Removable Paths Conjectures

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The following conjecture has been attributed to Lovász [8, p. 262].

(a) *For every integer $k > 0$ there exists a smallest integer $f(k)$ such that between any two vertices a, b of an $f(k)$-connected graph G there exists an induced a, b-path P such that $G - V(P)$ is k-connected.*

We know $f(1) = 3$ and $f(2) = 5$ from [4], where $f(1) \leq 3$ is a consequence of Tutte's Wheel Theorem [9]. For every $k > 2$, the existence of $f(k)$ is open, and no lower bound to $f(k)$ substantially larger than $k + 3$ is known. Furthermore, (a) is true when restricted to line graphs, where $f(k) \leq 2k + 5$ [4]. Finally, we can not expect P in (a) to be a *shortest* path [4].

(b) If we drop the condition to P in (a) of being *induced*, or

(c) replace $G - V(P)$ with $G - E(P)$ in (a), or

(d) do both (b) and (c),

we obtain, informally, three reasonable relaxations (b), (c), (d) of (a), each of which is open again. It is easily seen that (a) implies (b) and (c), and that (c) implies (d). All other possible implications are open, including the qualitative equivalence of all four versions.

Concerning (b), it is known that for any two vertices a, b of a 4-connected graph G there exists a path P such that $G - V(P)$ is 2-connected, unless $a \neq b$ are nonadjacent and $G - \{a, b\}$ is an induced cycle [2].

The possibly weakest version, (d), would follow from an affirmative answer to the following conjecture, which I've presented on several problem sessions since 1998.

(e) *For every integer $k > 0$ there exists a smallest integer $f(k)$ such that every $f(k)$-connected graph admits a spanning tree T such that $G - E(T)$ is k-connected.*

We know $f(1) \leq 4$ by Tutte's and Nash–Williams's Base Packing Theorem [10, 6]. Any 4-regular 4-connected graph without a hamiltonian path yields $f(1) \geq 4$. Recently, $f(2) \leq 12$ has been proved by a matroid argument [1]. For all $k > 2$, the existence of $f(k)$ is open.

Coming back to the original question (a) and the relaxation (c), there are two related results on removable cycles. It is known [7] that

(f) *for $k > 0$, every $(k+3)$-connected graph G has an induced cycle C such that $G - V(P)$ is k-connected*, and that

(g) *for $k > 0$, every $(k+2)$-connected graph G has an induced cycle C such that*

$G - E(P)$ *is* k*-connected* (implicitly in [5]).

If, in (f) or (g), we could prescribe an edge in C, at the expense of a larger connectivity bound, then (a) or (c), respectively, would follow. It is not even known if the prescription of a vertex is possible [3].

References

[1] T. Jordán, "On the existence of k edge–disjoint 2-connected spanning subgraphs", EGRES Technical Report 2004–05.

[2] K. Kawarabayashi, O. Lee, and X. Yu, "Nonseparating paths with two prescribed endvertices in 4-connected graphs", Annals of Combinatorics 9 (2005) 47–56.

[3] "Triangle Density and Contractibility", Combinatorics, Probability and Computing 14 (2005), 133–146.

[4] M. Kriesell, "Induced paths in 5-connected graphs", Journal of Graph Theory 36 (2001), 52–58.

[5] W. Mader, "Kreuzungsfreie a, b-Wege in endlichen Graphen", Abhandlungen Math. Sem. Univ. Hamburg 42 (1974), 187–204.

[6] C. St. J. A. Nash-Williams, "Edge-disjoint spanning trees of finite graphs", J. Lond. Math. Soc. 36 (1961), 445–450.

[7] C. Thomassen, "Nonseparating cycles in k-connected graphs", Journal of Graph Theory 5 (1981), 351–354.

[8] C. Thomassen, "Graph decomposition with applications to subdivisions and path systems modulo k", J. Graph Theory 7 (1983), 261–271.

[9] W. T. Tutte, "A theory of 3-connected graphs", Nederl. Akad. Wet., Proc., Ser. A 64 (1961), 441–455.

[10] W. T. Tutte, "On the problem of decomposing a graph into n connected factors", J. Lond. Math. Soc. 36 (1961), 221–230.