Question

Is it true that every connected graph distinct from K_3 of maximum degree d is $(2d+1,2)$-edge colourable?

Partial Answer

This is not true in general. In fact, it is not even true that only a finite number of graphs need to be excluded.

If $d = 2$ it is easy to see that the edge colouring does exist. If $d = 3$ the colouring also exists. If $d = 4$ the colouring exists provided that the graph is not K_5, $\overline{C_7}$, or $\overline{C_3 \cup C_4}$, which are all counterexamples. If $d = 5$ the colouring exists provided that the graph does not contain $\overline{H}$ as an induced subgraph, where H is the union of two isolated edges and a path of length two. Therefore, there are infinitely many counterexamples for $d = 5$.

These results on small d are implied by the following result of Nishizeki and Kashiwagi, which we apply for $M = 2G$, G a simple connected graph.

Theorem [1] *Every multigraph M of maximum degree Δ satisfies*

$$\chi'(M) \leq \max\{p(M), \lfloor 1.1\Delta + 0.8 \rfloor\},$$

where $p(M) = \max\{\lceil \frac{2|E(X)|}{(|X|-1)} \rceil : X \subseteq V(M), |X| \neq 1 \text{ and odd}\}$.

In addition to the results we have stated for small d, note that K_{d+1} is a counterexample for all even d. In fact, even if the problem is relaxed to ask for a $(kd + k - 1, k)$-edge colouring for some $k \geq 3$, the graphs K_{d+1} are still counterexamples for all even d.

References

[1] T. Nishizeki and K. Kashiwagi. On the 1.1 edge-coloring of multigraphs. *SIAM J. Disc. Math.* **3**(3) (1990), 391-410.