THE SPECTRAL RADIUS OF INFINITE GRAPHS

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1. Introduction

Recently some important results have been proved showing that the gap between the largest eigenvalue k of a finite regular graph of valency k and its second eigenvalue is related to expansion properties of the graph [1]. In this paper we investigate infinite graphs and show that in this case the expansion properties are related to the spectral radius of the graph.

First we introduce necessary notions for the spectrum of an infinite graph following the definitions of [7]. For an infinite graph Γ with vertex set V and finitely bounded valency, the adjacency operator A is well-defined on $l^2(V)$ and is bounded and self-adjoint. The spectrum of Γ is the approximate point spectrum of A in the space $l^2(V)$; that is $\lambda \in \text{Spec } A$ if and only if there is a sequence of unit vectors x^n such that

$$\|(A - \lambda I) x^n\| \to 0, \quad \text{as } n \to \infty.$$

The spectral radius of Γ, $\rho(\Gamma)$ is given by

$$\rho(\Gamma) = \sup \{|\langle x, Ax \rangle| \mid \|x\| = 1\},$$

which is also the maximum of the spectrum of A.

For a general graph Γ and a subset X of the vertices of Γ, we define ∂X to be the subset of edges of Γ incident with exactly one vertex of X. The usual distance between two vertices u and v of a graph will be denoted by $\partial(u, v)$. We can now define a measure of expansion in a graph, the *isoperimetric constant*, as

$$i(\Gamma) = \inf_{\substack{X \text{ finite} \\ X \subset V\Gamma}} \frac{|\partial X|}{|X|}.$$

2. The euclidean case

We begin by considering the case when the isoperimetric number is zero.

THEOREM 2.1. *If Γ is an infinite, connected, k-valent graph such that $i(\Gamma) = 0$, then $\rho(\Gamma) = k$.*

Proof. Since $i(\Gamma) = 0$, we can choose a sequence of subsets X_n, $n = 1, 2, \ldots$, of vertices of Γ such that

$$\frac{|\partial X_n|}{|X_n|} \to 0, \quad \text{as } n \to \infty.$$

Let x^n be the normalised characteristic vector of the subset X_n, for each n. Consider the vector $y^n = (A - kI) x^n$. It has a non-zero entry in coordinate i only if vertex i is

incident with an edge of ∂X_n. Note that each such incidence contributes $1/\sqrt{|X_n|}$ to the coordinate and that the total number of all contributions is exactly $2|\partial X_n|$. Hence the sequence of unit vectors satisfies

$$\|(A-kI)x^n\|^2 = \sum_{i=1}^{\infty} (y_i^n)^2$$

$$\leqslant \frac{k}{\sqrt{|X_n|}} \sum_{i=1}^{\infty} |y_i^n|$$

$$= 2k\frac{|\partial X_n|}{|X_n|} \to 0, \quad \text{as } n \to \infty, \qquad (2.1)$$

since each vertex can receive at most k contributions. Equation (2.1) tells us that k is in the spectrum of the operator A, and since $\|A\| = k$, we conclude that $\rho(\Gamma) = k$.

In any graph Γ and vertex $v \in V\Gamma$, let $B_i(v)$ $(i = 0, 1, 2, ...)$ denote the set of vertices w such that $\partial(v, w) \leqslant i$, and let

$$S_i(v) = B_i(v) \backslash B_{i-1}(v), \quad (i = 1, 2, ...).$$

The next theorem shows that when the spectral radius is strictly less than the valency, the sets $B_i(v)$ grow exponentially with i.

THEOREM 2.2. *If Γ is a regular graph of valency k and $\rho(\Gamma) < k$ then there is a constant $q > 1$ such that $|B_i(v)| > q^i$, for all $v \in V\Gamma$ and $i \geqslant 0$.*

Proof. By the previous theorem, $i(\Gamma) > 0$. Consider the set of edges $\partial B_i(v)$, for any fixed vertex $v \in V\Gamma$. Since there are at most k such edges incident with a vertex in $S_{i+1}(v)$ we have

$$|S_{i+1}(v)| > \frac{|\partial B_i(v)|}{k} \geqslant \frac{i(\Gamma)|B_i(v)|}{k}.$$

Hence

$$|B_{i+1}(v)| = |B_i(v)| + |S_{i+1}(v)|$$

$$\geqslant \left(1 + \frac{i(\Gamma)}{k}\right)|B_i(v)|.$$

Putting $q = (1 + i(\Gamma)/k)$ and observing that $|B_0(v)| = 1$, we obtain the result.

Let d be a non-negative real number. Godsil and McKay [5] define a graph Γ to have *dimension* d if for some vertex (and hence for all vertices) $v \in V\Gamma$, there are positive real constants C_1, C_2 such that

$$C_1 i^d \leqslant |B_i(v)| \leqslant C_2 i^d, \quad \text{for } i \in \{0, 1, 2, ...\}.$$

In their paper they show that if Γ is periodic in d independent directions (as in physical 'lattices'), then Γ has dimension d in the above sense.

THEOREM 2.3. *If Γ is a regular graph of valency k with dimension d, then $\rho(\Gamma) = k$.*

Proof. By the definition, if Γ has dimension d, there is no $q > 1$ for which $|B_i(v)| > q^i$ for all i. Hence the result follows from Theorem 2.2.

3. *The expanding case*

In this section we investigate the case when the isoperimetric constant is not zero. This corresponds to the non-euclidean geometry of riemannian manifolds, as studied for example by Gromov [6]. We prove analogous results to those proved by Alon and Milman [1] in the finite case, except that in the place of the second largest eigenvalue we find the spectral radius of the graph. We consider first a graph Γ with spectral radius given.

THEOREM 3.1.   *Let Γ be an infinite, connected, k-valent graph with spectral radius ρ. Then the isoperimetric number of Γ is bounded by the inequality*

$$i(\Gamma) \geqslant \frac{4(k-\rho)}{k}.$$

Proof.   Consider any finite subset X of the vertices V of Γ and let $x \in l^2(V)$ be given by $x_i = 1$, for $i \in X$, $x_i = \frac{1}{2}$, for $i \notin X$ but incident with an edge of ∂X and $x_i = 0$ otherwise. Clearly $\|x\|^2 > |X|$. Give each edge of Γ an arbitrary orientation, so that its 'positive' and 'negative' ends are the vertices e^+, e^- respectively. Let D be the incidence operator [2, Chapter 4] from oriented edges to vertices. Then we have

$$\|D^t x\|^2 = \sum_{e \in E\Gamma} (x_{e^+} - x_{e^-})^2.$$

The only edges giving a non-zero contribution are those incident with a vertex i with $x_i = \frac{1}{2}$, and there are at most $k|\partial X|$ such edges, so

$$\|D^t x\|^2 \leqslant \tfrac{1}{4}|\partial X| k.$$

We can use this equality to give a lower bound for (x, Ax):

$$\langle x, Ax \rangle = \langle x, (kI - DD^t)x \rangle$$
$$= k\|x\|^2 - \|D^t x\|^2$$
$$\geqslant k\|x\|^2 - \tfrac{1}{4}k|\partial X|$$
$$> k\|x\|^2 - \tfrac{1}{4}k \frac{|\partial X|}{|X|} \|x\|^2.$$

Hence we have a bound for ρ:

$$\rho > k - \tfrac{1}{4}k \frac{|\partial X|}{|X|},$$

for all finite X. This in turn gives us the required bound for $i(\Gamma)$:

$$i(\Gamma) = \inf_{X \text{ finite}} \frac{|\partial X|}{|X|}$$
$$\geqslant \frac{4(k-\rho)}{k}.$$

We now investigate the opposite situation, that is when the isoperimetric number is known to be positive. It turns out that in this case there is a result about the spectral radius analogous to Cheeger's inequality [3]. A similar result in a different context is given by Dodziuk [4].

THEOREM 3.2.   *Let Γ be an infinite, connected, regular graph of valency k with isoperimetric number $i = i(\Gamma)$. Then*

$$\rho(\Gamma) \leqslant k - \frac{i^2}{2k}.$$

Proof. It suffices to prove the inequality for finite induced subgraphs of Γ, since by [7] the spectral radius of an infinite graph is equal to the supremum of the spectral radii of finite subgraphs.

Let H be any finite induced subgraph of Γ and B be the adjacency matrix of H. Let λ be the maximal eigenvalue of H and let $x = (x_v)_{v \in VH}$ be a positive unit eigenvector for λ, so that $\langle Bx, x \rangle = \lambda$. Extend x to $l^2(V)$, where V is the set of vertices of Γ, by setting $x_v = 0$ for $v \notin VH$. For each edge $e = uv$, let $\delta^2(e) := |x_v^2 - x_u^2|$. We have

$$\Delta^2 := \left(\sum_e \delta^2(e) \right)^2$$

$$= \left(\sum_{uv \in E\Gamma} |x_u + x_v| \, |x_v - x_u| \right)^2$$

$$\leqslant \sum_{uv \in E\Gamma} (x_v + x_u)^2 \sum_{uv \in E\Gamma} (x_v - x_u)^2$$

$$\leqslant \sum_{uv \in E\Gamma} 2(x_v^2 + x_u^2) \left(\sum_{uv \in E\Gamma} (x_v^2 + x_u^2) - 2 \sum_{uv \in E\Gamma} x_v x_u \right)$$

$$= 2k \sum_{v \in V} x_v^2 \left(k \sum_{v \in V} x_v^2 - \langle Bx, x \rangle \right)$$

$$= 2k(k - \lambda).$$

It follows that $\lambda \leqslant k - \Delta^2/2k$. Our next step is to show that $\Delta \geqslant i = i(\Gamma)$ giving the required upper bound for the spectral radius of H and so also of Γ.

Since edges whose end vertices have the same coordinates in x have no influence on Δ, and there are only finitely many different coordinate values, it makes sense to denote all the values by a strict sequence $0 = y_0 < y_1 < y_2 < \ldots < y_n$. Let F_k be the subgraph of Γ induced on the vertices $V_k = \{ v \mid x_v \geqslant y_k \}$. Let the edges be oriented so that the value at the initial vertex is greater than or equal to the value at the terminal vertex and set $S_k = \{ e \in E\Gamma \mid x_{e^+} = y_k \}$. We can estimate Δ as follows:

$$\Delta = \sum_{e \in E\Gamma} |x_{e^+}^2 - x_{e^-}^2|$$

$$= \sum_{k=1}^n \left(\sum_{e \in S_k} (x_{e^+}^2 - x_{e^-}^2) \right)$$

$$= \sum_{k=1}^n \left(\sum_{e \in \partial V_k} (y_k^2 - y_{k-1}^2) \right).$$

Note that in the last line the value $(x_{e^+}^2 - x_{e^-}^2)$ has been split into parts $(y_k^2 - y_{k-1}^2)$, for $k = p+1, \ldots, q$, where $x_{e^+} = y_q$ and $x_{e^-} = y_p$, that is for k such that $e \in \partial V_k$. Hence since the isoperimetric number is i, we have $|\partial V_k| \geqslant i|V_k|$ and

$$\Delta = \sum_{k=1}^n |\partial V_k| (y_k^2 - y_{k-1}^2)$$

$$\geqslant i \sum_{k=1}^n |V_k| (y_k^2 - y_{k-1}^2),$$

$$= i \left(\sum_{k=1}^n y_k^2 |V_k| - \sum_{l=0}^{n-1} y_l^2 |V_{l+1}| \right)$$

$$= i \sum_{k=1}^n y_k^2 (|V_k| - |V_{k+1}|) \qquad \text{(as } y_0 = 0\text{)}$$

$$= i \sum_{x_v \neq 0} x_v^2$$

$$= i,$$

where we take $V_{n+1} = \varnothing$. The last inequality is the required bound on Δ and so the result follows.

Theorem 3.2 has as a corollary the converse of Theorem 2.1.

COROLLARY 3.3. *Let Γ be an infinite, connected, regular graph of valency k, for which $\rho(\Gamma) = k$. Then $i(\Gamma) = 0$.*

Proof. Set $\rho(\Gamma) = k$ in the inequality of the theorem and the result follows.

Theorem 3.2 can be generalised to non-regular infinite graphs with bounded valency. The same method of proof can be used by replacing k throughout by the maximum valency of Γ.

It is natural to ask whether the results are in any sense best possible. This is certainly not the case for all graphs as for example the infinite k-regular tree has spectral radius $2\sqrt{(k-1)}$ and isoperimetric number $k-2$. Hence for this graph neither inequality is tight provided $k > 2$. On the other hand it may well be that there are graphs which make one or other of the inequalities tight. It would be interesting to know for which values of k and ρ (if any) such examples exist.

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