ that are not attached to the "left". After these changes, the obtained set of bridges contains B_1°, B_m° and at most 8 additional bridges. (We have at most three bridges from the original $\mathcal{B}_\ell$, at most three from $\mathcal{B}_r$, and at most two bridges that are attached both to the "left" and "right".) Denote the resulting graph by $\tilde{M}$ and call it the *squashed millipede*. This way we reduce the size of M, while essentially preserving its embedding extension properties. To preserve also the interference of M with other bridges of K, we need to apply another change described below (the operation (SQ1) or (SQ2)).

Suppose that in the procedure of squashing we had $|\mathcal{B}_\ell| \geq 4$. Let $L_1 = B_\ell^{lo}$, $L_2 = B_\ell^{up}$, $R_1 = B_r^{lo}$, $R_2 = B_r^{up}$ be the bridges that participate in squashing of $\mathcal{B}_\ell$. By (M3) and (M4) it follows that $L_1 \cup L_2 \cup R_1 \cup R_2$ has at most one attachment out of e. We call this attachment the *left apex* of the millipede. Denote by l_i' the "leftmost" vertex of attachment on e of L_i and by r_i' the "rightmost" attachment on e of R_i $(i = 1, 2)$. Let $I = \bigcap_{i=1,2} (l_i', r_i') \subseteq e$ and let $D_\ell \subseteq G - E(K)$ be the union of all K-bridges that have an attachment in I. Similarly we define D_r with respect to $\mathcal{B}_r$ (if $|\mathcal{B}_r| \geq 4$). We also get the *right apex* of M. The set consisting of the left and the right apex (if defined) of M is the *apex* of M. Let $D = D_\ell \cup D_r$. The bridges in D will be used in defining the squashing operation on G. We distinguish two cases:

(SQ1) *$M^\circ \cup D$ admits a simple embedding in F.* We replace $M^\circ \cup D$ in the graph G by the squashed millipede $\tilde{M}$ and denote the obtained graph by $\tilde{G}$. It is easy to see that the embedding of K can be simple extended to $\tilde{G}$ if and only if it can be simple extended to G. Note that $\tilde{M}$ is contained in $\tilde{G}$. The operation of replacing G by $\tilde{G}$ and replacing M by $\tilde{M}$ is called *squashing* of the millipede M. It is important that any obstruction for simple embedding extensions to $\tilde{G}$ is also an obstruction for G.

(SQ2) *$M^\circ \cup D$ has no simple embedding in F.* In this case, $M^\circ \cup D$ contains a corner obstruction Ω (of bounded size) for extending the embedding of K to a simple embedding of $K \cup M^\circ \cup D$. Such an obstruction can be obtained in linear time as follows. If D contains a bridge B' that is not attached only to I and to the apex of the millipede, then $\Omega = B' \cup L_1 \cup L_2 \cup R_1 \cup R_2$ is an obstruction of bounded size. Otherwise, let $\{x, y\}$ be the apex of the millipede. If $x \neq y$, let $K' = K \cup xy$. If $x = y$, we add an edge xz instead of xy where z is a vertex of K in the part of $\partial F - e$ that does not include x. Results of [8] can be applied to get the required corner obstruction Ω contained in $M^\circ \cup D$ by solving a 2-prism embedding extension problem for $K' \cup M^\circ \cup D$ extending the obvious embedding of K. (Note that solving this auxiliary 2-prism problem also enables us to distinguish between (SQ1) and (SQ2).) We may assume that every K'-bridge in $K' \cup M^\circ \cup D$ has a simple embedding in F. Then it admits upper and lower embeddings. Results of [8, Theorem 7.1] imply that in this case Ω contains at most 4 bridges. For convenience, the replacement of M by Ω is also called *squashing* of M. In this case, $\tilde{G} = G$, and M is replaced by the obstruction Ω of bounded size that obstructs stronger than M.

Proposition 2.2 *Let M be a millipede. Then the graph $\tilde{G}$ obtained after squashing of M admits essentially the same simple embedding extensions as G. More precisely, every simple embedding extension to G gives rise to a simple embedding of $\tilde{G}$ that coincides with the embedding of G on $G \cap \tilde{G}$. Conversely, having a simple embedding of $\tilde{G}$, the embedding of $G \cap \tilde{G}$ can be extended to a simple embedding of G.*

Proof. If $G = \tilde{G}$, there is nothing to prove, so assume that $m \geq 9$. The case (SQ1) is easy to verify, and we leave the details to the reader. In the case (SQ2), $\Omega \subseteq G - E(K)$. Since Ω is an obstruction for any simple embedding extensions, the claims of the proposition are vacuously satisfied. □

Proposition 2.2 enables us (after squashing) to pretend that M is of bounded size (since M turns into $\tilde{M}$). If we use squashing, the new edges f_1, f_2 have to be replaced by $B_2^\circ \cup \cdots \cup B_{m-1}^\circ$ at the very end.

Let Ω be a corner obstruction. We say that Ω is of *class 0*, if it is of bounded size and does not contain millipedes. For $k > 0$, Ω is of *class* k if it contains a millipede $M = B_1^\circ \cup \cdots \cup B_m^\circ$ (with respect to some choice of allowed embeddings of B_1° and B_m°) such that $\tilde{\Omega}$ obtained after squashing the millipede M is of class $k-1$. In particular, every millipede is of class 1. Finally, Ω is said to be *nice* if it is of class k for some bounded $k \geq 0$. Proposition 2.2 shows that nice obstructions have bounded number of different simple embeddings.

Suppose that we have an obstruction Ω_0 of class k. Let $\tilde{G}$ be the graph obtained by successive squashing of k millipedes in Ω_0. Denote by $\tilde{\Omega}_0$ the corresponding squashed obstruction Ω_0. Suppose that Ω_1 is an obstruction of class l in $\tilde{G}$. Then $\tilde{\Omega} = \tilde{\Omega}_0 \cup \Omega_1$ determines an obstruction Ω in G (after replacing squashed parts by millipedes) which is of class $k + l$ or less. We will use such a recursive construction of obstructions in our algorithms and simply refer to them as a *union* of Ω_0 and Ω_1 (or $\tilde{\Omega}_0$ and Ω_1). The operation of obtaining the union will also be called *combining* of obstructions.

3 2-restricted embeddings

Let $K \subseteq G$ be 2-cell embedded in some orientable surface. Denote by $\mathcal{B}$ a set of K-bridges in G. Suppose that $\mathcal{B}$ contains no local bridges and that for every $B \in \mathcal{B}$ at most two simple embeddings in faces of K are allowed. The following result is proved in [6].

Lemma 3.1 *([6]) Let G, K, $\mathcal{B}$ be as above, and let $Q = \bigcup\{B \mid B \in \mathcal{B}\}$. There is a linear time algorithm which either finds an embedding extension of K to $K \cup Q$, or returns a nice obstruction $\Omega \subseteq Q$ for those embedding extensions that use only allowed embeddings of bridges in $\mathcal{B}$.*

The same result holds for non-orientable surfaces [6]. However, the notion of millipedes (and hence of nice obstructions) must be slightly extended in such a case.

We will apply the algorithm of Lemma 3.1 in the case when the embedding of K has a 2-singular face F. In such a case there may exist bridges with up to 4 different simple embeddings. In order to be able to apply the 2-restricted embedding algorithm, we choose one of the singular branches, say f, and prohibit bridges to be attached to one of its appearances on ∂F. Suppose that a nice obstruction which contains a millipede M based on the other singular branch e has been obtained and that (SQ1) has been performed for squashing. In our applications we later test all simple embeddings of M (including those embeddings that have bridges attached to the "forbidden" appearance of f). If two bridges B_i°, B_j° of M° with indices i, j of the same parity are attached to different appearances of f, then the corresponding embedding of the squashed millipede $\tilde{M}$ is not simple. However, the embedding of $\tilde{M}$ can be replaced by a simple embedding of $\tilde{M}$ without affecting the rest of the embedded graph since the case (SQ1) was used to squash M. Therefore, it is enough to consider only simple embeddings of $\tilde{M}$ when trying to find a simple embedding extension of M to G.

4 Planarity

There are well known linear time algorithms which for a given graph determine whether the graph is planar or not. The first such algorithm was obtained by Hopcroft and Tarjan [4]. Extensions of planarity testing algorithms also return an embedding (rotation system) if the input graph is planar [1], or find a small obstruction – a subgraph homeomorphic to K_5 or $K_{3,3}$ – if the graph is non-planar [11, 12]. Recall that a subgraph of G homeomorphic to K_5 or $K_{3,3}$ is called a *Kuratowski subgraph* of G.

Lemma 4.1 *There is a linear time algorithm that, given a graph G, either exhibits an embedding of G in the plane, or finds a Kuratowski subgraph of G.*

We refer to the algorithm of Lemma 4.1 as *planarity testing*. The following extension of planarity testing is presented in [8]:

Lemma 4.2 *([8]) Let F be a non-singular face of 2-cell embedded subgraph K of G, and let $\mathcal{B}$ be a set of K-bridges in G. There is a linear time algorithm which either finds a simultaneous embedding of $\mathcal{B}$ in F, or returns an obstruction of bounded size for such embeddings.*

By using Lemma 3.1, we can solve the embedding extension problem when the face F contains a singular branch or a singular vertex (and no other singular parts):

Lemma 4.3 *Let F be a singular face of 2-cell embedded subgraph K of G and let $\mathcal{B}$ be a set of K-bridges in G. Suppose that the only singular piece of F is a branch e of K such that $F \cup e$ is homeomorphic to the cylinder, or it is a vertex which appears at most twice on ∂F. There is a linear time algorithm which either finds a simultaneous simple embedding of $\mathcal{B}$ in F, or returns a nice corner obstruction for such embeddings.*

More involved is the situation when we have a set $\mathcal{B}$ of bridges embeddable in more than just a single (singular) face F. We will need the following special case of such a situation.

Lemma 4.4 *Let F be a 2-singular face of 2-cell embedded subgraph K of G and let $\mathcal{B}$ be a set of K-bridges in G. Suppose that the singular pieces of F, branches e and f, appear on ∂F in interlaced order, e, f, e^-, f^-. Let P be either a path, or the union of two paths embedded in F as shown in Figure 3. Suppose that all bridges in $\mathcal{B}$ are of bounded size (including those containing P). Suppose also that there are no local K-bridges in G. Then there is a linear time algorithm that either finds a simple embedding of $\mathcal{B}$ in F extending the embedding of $K \cup P$, or discovers a nice obstruction for such embedding extensions.*

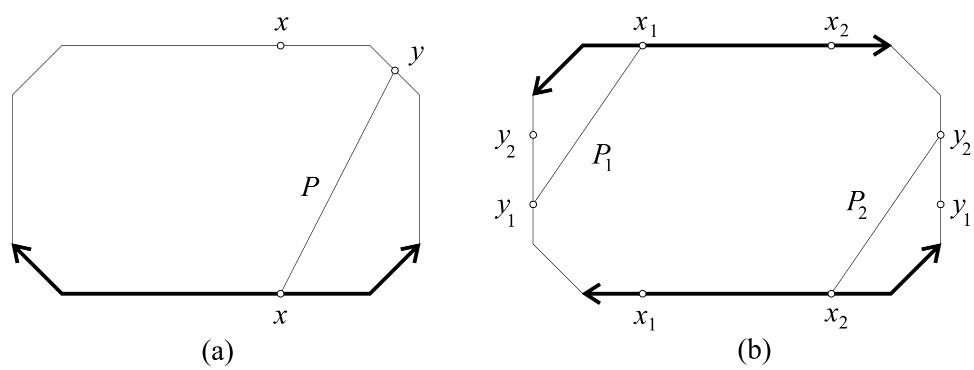

Figure 3: Removal of 2-singularity by one or two paths.

Remark. The path P represented in Figure 3(a) is attached to vertex y on a non-singular part of ∂F and to vertex x such that P leaves at most 1-singular faces. (Admissible region for x is shown bold. Arrows indicate that the corresponding endpoint of the bold segment is not included.) In case of Figure 3(b), P consists of two paths P_1 and P_2 that together leave at most 1-singularity. Vertices y_1, y_2 are both on f as shown, while x_1 and x_2 can appear anywhere on the corresponding bold segments.

Proof. If there are no K-bridges with three essentially different simple embeddings extending the embedding of P in F, then we do the following. For every simple embedding of the bridge(s) containing P that extends the embedding of $K \cup P$, we apply Lemma 3.1. (Note that there are at most four cases.) We either get a simple embedding, or up to 4 nice obstructions for such extensions. The combination of these obstructions is the desired corner obstruction.

Suppose now that $\mathcal{B}$ contains bridges that admit more than two simple embeddings extending the embedding of $K \cup P$. Let $B \in \mathcal{B}$ be one of such bridges. Then B is attached to e and the only attachment on e is the vertex x (x_1 or x_2 in case (b)). In particular, $x \in V(e)$ (respectively, x_1 or x_2 is on e). It is possible that B is attached only to x and y (x_i and y_j, $i, j \in \{1, 2\}$, respectively). In that case, we can assume that B is the only such bridge in case (a) and that there are at most four such bridges in case (b). Otherwise, B is necessarily attached to f, and in case (a), it may have one further attachment, the vertex $y \notin V(e \cup f)$. Note that B has exactly three simple embeddings. (We say that B is *3-embeddable*.) Let $\mathcal{B}_0 \subseteq \mathcal{B}$ be the set of bridges consisting of the bridge(s) containing P, together with 3-embeddable bridges that are attached to y. If $\mathcal{B}_0$ contains more than five bridges, any six of them form an obstruction for simple embeddings which is of bounded size. Thus, we assume that this is not the case. For every simple embedding of bridges in $\mathcal{B}_0$ extending the embedding of $K \cup P$ we continue with the procedure described in the sequel. This procedure will find a nice obstruction for simple extensions of the embedding of $K \cup \mathcal{B}_0$ (or stop after constructing a required simple embedding of $K \cup \mathcal{B}$). Finally, combination of all obtained obstructions is the desired corner obstruction.

After fixing an embedding of $K \cup \mathcal{B}_0$, we may still have 3-embeddable bridges. Their only attachments are the vertex x (or x_1, or x_2) on e and one or more vertices on f.

Let us first suppose we have case (a). As the first step we try to find an obstruction for embedding extensions where all 3-embeddable bridges attach to x at the same side as P (Lemma 3.1). We either stop after obtaining an embedding extension, or else we get a nice obstruction Ω_1.

Now, for every embedding of Ω_1 extending the embedding of $K \cup \mathcal{B}_0$, there is a bridge $Q \in \Omega_1 \cup \mathcal{B}_0$ embedded in the singular part F_1 of F such that Q separates the two appearances of x on ∂F_1. Otherwise, we could re-embed Ω_1 so that the 3-embeddable bridges are attached to x only at the "lower" side of e. Consequently, we can find an obstruction for extensions of this embedding to $K \cup \mathcal{B}$ (or discover an embedding) by applying Lemma 3.1. This lemma can be applied since every remaining bridge has at most two embeddings with a possible exception of an edge which could have 3 embeddings. All such edges (corresponding to different embeddings of Ω_1) can be added to Ω_1 and then we can assume, their embedding is fixed. If we do not get an embedding extension, we combine all obtained nice obstructions with $\Omega_1 \cup \mathcal{B}_0$ into a required nice obstruction Ω.

It remains to explain the corresponding procedure in case (b). If x_2 is to the left of x_1, then there are no 3-embeddable bridges, and it is easy to get a solution. Otherwise, 3-embeddable bridges are of two classes:

(i) Attached to f above y_2, and attached to x_1.

(ii) Attached to f below y_1, and attached to x_2.

If $x_1 = x_2$, then we can proceed as in case (a). Let Ω_1 be a nice obstruction for embedding extensions having all bridges of class (i) in F_1 lower-embedded, and all bridges of class (ii) in F_1 upper-embedded. Every embedding of Ω_1 has a bridge in the central face F_1 separating the two appearances of $x_1 = x_2$. We conclude as above.

Suppose that $x_1 \neq x_2$. Consider nine cases where we allow bridges of class (i) and bridges of class (ii) to have only two of the three possible embeddings (for all bridges in the same class of the same type). In each of these restricted problems, any other bridge has at most two simple embeddings. Therefore, Lemma 3.1 can be used. If we do not get an embedding extension, we obtain nine nice obstructions $\Omega_1, \ldots, \Omega_9$ for such restricted problems. Let Ω' be the combination of $\Omega_1, \ldots, \Omega_9$. Every simple embedding of Ω' extending the embedding of $\mathcal{B}_0$ has either bridges of class (i) or bridges of class (ii) embedded in all three ways (obviously not both at the same time). Moreover, this embedding has only non-singular faces (otherwise we could get rid of three distinct ways of embedding bridges of the same class). Therefore, there is at most one 3-embeddable bridge (attached only to two vertices), and for every embedding of such a bridge we use Lemma 3.1. The combination of $\mathcal{B}_0 \cup \Omega'$ with all obtained obstructions will be a final corner obstruction. □

To apply Lemma 4.4, we will use the following obvious result.

Lemma 4.5 *Let F be a 2-singular face and $\mathcal{B}$ a set of K-bridges simple embedded in F such that we get at most 1-singular faces. Then $\mathcal{B}$ contains path(s) that are embedded as requested in Lemma 4.4. Moreover, such path(s) can be found in linear time.*

5 Weak 2-singularity

Suppose that $K \subseteq G$ is 2-cell embedded in some surface. A face F of K is *weakly 2-singular*, if it contains on its border a singular branch, say e, and a singular main vertex x which is not an endpoint of e and appears on ∂F exactly twice. Moreover, it is required that there are no other singular vertices or branches on ∂F (see Figure 4). A face F is *at most weakly 2-singular* if every vertex appears on ∂F at most twice, and there are a branch e and a vertex x on ∂F that contain all singular vertices and edges of ∂F. (This also includes at most 1-singular faces.)

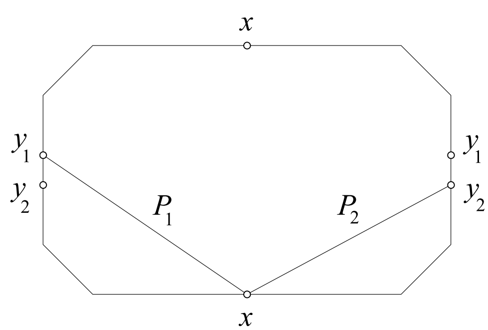

Figure 4: A weakly 2-singular face.

Lemma 5.1 *Let F be a 2-singular face of 2-cell embedded subgraph K of G and let $\mathcal{B}$ be a set of K-bridges in G. Suppose that the singular pieces of F, branches e and f, appear* *on ∂F in interlaced order e, f, e^-, f^-. Let P_1, P_2 be paths embedded in F as shown in Figure 4. Suppose that all bridges in $\mathcal{B}$ are of bounded size (including those containing P_1, P_2). Suppose also that there are no local K-bridges in G. Then there is a linear time algorithm that either finds a simple embedding of $\mathcal{B}$ in F extending the embedding of $K \cup P_1 \cup P_2$, or discovers a nice obstruction for such embedding extensions.*

Remark. Paths P_1, P_2 in Figure 4 are attached to distinct vertices y_1, y_2 (respectively) on f and to vertex x on e such that $P_1 \cup P_2$ leaves weakly 2-singular face. Note that y_1 is closer to the upper appearance of e than y_2. As in Lemma 4.5, paths P_1 and P_2 with these properties can be found (in linear time) whenever a set of bridges simple embedded in a 2-singular face leaves a weakly 2-singular subface.

Proof. Let $\mathcal{B}_0$ be the union of bridges that contain P_1 and P_2. If simple embedding of $\mathcal{B}_0$ extending the embedding of $P_1 \cup P_2$ is at most 1-singular, we apply Lemma 4.4. Otherwise, let $\mathcal{B}_1 \subseteq \mathcal{B} \backslash \mathcal{B}_0$ be the set of bridges that are attached to e only at x and have an attachment on f that is above y_2 (possibly at y_2). Similarly, let $\mathcal{B}_2 \subseteq \mathcal{B} \setminus (\mathcal{B}_0 \cup \mathcal{B}_1)$ be the set of bridges attached to f only at y_1 and attached to e only to the left of x (including x). Note that every other bridge has at most two simple embeddings extending the embedding of $\mathcal{B}_0$. First, we try to embed $\mathcal{B}$ such that no bridge from $\mathcal{B}_1$ is upper-embedded and no bridge from $\mathcal{B}_2$ is attached to the right appearance of y_1. To solve this problem, we can apply Lemma 3.1. Let Ω_1 be the obtained obstruction. We claim that every simple embedding of $\mathcal{B}_0 \cup \Omega_1$ extending the embedding of $\mathcal{B}_0$ is at most 1-singular. Clearly, every such embedding has either a bridge $B_1 \in \mathcal{B}_1$ that is upper-embedded, or a bridge $B_2 \in \mathcal{B}_2$ that is attached to the right. In the latter case, $B_2 \cup P_1$ obviously leaves at most 1-singular faces. In the former case, if the embedding of $\mathcal{B}_0 \cup \Omega_1$ is not at most 1-singular, then there is a face containing two appearances of x. Hence, the upper-embedded bridges from Ω_1 attached to e only at x can all be re-embedded into lower-embedding. This gives a "forbidden" embedding of Ω_1, which is a contradiction.

By the above claim we can use Lemma 4.4 to get nice corner obstructions (or an embedding) for each simple embedding of $\mathcal{B}_0 \cup \Omega_1$. Their combination is a required obstruction. □

6 Simple embeddings

Let K be a subgraph of G such that no K-bridge in G is local. Suppose that K is 2-cell embedded in some surface such that there is a face F of K with singular branches e and f which appear on ∂F interlaced: e, f, e^-, f^-, the second time in the opposite direction. Let $\mathcal{B}$ be a set of K-bridges in G and let $Q = \bigcup\{B \mid B \in \mathcal{B}\}$. In this section we describe an algorithm which decides if it is possible to embed all members of $\mathcal{B}$ in F such that every bridge has a simple embedding.

Let $\partial F = aebfce^-df^-$. The open segments a, b, c, d of ∂F are assumed to be non-singular and pairwise disjoint. The segments $\alpha = f^-ae$, $\beta = ebf$, $\gamma = fce^-$, and $\delta = e^-df^-$ are called the *corners* of F (see Figure 5). We say that a bridge $B \in \mathcal{B}$ is (simple) *embedded in the corner* α if it is embedded in F so that it is attached only to the appearances of e

and f that participate in α. Note that this does not exclude B being attached to segments b, c, or d. Simple embeddings in other corners are defined analogously.

Theorem 6.1 *Let G, K, F, $\mathcal{B}$, and Q be as above. There is a linear time algorithm that either finds a simple embedding extension of K to $K \cup Q$ in F, or returns a nice corner obstruction $\Omega \subseteq Q$ for such embedding extensions.*

Proof. First of all, we determine for each of the bridges $B \in \mathcal{B}$ all of its simple embeddings in F. If $B \in \mathcal{B}$ has no simple embedding in F, then $\Omega = B$ can be taken as a corner obstruction of bounded size (Lemma 2.1). From now on we will assume that every $B \in \mathcal{B}$ admits a simple embedding and that the list of admissible corners is determined. Recall that, according to Lemma 2.1, all bridges are of bounded size.

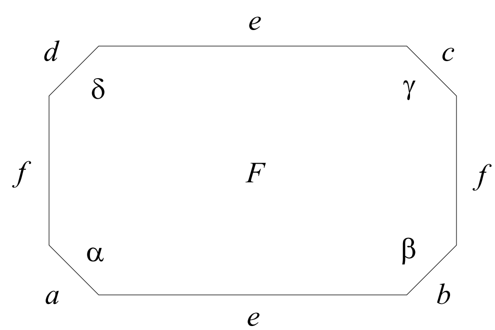

Figure 5: 2-singular face of K.

Suppose that there is a bridge $B \in \mathcal{B}$ attached to a vertex of a and to a vertex of c. Then every simple embedding of B removes the singularity of F. By applying Lemma 4.4 we either obtain a simple embedding, or we get a required nice obstruction. Similarly, when there is a bridge $B \in \mathcal{B}$ attached to a and b. Of course, the same method applies for bridges joining b and d, or b and c, etc. From now on we may assume that there are no bridges attached to two of the segments among a, b, c, or d.

Next, try to embed all bridges attached to a into the corner α, and similarly the bridges at b, c, d into their corners β, γ, δ, respectively. We can use planarity testing to get such an embedding (for all corners simultaneously). If the test fails, we get an obstruction Ω' of bounded size such that every embedding of Ω' in F contains a K-bridge B in $K \cup \Omega'$ which is not embedded in its corresponding corner. We may assume that B is attached to a. Let P be a path from a to an attachment of B which is not in the corner α. Apply Lemma 4.4 to get a simple embedding or a corner obstruction for extending the embedding of Ω'. Finally, the union of obtained obstructions (for different embeddings of Ω') completes our task.

We may henceforth assume that the above planarity test was successful. Throughout the following steps, the bridges in $\mathcal{B}$ will be divided into three classes:

(a) *embedded bridges*,

(b) *labeled bridges*,

(c) bridges that have not yet been processed by the algorithm.

Bridges of particular classes will satisfy some additional properties. Each of the bridges in the above classes (a) and (b) has been assigned to a corner (α, β, γ, or δ) in which it is supposed to be (simple) embedded. All embedded bridges can be simultaneously embedded in their corresponding corners. Let a_e be the segment on ∂F containing a and the part of the singular branch e from a to the "rightmost" attachment on e of embedded bridges in the corner α. Similarly, let a_f be the segment of f from the "uppermost" attachment on f (of bridges of class (a) embedded in the corner α) to a together with a. If a bridge $B \in \mathcal{B}$ has an attachment in the interior of the segment a_e, then it is either embedded in the corner α, or it is a bridge of class (a) or (b) assigned to δ. Similarly, bridges attached in the interior of a_f are either embedded in α, embedded in β, or labeled and assigned to β. During the algorithm, the similar property will hold also for the corners β, γ, and δ. It is a simple consequence of these requirements that no bridge of class (c) interferes with already embedded bridges. Note also that a bridge attached to the interiors of both a_e and a_f is already embedded (i.e., of class (a)). Some bridges assigned to a corner, say α, can also be embedded in a corner different than α. However, every such embedding together with the embedded bridges leaves at most 1-singularity of F.

Furthermore, every labeled bridge has been labeled because of some of the requirements given above. For example, if B is a labeled bridge assigned to β, then it has an attachment in the interior of a_f or in the interior of c_e (corresponding to the corner γ). Properties of labeled bridges given above imply that no labeled bridge assigned to a corner, say α, interferes with the already embedded bridges in α.

Bridges attached to a, b, c, or d have been (simple) embedded in their corresponding corners. Initially, these bridges are of class (a) (i.e., they are considered as embedded). This defines the segments a_e, a_f, b_e, b_f, etc., and all remaining bridges attached to interiors of these intervals are labeled and assigned to appropriate corners. It may happen that some bridge would be assigned to two different corners. In such a case, we proceed as in *Cycling procedure* when the planarity testing fails, with Ω' being one of the bridges assigned to two corners (and $i = 2$). Otherwise, every labeled bridge is assigned to exactly one of the corners. It is easy to see that in this case no labeled bridge, if embedded in its corner, interferes with any of the embedded bridges. (But it may interfere with some other labeled bridges.) All other bridges of $\mathcal{B}$ are unlabeled (of class (c)). At this point we also define B_1 as follows. Consider bridges of class (a) embedded in their corners, and let B_1 be the set of those bridges which are attached to e or to f and have at least one of their edges on the boundary of the "central" subface of F. Note that B_1 contains at most 8 bridges (see Figure 6). If none of the bridges is attached to the interior of a, b, c, or d then classes (a) and (b) remain empty and the subgraph B_1 is undefined.

The algorithm will in a general step choose one of the corners, say α, and try to embed all labeled bridges assigned to α in this corner. If successful, the segments a_e and a_f will be updated and some bridges of class (c) will become labeled and assigned to corners β or δ. If not successful, or if the already embedded bridges (including the new ones) leave at most 1-singularity, we will be able to construct a nice corner obstruction Ω_0 with the property that any simple embedding of Ω_0 in F is at most 1-singular. If none of such embeddings

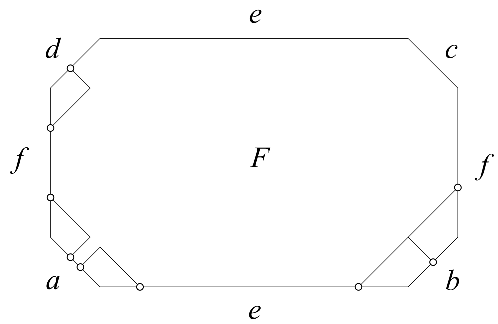

Figure 6: An example of B_1 composed of four bridges.

of Ω_0 can be extended to a simple embedding of $K \cup Q$, then we will get the required corner obstruction.

We let $i = 2$ and proceed with the *Cycling procedure* explained below.

Cycling procedure. Let us now give the details of the general step. At the ith step, the next corner from the circular sequence α, γ, β, δ, is selected. We assume that in the selected corner there is at least one labeled bridge. Otherwise, we select the next corner. (In this case, we do not increment the step counter i in order that the bridges B_i are always defined. If none of the corners contains labeled bridges, we stop this procedure and continue with the part called *Harmless embedding* that is explained below.) Suppose that the selected corner is α. We take all labeled bridges in this corner and try to embed them simultaneously in α using Lemma 4.2. If the test fails, we find an obstruction Ω' consisting of one or two bridges. For every simple embedding of Ω' extending the embeddings of bridges of class (a), Ω' leaves at most 1-singularity in F. Applying Lemma 5.1 or Lemma 4.4 on bridges of classes (b) and (c) we either get a simple embedding extension, or get a nice obstruction Ω'' for embedding extensions. (In applying Lemma 5.1, or Lemma 4.4, already embedded bridges define the boundary of the face F, while the path P is a part of a bridge in Ω'.) The union B_i of Ω' and these obstructions (for different embeddings of Ω') has the property that the original simple embedding of $B_1 \cup \cdots \cup B_{i-1}$ cannot be extended to a simple embedding of $B_1 \cup \cdots \cup B_{i-1} \cup B_i$. In this case we proceed with the part of the algorithm called *Obstruction compression* described in the sequel.

Otherwise, the obtained simple embedding of labeled bridges in α is used as follows. These bridges become unlabeled and embedded in the corner α. Also, the "outermost" bridge is denoted by B_i. According to the change of a_e and a_f, we get new labeled bridges assigned to the corners β and δ. Then we increment the counter i. If some bridge is attempted to be assigned to two corners (including labeled bridges assigned to a different corner than previously), we take the offending bridge as Ω'. Then we proceed as above (obtain B_i and continue with *Obstruction compression*). Another possibility is when $B_1 \cup \cdots \cup B_{i-1}$ removes the 2-singularity. In this case we try to extend the obtained embedding to $K \cup Q$ by using Lemma 5.1 or Lemma 4.4. We may assume that a nice corner obstruction is obtained. Denote it by B_i and proceed with *Obstruction compression*. If none of the above cases occurs, we return to the beginning of the *Cycling procedure*. It is easy to see that the amount of time spent by the *Cycling procedure* is linear in the number of bridges that it embeds or labels.

Harmless embedding. We reach this case when there are no labeled bridges left. Denote by $\mathcal{B}_0$ the embedded bridges. The bridges in $\mathcal{B}_0$ embedded in their corresponding corners do not interfere with any of the remaining bridges, and they can be simultaneously embedded in their corners. Thus, it is clear that the embedding of K can be extended to a simple embedding of $K \cup Q$ if and only if the embedding of K can be extended to a simple embedding of $K \cup Q'$, where $Q' := \bigcup\{B \mid B \in \mathcal{B} \setminus \mathcal{B}_0\}$. Therefore, we may repeat the above procedure with the smaller set of bridges $\mathcal{B}' := \mathcal{B} \setminus \mathcal{B}_0$. There is a slight difference, though. Now we do not have any bridges attached to the interiors of a, b, c, or d. (It may happen that we have this situation from the very beginning in which case the first traversal of the *Cycling procedure* was void.) If $\mathcal{B}' = \emptyset$ then we have a simple embedding. Otherwise, let B_1 be a bridge in $\mathcal{B}'$. If possible, we choose B_1 so that it has at least three vertices of attachment. The bridge B_1 has at most four different simple embeddings. For each of them, we will repeat the following procedure. Let us consider the case when B_1 will be embedded into the corner α. Define a_e and a_f as before. We label and assign to the corner α all bridges of $\mathcal{B}'$ that are attached to an interior point of a_e and to an interior point of a_f. In addition, we also label all bridges that have all vertices of attachment in $a_e \cup a_f$ (boundary vertices included). Using planarity testing (Lemma 4.2), we check if all labeled bridges (including B_1) can be simultaneously embedded in α. If so, then we repeat *Cycling procedure* on the set $\mathcal{B}'$ starting with $i = 1$, where none of the bridges is embedded, and the labeled bridges are selected as above and assigned to the chosen corner α.

The above planarity test may have failed. In this case we get an obstruction Ω' consisting of one or two bridges. It is easy to see that no simple embedding of $B_1 \cup \Omega'$ with B_1 in α is 2-singular. For every such simple embedding of $B_1 \cup \Omega'$ we apply Lemma 5.1 (or Lemma 4.4). We either get a simple embedding extension from K to $K \cup Q'$ (and hence also an embedding extension to $K \cup Q$), in which case we are done, or we find a nice obstruction Ω'' for simple embeddings of $\mathcal{B}'$ under the condition that $B_1 \cup \Omega'$ is embedded as selected in this case. The union of all such obstructions Ω'' for different embeddings of $B_1 \cup \Omega'$ is an obstruction for simple embeddings of $\mathcal{B}'$ under the condition that B_1 is embedded in the corner α. We get the same outcome for the other three corners in which B_1 is chosen to be (simple) embedded. Obtaining an embedding in any of those cases, we stop. Otherwise, the union of obtained obstructions is a nice obstruction for all simple embedding extensions. Note that in some cases, obstructions are reached in *Cycling procedure*. If they are "long", the resulting obstruction Ω^* produced in *Obstruction compression* does not contain B_1. In such a case we do not need to combine Ω^* with other three obstructions.

It remains to see how to implement the above procedure in linear time. The odd case is when in one or more cases for the initial embedding of B_1, we continue working with *Cycling procedure*. It can happen that this procedure stops with *Harmless embedding*. In that case we can embed some of the bridges and continue with the remaining bridges. The danger lies in the fact that we might have spent linear amount of time for one of the previous cases for the initial embedding of B_1 that was not successful, and later embed only a constant number of bridges. This can give overall quadratic time complexity. A solution to this problem is as follows. We perform in parallel all four cases (for the selection of the initial corner of B_1) and stop all of them if *Harmless embedding* is reached in any of them. It turns out that this parallelism needs not to be used more than once at the same time. Let us remark that parallelism used above can be implemented in the chosen model of computation (RAM) without increasing time complexity (see [5] for more details).

Obstruction compression. Let $k := i$ (where i is the step counter in *Cycling procedure*). Recall that we have a sequence B_1, B_2, $\dots$, B_k and for each $j = 2, 3 \dots, k-1$, the bridge B_j is embedded in the corresponding corner α_j. (If B_1 is obtained in the part *Harmless embedding*, then also α_1 is well defined. Otherwise, B_1 can be composed of several bridges. To unify presentation, we say also in this case that B_1 is in α_1 if the bridges in B_1 are embedded as chosen by the algorithm.) Moreover, the embedding of $B_1 \cup \cdots \cup B_{k-2}$ still has a 2-singular face and the embedding of $B_1 \cup \cdots \cup B_{k-1}$ cannot be extended to a simple embedding of $B_1 \cup \cdots \cup B_{k-1} \cup B_k$. Our goal is to find a nice obstruction Ω with the property that there is no simple embedding of Ω with B_1 in α_1 (if $B_1 \subseteq \Omega$). If $k \leq 15$, let $L = B_1 \cup \cdots \cup B_k$. By the above, every simple embedding of L which has B_1 in α_1 is at most 1-singular. Thus, L contains path(s) as required in Lemma 4.4. Applying this lemma we either stop by finding an embedding extension, or get a nice obstruction for extending the chosen embedding of L. The combination of obstructions for all possible embeddings of L is the required obstruction Ω. The non-trivial case is when $k > 15$. In this case we will obtain Ω as described below.

In the sequel we will use the following notation. We write $(B_i, \alpha_i) \to (B_j, \alpha_j)$ if every simple embedding of $B_i \cup B_j$ with B_i embedded in α_i and B_j embedded in a corner distinct from α_j is at most 1-singular. (Since B_i is embeddable in α_i by construction, this implies that B_j is attached to a vertex that is "covered" by B_i in the corner α_i.) Note that for every j, $2 \leq j \leq k-1$, there is an $i < j$ such that $(B_i, \alpha_i) \to (B_j, \alpha_j)$. Moreover, $i \geq j-3$ because of our "cycling" choice of corners in *Cycling procedure*. We will occasionally use the following obvious fact: if $(B_i, \alpha_i) \to (B_j, \alpha_j)$ and for $l \geq i$ we have $\alpha_l = \alpha_i$, then $(B_l, \alpha_l) \to (B_j, \alpha_j)$. Let $\Theta = \{B_{k-14}, B_{k-13}, \dots, B_{k-2}\}$. Since $k > 15$, every bridge in Θ is attached only to e and f, and as there are no local bridges, it is indeed attached to both branches. We distinguish two cases.

Case (A): *There are indices i and j, $k-14 \leq i < j \leq k-3$, such that $(B_i, \alpha_i) \to (B_j, \alpha_j)$ and $(B_j, \alpha_j) \to (B_i, \alpha_i)$.* Then $j - i \leq 3$. Note that α_i and α_j are adjacent corners. Bridges $B_i \cup B_j$ have three classes of simple embeddings as listed below. Other possibilities are excluded since $j < k-1$, and thus B_i, B_j embedded in their corners do not remove the 2-singularity. The classes of simple embeddings are:

(i) *Embeddings where $B_i \cup B_j$ leaves at most 1-singularity of F.* For every such possibility we use algorithms of Section 4 to check if this embedding can be extended. If we succeed, then we are done. Otherwise we get an obstruction for embedding extensions. Denote by Ω_1 the union of obtained obstructions for all such embeddings of $B_i \cup B_j$.

(ii) *Assuming that $\alpha_i = \alpha$, $\alpha_j = \beta$, the bridges B_i and B_j are embedded in corners δ and γ, respectively* (see Figure 7).

Let y_i be the vertex of attachment of B_i on f that is as close to a as possible. Define similarly y_j. Let y be the lower of y_i, y_j, and let f_y be the segment of f from a to y. The segment f_y of f remains singular after the embedding of B_i and B_j. The bridges $\mathcal{B}_y$ of K attached to f_y (including y) can all be simultaneously embedded under

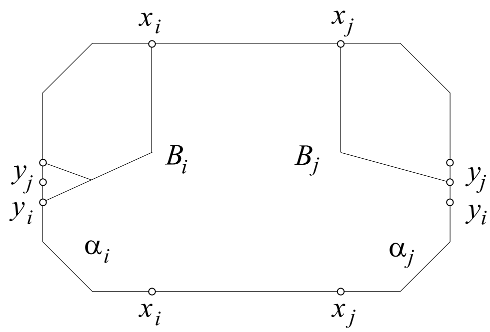

Figure 7: An embedding in Case (A)(ii).

the bridges B_i and B_j with respect to their embeddings in α and β, respectively. This is easy to see since $(B_i,\alpha) \to (B_j,\beta)$, $i < j < k-1$, and since we have not reached a contradiction when considering B_i and B_j. Thus, if a simple embedding of some bridge $B \notin \mathcal{B}_y$ is obstructed by bridges in $\mathcal{B}_y$ embedded in their corners, it is also obstructed by the chosen embeddings of B_i in δ and B_j in γ. Therefore, we may assume without loss of generality that all bridges from $\mathcal{B}_y$ are embedded in the corners α and β as determined by *Cycling procedure*. Therefore we have essentially removed 2-singularity. Using the algorithms of Section 4, we either get a simple embedding extension, or we get a nice corner obstruction Ω_2 for such extensions.

(iii) *Embedding where B_i is in α_i and B_j is in α_j.* We are going to show how to obtain a nice corner obstruction $\overline{\Omega} \supseteq B_i \cup B_j \cup B_{j+1}$ such that under every simple embedding of $\overline{\Omega}$ extending the embedding of $B_i \cup B_j$, either B_{j+1} is embedded in its corner α_{j+1}, or the 2-singularity is removed. Consider a simple embedding of B_{j+1} extending the embedding of $B_i \cup B_j$. Clearly, $\alpha_{j+1} \neq \alpha_j$. Suppose that $\alpha_i = \alpha$ and that $\alpha_j = \beta$. Let $r \le j$ be the largest index such that $(B_r,\alpha_r) \to (B_{j+1},\alpha_{j+1})$. If $i = r$ or $j = r$, then we can take $\overline{\Omega} = B_i \cup B_j \cup B_{j+1}$. (Similarly if $r = 1$.) Otherwise, we will consider the three possibilities for α_{j+1}.

— If $\alpha_{j+1} = \gamma$, then $r \ge j-2$. Since $j+1 \le k-2$, one can show that $\alpha_r \in \{\beta,\delta\}$. If $\alpha_r = \beta$, then $r = j$. The next case is when $\alpha_r = \delta$. If $(B_i,\alpha_i) \to (B_r,\alpha_r)$, then B_r is either embedded in α_r, or 2-singularity is removed. Consequently, the same conclusion applies to B_{j+1}, whenever B_r is in α_r. Thus, one can take $\overline{\Omega} = B_i \cup B_j \cup B_r \cup B_{j+1}$. The only other possibility is that $(B_{j+1},\alpha_{j+1}) \to (B_r,\alpha_r)$. In this case B_r and B_{j+1} are in the same relation as B_i and B_j. For their embeddings we also have cases (i)–(iii). For (i) and (ii) we already know what to do. (It is easy to verify that (i) and (ii) are valid for this case even though it can happen that $j+1 = k-2$.) In case (iii), B_{j+1} is embedded in α_{j+1} as required. These three cases yield a nice obstruction $\overline{\Omega}$ such that every simple embedding of $\overline{\Omega}$ has B_{j+1} embedded in α_{j+1}.

— Another possibility is $\alpha_{j+1} = \delta$. This case is similar to the above.

— The remaining choice is $\alpha_{j+1} = \alpha$. We may assume that $\alpha_r = \delta$. (If $\alpha_r = \beta$, then also $(B_j,\alpha_j) \to (B_{j+1},\alpha_{j+1})$.) If $(B_i,\alpha_i) \to (B_r,\alpha_r)$, then we are done again since $(B_r,\alpha_r) \to (B_{j+1},\alpha_{j+1})$. Otherwise, $(B_s,\alpha_s) \to (B_r,\alpha_r)$, where $r-3 \le s < r$ and $\alpha_s = \gamma$. (Similarly if $s=1$.) We conclude as above.

Having $\overline{\Omega}$, we may consider only simple embeddings where B_i, B_j, B_{j+1} are in their corners. We intend to extend $\overline{\Omega}$ in such a way that $B_{j+2} \subseteq \overline{\Omega}$ and every simple embedding of $\overline{\Omega}$ with B_i in α_i, B_j in α_j has B_{j+2} in α_{j+2}, or the 2-singularity is removed. If $(B_t,\alpha_t) \to (B_{j+2},\alpha_{j+2})$ for some $t \in \{i,j,j+1\}$, we are done. Otherwise, we have $t=j-1$ and $i \in \{j-3,j-2\}$.

If $i=j-2$, then $\alpha_{j-1} \ne \alpha_i,\alpha_j$. By our assumptions, $\alpha_i=\alpha$, $\alpha_j=\beta$. Let $s<j-1$ be the largest index such that $(B_s,\alpha_s) \to (B_{j-1},\alpha_{j-1})$. If $\alpha_s \in \{\alpha_i,\alpha_j\}$, then either $(B_i,\alpha_i) \to (B_{j-1},\alpha_{j-1})$ or $(B_j,\alpha_j) \to (B_{j-1},\alpha_{j-1})$ (and we are done by adding B_{j-1} and B_{j+2} to $\overline{\Omega}$). Otherwise, $\{\alpha_s,\alpha_{j-1}\} = \{\gamma,\delta\}$. If B_s has been forced in α_s from α or β, then (B_i,α_i) or (B_j,α_j) also "implies" (B_s,α_s) and we can stop with $\overline{\Omega}$ extended by B_s, B_{j-1} and B_{j+2}. Same if $s=1$. In the remaining case, we have $(B_{j-1},\alpha_{j-1}) \to (B_s,\alpha_s)$. These two pairs thus "imply" each other and by the same conclusions as above we get an obstruction $\tilde{\Omega} \supset B_s \cup B_{j-1}$ such that every simple embedding of $\tilde{\Omega}$ either removes the 2-singularity, or has B_s, B_{j-1} embedded in their corners. Note that only parts (i) and (ii) of Case (A) are used for this purpose, since in (iii) we have the two bridges in the desired corners. Finally, we extend $\overline{\Omega}$ by adding $\tilde{\Omega}$.

Suppose now that $i=j-3$. Clearly, $\alpha_{j-1} \ne \alpha_j = \beta$. Since $(B_i,\alpha_i) \to (B_j,\alpha_j)$ and since bridges are enumerated when their corner is considered, we have $\alpha_{j-1} \ne \alpha_i = \alpha$. Thus $\alpha_{j-1} \in \{\gamma,\delta\}$. By the same arguments as above, we get $(B_s,\alpha_s) \to (B_{j-1},\alpha_{j-1})$ with $\{\alpha_s,\alpha_{j-1}\} = \{\gamma,\delta\}$ and conclude in the same way.

Now, that we have three consecutive bridges B_j, B_{j+1}, B_{j+2} embedded in their respective corners, all the subsequent bridges $B_{j+3},\dots,B_{k-1}$ also go into their corners (or we get at most 1-singularity).

Let $\Omega' = \overline{\Omega} \cup B_{j+2} \cup \dots \cup B_{k-1} \cup B_k$. Embeddings of Ω' with B_i, B_j embedded in their corners are at most 1-singular. Thus, we get a nice obstruction (or we get a simple embedding of $K \cup Q$) for each such embedding by the results of Section 4. Let Ω_3 be the union of these obstructions.

Finally, let Ω be the union of Ω_1, Ω_2, and Ω_3. It is clear that Ω is a nice obstruction for simple embedding extensions from K to $K \cup Q$ (under the condition that B_1 is in α_1 if $B_1 \subseteq \Omega$).

Case (B): *Suppose now that among $B_{k-14},\dots,B_{k-3}$ no two bridges "imply" each other.* Then we claim that there is a subsequence of $B_{k-14},\dots,B_{k-2}$, say Q_1, Q_2, Q_3, Q_4, with the following property. If β_i is the corner of Q_i, $i=1,2,3,4$, then $(Q_1,\beta_1) \to (Q_2,\beta_2) \to (Q_3,\beta_3) \to (Q_4,\beta_4) \to (Q_1,\beta_1)$. Let $(Q_4,\beta_4) := (B_{k-3},\alpha_{k-3})$. Then for some $i \ge (k-3)-3$, $(B_i,\alpha_i) \to (Q_4,\beta_4)$. Let $(Q_3,\beta_3) := (B_i,\alpha_i)$. Similarly, we have $(B_\ell,\alpha_\ell) \to (B_i,\alpha_i)$ for some ℓ, $i > \ell \ge (k-3)-6$. By excluding the possibility of $(B_{k-3},\alpha_{k-3}) \to (B_i,\alpha_i)$, we have $\alpha_\ell \ne \alpha_{k-3}$. Let $(Q_2,\beta_2) := (B_\ell,\alpha_\ell)$. Note that β_2, β_3, β_4 are distinct corners. Similarly, we get $Q_1 := B_j$ in the fourth corner, $\beta_1 \ne \beta_2,\beta_3,\beta_4$. Let $(Q_0,\beta_0) = (B_s,\alpha_s)$ where $s<j$ is the largest index such that $(B_s,\alpha_s) \to (B_j,\alpha_j)$. Since our *Cycling procedure*

takes corners in the cycling sequence $\alpha, \gamma, \beta, \delta$, the difference $i - \ell$ can be equal to 3 only if $(\beta_2, \beta_3) = (\alpha, \delta)$ or $(\beta_2, \beta_3) = (\beta, \gamma)$. Extending this argument to other differences and noting that β_0, β_1, β_2, β_3, β_4 are consecutive corners, we see that $(k-3) - j \leq 8$. Since $k > 15$ and by excluding Case (A), we have $\beta_0 = \beta_4$. As $(Q_0, \beta_4) \to (Q_1, \beta_1)$, we also have $(Q_4, \beta_4) \to (Q_1, \beta_1)$, and the claim is proved.

Consider the simple embeddings of $Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ in F. By their property of cyclically "implying" each other, all embeddings where at least one of these bridges is in its corner, are either at most 1-singular, or all of them are in their corners. The former possibilities are handled by algorithms of Section 4. We may assume that in each case we get a nice obstruction. (Having an embedding extension, we can stop.) The other case is when Q_1, Q_2, Q_3, Q_4 are in their corners. Recall that $Q_1 = B_j$ and consider the bridge B_{j+1}. Since $(B_r, \alpha_r) \to (B_{j+1}, \alpha_{j+1})$ for some $r \leq j$, we also have $(Q_t, \beta_t) \to (B_{j+1}, \alpha_{j+1})$ for some $t \in \{1, 2, 3, 4\}$. Thus, every simple embedding of B_{j+1} extending the chosen embedding of Q_1, Q_2, Q_3, Q_4 either leaves at most 1-singularity, or has B_{j+1} in α_{j+1}. Similarly for $B_{j+2}, \ldots, B_{k-1}$.

It is easy to see that if none of Q_1, Q_2, Q_3, Q_4 is in its corner, then the 2-singularity is removed. We either embed the remaining bridges (Section 4), or we get a nice obstruction for simple embedding extensions. Recall that extensions of simple embedding of $B_j, \ldots, B_{k-1}$ (each in its corner) are obstructed by B_k. We let Ω be the union of $B_j \cup B_{j+1} \cup \cdots \cup B_{k-1} \cup B_k$ together with all nice obstructions obtained above for the simple embeddings of $B_j, \ldots, B_{k-2}$ where at least one of these bridges is not in its corner. It is clear that Ω is an obstruction for simple embeddings.

One can easily see that the presented proof yields an algorithm with linear time complexity. □

Theorem 6.1 has a "weaker" version: Suppose that the assumptions of Theorem 6.1 are satisfied. There is a linear time algorithm that either finds an embedding extension (possibly not simple) of K to $K \cup Q$, or returns a nice obstruction $\Omega \subseteq Q$ for simple embeddings, i.e., Ω has no simple embedding in F. This version is easier to implement and may be sufficiently strong for some applications. The algorithm follows the proof of Theorem 6.1 except that applications of Lemmas 4.3 and 4.4 are replaced by their "weaker" versions (with simpler algorithms), where we allow to return embeddings that are not simple.

Remark. Nice obstructions produced by our corner algorithm have bounded size up to a bounded number of millipedes that they may contain. Millipedes can occur every time a subset of bridges removes 2-singularity and the algorithms of Section 4 or 5 are applied to get an obstruction for the corresponding simple embedding extension problem. However, if we have two millipedes based on the same branch e or f, they can be replaced by a single millipede together with a bounded number of additional bridges without loosing obstructing properties. Consequently, every nice obstruction can be changed in a corner obstruction which contains at most two millipedes. Moreover, this can be done in linear time.

7 Obtaining a small obstruction

Theorem 6.1 gives a nice obstruction Ω for simple embedding extensions in a 2-singular face. One can easily reduce Ω (in linear time) into an obstruction which is minimal in the sense that no K-bridge B participating in Ω is redundant (i.e., $\Omega\backslash B$ admits a simple embedding).

Although minimal, Ω can be arbitrarily large. From the algorithmic point of view, obstructions of bounded size are more convenient. For example, in [9] we need obstructions of bounded size, but we allow K to be changed. The proof of Theorem 7.1 below describes how this can be achieved in linear time.

Let W_0 be a set of vertices of K. A subgraph K' of G is called a *relative* of K with respect to W_0 if the following holds:

(a) K' is homeomorphic to K and is obtained from K by replacing e and f by new branches e' and f', respectively, joining the same main vertices.

(b) All vertices of $W_0 \cap e$ (respectively $W_0 \cap f$) and all their neighbors on e (f) appear on e' (f') in the same order as they appear on e (f).

Theorem 7.1 *Let G, K, F, $\mathcal{B}$, and Q be as in Theorem 6.1. Suppose that Q has no simple embedding in F. There is a function $\mathbf{c} : \mathbf{N} \to \mathbf{N}$ such that the following holds. If W_0 is a set of vertices of K, then there is a relative K' of K with respect to W_0 such that the modified embedding extension problem admits an obstruction Ω_0 of branch size $b(\Omega_0) \le \mathbf{c}(|W_0|)$. Moreover, there is an algorithm of time complexity $O(\mathbf{c}(|W_0|)\,|V(G)|)$ that finds K' and Ω_0.*

Proof. Applying Theorem 6.1 we get a nice obstruction Ω for simple extensions. Replace the bridges in Ω that correspond to squashed millipedes by the original millipedes. Suppose that Ω contains a millipede $M = B_0^\circ \cup \cdots \cup B_m^\circ$ based on e. We may assume that M has been squashed using (SQ1).

Assume first that $W_0 \cap e = \emptyset$. We will use the notation introduced in Section 2. Recall that M is composed of $\mathcal{B}_\ell$, $\mathcal{B}_r$ and at most two additional bridges that are attached to both sides of F. We will change e into a branch e' contained in $e \cup \mathcal{B}_\ell$ such that the corresponding embedding of the new graph $K' = K - e + e'$ admits a corner obstruction that coincides with Ω outside $\mathcal{B}_\ell$ and such that $\mathcal{B}_\ell$ is replaced by at most four K'-bridges. If $|\mathcal{B}_\ell| < 4$, then no change of K is required. Otherwise, we have $I = \bigcap_{i=1,2} (l_i', r_i') \subseteq e$ (cf. Section 2 for details). Note that $\mathcal{B}_\ell$ has exactly two simple embeddings in F. For every bridge $B_i^\circ \in \mathcal{B}_\ell$, let P_i be a path in B_i joining l_i and r_i (the leftmost and the rightmost vertex of attachment of B_i° on e, respectively). Select j such that $B_j^\circ \in \mathcal{B}_\ell$ and l_j is equal to the left endpoint of I. Moreover, let k be the largest index such that $B_k^\circ \in \mathcal{B}_\ell$ is attached entirely to I and is embedded in the same way as B_j°. Now replace the segment of I between l_j and r_k by the paths P_j, $P_{j+2}, \ldots, P_k$ joined together by the segments $(r_j, l_{j+2}), \ldots, (r_{k-2}, l_k)$ of I.

Applying this change of I, all bridges in $\mathcal{B}_\ell$ attached to I and embedded differently than B_j° merge into a single bridge B. Now a new corner obstruction of bounded size can be obtained as follows. Instead of B we take its subgraph $\tilde{B}$ (see Lemma 2.1), and instead of all bridges $B_j^\circ, \ldots, B_k^\circ$ we take just $B_j^\circ \setminus P_j$ and $B_k^\circ \setminus P_k$. This way, $\mathcal{B}_\ell$ is replaced by at most four bridges. This construction and (SQ1) imply that $\Omega \setminus \mathcal{B}_\ell$ together with these four bridges form a corner obstruction equivalent to Ω.

If $W_0 \cap e \neq \emptyset$, we perform the same operation as described above on each segment σ of $e - W_0$, considering only those bridges of $\mathcal{B}_\ell$ whose attachments to e are contained in σ. The final number of bridges left from $\mathcal{B}_\ell$ is $O(|W_0|)$.

Applying the same changes to $\mathcal{B}_r$ and then also to other millipedes contained in Ω, we obtain a relative of K and a desired corner obstruction of bounded size. □

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