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Branched Coverings*

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Abstract. Basic results on combinatorial branched coverings between relative geometric cycles are given. It is shown that every geometric n-cycle is a branched covering over S^n. If the downstairs space of a branched covering is locally simply connected then the branched set is a pure subcomplex of codimension 2. Finally, several Hurwitz-like theorems on existence and representation of branched coverings between relative geometric cycles are derived.

1. Introduction

Branched coverings (of manifolds) are extensively studied but no common definition of this notion is accepted. It seems that the most general approach was made by Fox [7]. Many authors use his definition of branched coverings, while many others consider only mappings where the branched set is a submanifold of codimension two. As we see, there are two reasons for this. First, there are no powerful methods to describe branched coverings in higher dimensions. In Section 5 we offer such a description which is an obvious generalization of the well-known Hurwitz existence theorem for Riemann surfaces (see, e.g., [2]) and the results of Heegaard [11] for describing branched coverings of 3-manifolds with a union of disjointed curves as the branched set.

The second reason why mathematicians sometimes consider only branched coverings with the branched set a submanifold is because such coverings are sufficient for most topological applications. The most common use is in attempts to classify manifolds. By a well-known classical theorem of Alexander [1], each closed orientable PL n-manifold can be obtained as a branched covering over S^n. In dimension 3, which is of special interest, the branched coverings of S^3 of degree 3 or less and with the branched set a single closed curve are sufficient.

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The results of this paper show that the most natural category for considering branched coverings consists of (relative) geometric cycles. In view of our combinatorial interest, we consider only combinatorial branched coverings, i.e., only those which are simplicial maps between simplicial pseudocomplexes.

Most of the results in this paper can be considered as "folklore" properties of branched coverings, at least in a lesser generality when restricted to combinatorial manifolds. However, we did not find any clear presentation of them, neither in papers, nor in textbooks. Besides that, we believe that the present results might be of interest in a combinatorial approach to branched coverings and manifold topology. An interesting field is the growing theory of crystallizations developed by Pezzana and others [3]-[6], [8], [13], [14], [16]. A *crystallization* is a simplicial pseudocomplex of dimensions n with $n+1$ vertices (0-simplices). As implied by our Theorem 3.1, a crystallization is a very special branched covering space over the n-sphere (in fact, a crystallization of S^n obtained by taking two n-simplices and pairwise identifying all $n+1$ of their $(n-1)$-faces).

In Section 3 we give a short proof of a generalization of the theorem of Alexander. By Proposition 3.3, every orientable geometric n-cycle with a proper $(n+2)$-coloring of the vertex set has a stellar subdivision which is a (simplicial) branched covering over the boundary of the $(n+1)$-simplex, $\partial\Delta^{n+1} \approx S^n$, with the branched set a subpseudocomplex of the $(n-2)$-skeleton of $\partial\Delta^{n+1}$. Moreover, Theorem 3.1 implies that the barycentric subdivision of every orientable geometric n-cycle is a branched covering over S^n with the branched set a subcomplex of the $(n-2)$-skeleton of Δ^n.

In Section 4 we consider the branched set of a branched covering. We show that the branched set of a branched covering $f: M \to N$ is purely $(n-2)$-dimensional if the space N is locally simply connected, in particular if N is a manifold. This is not true in general.

There are two classical Hurwitz theorems on the existence and classification of branched coverings of closed surfaces, see, e.g., [2]. Let M and N be closed surfaces. If f: $M \to N$ is an n-fold branched covering with branched set B, there is a homomorphism $\varphi(f)$: $\pi_1(N-B, *) \to S_n$, determined up to conjugacy, obtained by choosing a 1-1 correspondence between $f^{-1}(*)$ and the set $\{1, 2, \ldots, n\}$ and assigning to a loop γ in $\pi_1(N-B, *)$ the permutation induced by lifting γ to M. Conversely, given a representation φ: $\pi_1(N-B, *) \to S_n$ there is a degree n branched covering $M \to N$ with the branched set contained in B. The surface M is connected if and only if the image of φ is a transitive subgroup of S_n.

Section 5 is devoted to representations of branched coverings in any dimension. We derive several Hurwitz-like theorems on the existence and representation of branched coverings between relative geometric cycles.

In the last section the representations of regular branched coverings are considered.

2. Basic Definitions

A *pseudocomplex* is a cell complex in which each cell, considered with all its faces, is abstractly isomorphic to the closed simplex of the same dimension. This notion represents a generalization of a simplicial complex since we allow that two distinct cells of a pseudocomplex intersect in the union of cells (simplices). The cells of the pseudocomplex are called simplices, and the usual notions of combinatorial topology (e.g., the star and the link of a simplex, the simplicial map, etc.) are carried to pseudocomplexes in the obvious way.

A particular class of pseudocomplexes is important for our purposes. A pseudocomplex K is a *relative geometric n-cycle* if:

(1) K is n-dimensional and *pure*, i.e., each simplex of K is contained in an n-simplex,
(2) the open star of each simplex of K is *strongly connected*, i.e., any two n-simplices in the open star of a simplex A can be joined by a sequence of n-simplices, all of them containing A, such that any two consecutive simplices intersect in an $(n-1)$-simplex which contains A, and
(3) every simplex of codimension 1 is contained in at most two n-simplices.

The *boundary* of the relative n-cycle K, denoted by ∂K, is the subpseudocomplex of K induced by the $(n-1)$-simplices which are contained in only one n-simplex of K. The n-cycles with empty boundary are called *geometric cycles.*

Let M and N be relative n-cycles and assume that N is connected. A simplicial mapping $p\colon M \to N$ is a *branched covering* if:

(1) p is *nondegenerate*, i.e., p preserves the dimension of simplices,
(2) p is *nonsingular*, i.e., any two distinct n-simplices of M having a common $(n-1)$-face are mapped to distinct n-simplices in N, and
(3) $p(\partial M) = \partial N$.

Note that every branched covering is onto. Note also that we do not require that the upstairs space M is connected.

The set of simplices in M at which p fails to be a local isomorphism is called the *singular set* of p. It is a subpseudocomplex of dimension at most $n-2$. The image of the singular set is called the *branched set* of p. It will be denoted by B_p, or simply B if no confusion arises. The preimage $p^{-1}(B_p)$ is the *branch cover* of p.

To describe relative n-cycles and branched coverings we shall make use of dual graphs and we proceed by basic definitions concerning graphs. We consider finite, undirected graphs with multiple edges and half-edges, where a half-edge is an edge having only one end vertex (sometimes is called free edge). By $V(G)$ and $E(G)$ we denote the vertex-set and the edge-set of the graph G, respectively. Each edge of G which is not a half-edge gives rise to two oppositely oriented arcs, and each half-edge determines one arc. In this way we get the set $D(G)$ of *arcs* of G. The *initial vertex* of the arc $e \in D(G)$ is denoted by $i(e)$, the *terminal vertex* by $t(e)$, and the *opposite arc* of e by $r(e)$. The arc e is a half-edge if and only if $r(e) = e$. For a half-edge e, $i(e) = t(e)$.

For each $v \in V(G)$ we define the *star* of v, denoted by $\mathrm{st}(v, G)$, as the set of all arcs emanating from v. A graph G is *k-regular* if $|\mathrm{st}(v, G)| = k$ for each $v \in V(G)$.

Let H be a connected graph. A graph map $p: G \to H$ is a (graph) *covering projection* if for each vertex $v \in V(G)$, the induced mapping $\mathrm{st}(v, G) \to \mathrm{st}(p(v), H)$ is a bijection where half-edges and only half-edges are mapped to half-edges. Thus, the deletion of half-edges in both graphs gives rise to a covering in the usual sense. Since the downstairs graph is required to be connected, every covering projection is onto.

The *dual graph* K^* of the relative geometric n-cycle K is defined as follows. It has n-simplices of K as its vertices, and two arbitrary vertices are joined by one edge for each common $(n-1)$-face of the corresponding two n-simplices. Furthermore, for each boundary $(n-1)$-simplex we add a half-edge to the corresponding vertex. The dual graph of a relative n-cycle is $(n+1)$-regular and its connected components correspond to (strongly) connected components of the n-cycle.

A nondegenerate map $f: K \to L$ between relative cycles induces a map $f^*: K^* \to L^*$ between the dual graphs. The following is clear by definitions.

Proposition 2.1. *The nondegenerate map $f: K \to L$ is a branched covering if and only if the dual map f^* is a graph covering projection.*

3. Theorem of Alexander

A well-known theorem of Alexander [1] states that each closed orientable PL manifold of dimension n can be obtained as a branched covering space over S^n. We generalize this to geometric cycles. Let Φ_n be the pseudotriangulation of S^n which is obtained from two n-simplices by pairwise identifying all their $(n-1)$-faces. Recall that a *proper k-coloring* of K is an assignment of colors $1, 2, \ldots, k$ to the vertex set of K such that no two adjacent vertices receive the same color.

Theorem 3.1. *Each orientable geometric n-cycle without boundary which has a proper $(n+1)$-coloring of the vertex set is a branched covering over Φ_n with the branched set a subcomplex of the $(n-2)$-skeleton of Φ_n.*

Proof. In Corollary 4.6 of [15] it is shown that the orientable n-cycle K admits an $(n+1)$-coloring if and only if there is a nondegenerate and nonsingular simplicial mapping into the pseudocomplex Φ_n. If K has no boundary this is a branched covering. □

Recall that the barycentric subdivision of a cell complex has an $(n+1)$-coloring of the vertex set. This implies the following corollary.

Corollary 3.2. *Each orientable geometric n-cycle without boundary is a (topological) branched covering over S^n with the branched set a subcomplex of the $(n-2)$-skeleton of Δ^n.*

For PL manifolds, Corollary 3.2 was strengthened by Pezzana [16] who proved that every PL manifold M admits a crystallization which has only $n+1$ vertices (n is the dimension of M), and thus an $(n+1)$-coloring. Consequently, this gives a very special branched covering of M to S^n. See also [4] and [5] for additional generalizations in this direction.

The following result, which is topologically only a weaker version of Theorem 3.1, may be used for constructing branched coverings.

Proposition 3.3. *Each orientable geometric n-cycle without boundary which admits a proper $(n+2)$-coloring of its vertex-set has a stellar subdivision which is a branched covering over $\partial\Delta^{n+1} \approx S^n$.*

Proof. Orient K and $\partial\Delta^{n+1}$. Note that the coloring of $K^{(1)}$ determines a nondegenerate simplicial mapping $f\colon K \to \partial\Delta^{n+1}$ (consider the 0-simplices of $\partial\Delta^{n+1}$ as colors). Let T be the set of all n-simplices of K such that f does not preserve their orientation. Use the stellar subdivision at each n-simplex of T, thus obtaining the complex K_1. The introduced vertex in a simplex A of T can be colored by the color which is not used on A. The obtained coloring of K_1 gives rise to a nondegenerate map $K_1 \to \partial\Delta^{n+1}$ which is easily seen to be nonsingular and hence a branched covering. □

4. The Branched Set of a Branched Covering

Branched coverings of n-manifolds whose branched set is a pure $(n-2)$-dimensional subcomplex tamely embedded in the interior are of special interest in topology [7]. In general, it is not true that the branched set is pure of codimension two. Consider the following example. Let N be a relative n-cycle and let M be any (unbranched) cover of N. Denote by $\sum X := X * S^0$ the suspension of X. The covering projection $f\colon M \to N$ can be extended to the branched covering $\sum f\colon \sum M \to \sum N$. It is easy to see that the branched set of $\sum f$ is equal to both suspension points (if f is not an isomorphism), and hence it is of dimension 0.

We shall show that the above irregularity cannot appear if N is *locally simply connected*, i.e., if for each simplex A in the barycentric subdivision N' of N, if $\dim(A) < n-2$ then $\mathrm{link}(A, N')$ is simply connected. Note that every combinatorial manifold is locally simply connected.

Theorem 4.1. *Let M and N be relative n-cycles and N locally simply connected. Then the branched set of any branched covering $f\colon M \to N$ is a pure $(n-2)$-dimensional subpseudocomplex of N.*

Proof. The branched covering f induces the branched covering $f'\colon M' \to N'$ of first barycentric subdivisions. Note that M' and N' are simplicial complexes and that the branched set of f' has the same underlying topological space as the branched set of f. Hence it suffices to prove that $B_{f'}$ is pure $(n-2)$-dimensional.

Let A be a simplex in $B_{f'}$, and assume that $\dim(A) < n-2$ and that A is not contained in any other simplex from $B_{f'}$. Take an arbitrary simplex $A^{\sim}$ in $f'^{-1}(A)$ and any simplex B in $\operatorname{link}(A, N')$. Since M' and N' are simplicial complexes and the join $A * B$ is not in the branched set, f' induces an isomorphism between $\operatorname{link}(A^{\sim} * B^{\sim}, M')$ and $\operatorname{link}(A * B, N')$ for any $B^{\sim}$ in $\operatorname{link}(A^{\sim}, M') \cap f'^{-1}(B)$. But since

$$\operatorname{link}(A^{\sim} * B^{\sim}, M') = \operatorname{link}(B^{\sim}, \operatorname{link}(A^{\sim}, M'))$$

and

$$\operatorname{link}(A * B, N') = \operatorname{link}(B, \operatorname{link}(A, N')),$$

the above isomorphism is also an isomorphism between $\operatorname{link}(B^{\sim}, \operatorname{link}(A^{\sim}, M'))$ and $\operatorname{link}(B, \operatorname{link}(A, N'))$. Thus, the restriction of f' to $\operatorname{link}(A^{\sim}, M')$ is a covering projection to $\operatorname{link}(A, N')$. From the local simple connectivity of N it follows that $\operatorname{link}(A, N')$ is simply connected, and hence every connected cover of it is isomorphic to it. Since links in relative geometric cycles are connected, this implies that $\operatorname{link}(A^{\sim}, M')$ is isomorphic to $\operatorname{link}(A, N')$. But then $A^{\sim}$ is not in the singular set of f'. It follows that A is not in $B_{f'}$. The proof is completed. □

Denote by $\rho(A, K)$ the number of n-simplices of the pseudocomplex K which contain the simplex A. An immediate consequence of Theorem 4.1 is the following proposition.

Proposition 4.2. *Let M and N be relative n-cycles and N locally simply connected. The singular set of a branched covering f: $M \to N$ is the full subcomplex on those $(n-2)$-simplices A of M for which $\rho(A, M) \neq \rho(f(A), N)$.* □

Proposition 4.2 gives us a simple criterion to check if a given branched covering is a covering projection. We must verify the local homeomorphism property only on the neighborhoods of $(n-2)$-simplices.

At the end of this section we point out an obvious conjecture. Note that the branched set of branched coverings over the 3-sphere is a graph without isolated vertices, and it is easily seen that it has no vertices of degree 1.

Conjecture 4.3. *Every graph without vertices of degree* 0 *or* 1 *is a branched set of a branched covering f: $M \to S^3$ where M is a* 3*-manifold.* □

5. Simplicial Schemes and Representations of Branched Coverings

A convenient description of relative cycles, especially good when working with branched coverings, is the concept of simplicial schemes [15]. Let G be a regular graph. A *presimplicial scheme* g on G is a function which assigns to every arc $e \in D(G)$ a bijective map g_e: $\operatorname{st}(i(e), G) \to \operatorname{st}(t(e), G)$ such that the following conditions are satisfied:

(SS1) for each $e \in D(G)$, $g_e(e) = r(e)$, and
(SS2) for each $e \in D(G)$, $g_{r(e)} = g_e^{-1}$.

Let $W = f_1 f_2 \cdots f_d$ be a walk in G. Denote by g_W the composition $g_{f_d} \cdots g_{f_2} g_{f_1}$. An arc $e \in \mathrm{st}(i(f_1), G)$ is said to *avoid the walk* W (with respect to the presimplicial scheme g) if $e \neq f_1$ and, for $j = 1, \ldots, d-1$, $g_{f_1 \cdots f_j}(e) \neq f_{j+1}$.

A presimplicial scheme g is a *simplicial scheme* if, in addition to (SS1) and (SS2), the following condition is also satisfied:

(SS3) for each closed walk W and each arc $e \in \mathrm{st}(i(W), G)$ which avoids W, $g_W(e) = e$.

Relative geometric cycles and simplicial schemes are related by the following construction. Let G be an $(n+1)$-regular graph and g be a simplicial scheme on G. Then we construct an n-dimensional pseudocomplex, denoted by $K = K(G, g)$, as follows. For each vertex $v \in V(G)$ take an n-simplex A_v and choose a 1-1 correspondence between the arcs in $\mathrm{st}(v, G)$ and vertices of A_v. For each arc e of $\mathrm{st}(v, G)$, let F_e be the $(n-1)$-face of A_v which is opposite the vertex of A_v corresponding to e. Finally, we identify for each arc $e \in D(G)$ the $(n-1)$-simplices F_e and $F_{r(e)}$ in such a way that for each arc $f \in \mathrm{st}(i(e), G) - \{e\}$, the vertex of $A_{i(e)}$ corresponding to f is identified with the vertex of $A_{t(e)}$ which corresponds to $g_e(f)$.

It is easy to see that $K(G, g)$ is a relative geometric n-cycle, and its dual graph is isomorphic to G. Conversely, if M is a relative cycle then we may define a simplicial scheme g on the dual graph M^* of M such that $K(M^*, g)$ is isomorphic with M [15]. The simplicial scheme g is obtained as follows. Each arc e of M^* corresponds to a pair (A, B) where A is an n-simplex and B is an $(n-1)$-face of A. Let $F_e := B$ and let V_e be the vertex of A opposite B. Then $g_e(e) = e$, and, for $f \neq e$, $g_e(f)$ is the arc of $\mathrm{st}(t(e), M^*)$ with $V_{g_e(f)} = V_f$.

Thus regular graphs and simplicial schemes are in a natural bijective correspondence with relative cycles. Moreover, the branched coverings correspond to graph coverings between the dual graphs.

Let $p\colon G \to H$ be a graph covering and $e \in D(G)$ an arc in G with initial vertex u and terminal vertex v. A simplicial scheme h on H determines the local map $g_e\colon \mathrm{st}(u, G) \to \mathrm{st}(v, G)$ such that $h_{p(e)} p \,|\, \mathrm{st}(u, G) = p g_e$. The family $\{g_e \,|\, e \in D(G)\}$ is a simplicial scheme on G which is called the *lift* of h to G. Interested readers will find the proofs of the above-stated facts in [15], where the following theorem is also given.

Theorem 5.1. *Let K be a relative geometric cycle and let (G, g) be its dual graph with the corresponding simplicial scheme. Every branched covering $K^{\sim} \to K$ over K is uniquely determined by a graph covering $G^{\sim} \to G$. The simplicial scheme $g^{\sim}$ on $G^{\sim}$ which determines $K^{\sim}$ is just the lift of g to $G^{\sim}$, and $K^{\sim} = K(G^{\sim}, g^{\sim})$.*

Theorem 5.1 provides a convenient way of describing branched covering projections between relative cycles. All the information we need is the graph covering projection between dual graphs.

It is known [10] that every d-fold graph covering over the graph G can be described by a *permutation voltage assignment* on G. This is a function $\varphi\colon D(G) \to S_d$ (where S_d denotes the symmetric group on d letters) which assigns inverse group elements to opposite arcs, and assigns the identity permutation to each half-edge. The pair (G, φ) is then called the *voltage graph*. We may regard φ as a 1-*cocycle* with values in S_d, and we write $\varphi \in Z^1(G; S_d)$. (In fact, φ is a cocycle on the graph G with deleted half-edges, but, nevertheless, we shall not hesitate to use the terminology introduced above.)

A voltage graph (G, φ) determines the *derived graph* G^φ which is a covering graph over G and is defined as follows. The vertex set $V(G^\varphi)$ is $V(G) \times \{1, 2, \ldots, d\}$ and for each arc $e \in D(G)$ and $k \in \{1, \ldots, d\}$ there is an arc (e, k) with initial vertex $(i(e), k)$ and terminal vertex $(t(e), \varphi(e)(k))$.

Let (G, φ) be a voltage graph, and let $u \in V(G)$. If γ is a closed walk in G based at u then the product of voltages on the arcs on γ is denoted by $\varphi(\gamma)$. Denote by $\Gamma_u(\varphi)$ the set of voltages $\varphi(\gamma)$ taken over all the closed walks based at u. Clearly, $\Gamma_u(\varphi)$ is a subgroup of S_n, and it is called the *local group* based at u (sometimes the term monodromy group is also used). Note that any two local groups (with respect to the same voltage assignment) are conjugate in S_n. The derived graph G^φ of the voltage graph (G, φ) is connected if and only if the local group is transitive in S_n.

Two voltage graphs (G, φ) and (G, ϕ) are called *equivalent* if there is a 0-cochain ξ: $V(G) \to S_d$ such that for each $e \in D(G)$

$$\phi(e) = \xi^{-1}(i(e))\varphi(e)\xi(t(e)).$$

Clearly, the derived graphs G^φ and G^ϕ are isomorphic (as covering spaces over G) if and only if the two voltage graphs are equivalent.

The equivalence of voltage graphs induces an equivalence relation on $Z^1(G; S_d)$. We denote the quotient set by $H^1(G; S_d)$ and we call it the 1-*cohomology set* of G with values in S_d. Note that usually only the cohomology with values in Abelian groups is considered. In this case the 1-cohomology set admits the structure of a group.

The following result is an easy consequence to Theorem 5.1. Recall that the isomorphism of (branched) coverings means a combinatorial isomorphism which preserves fibers.

Proposition 5.2. *The isomorphism classes of d-fold covering graphs over the graph G, and, consequently, the isomorphism classes of d-fold branched covering spaces over $K = K(G, g)$ are in a bijective correspondence with the* 1-*cohomology sets in* $H^1(G; S_d)$.

There is another, sometimes more convenient, description of covering graphs and hence also of branched coverings of relative geometric cycles. Let T be spanning tree of G. For every voltage assignment on G there is an equivalent one which is trivial on T [10]. Obviously, the equivalence between voltage assignments which are trivial on T is just conjugacy. On the other hand, the edges of $G - T$ are in a natural 1-1 correspondence with generators of the fundamental group $\pi_1(G, *)$ where $*$ is any vertex of G. Thus the voltage assignment φ induces a homomorphism φ_T: $\pi_1(G, *) \to S_d$, determined up to conjugacy. Conversely, every such homomorphism gives rise to a voltage assignment on the arcs of $G-T$ in the obvious way. We may summarize this as follows.

Proposition 5.3. *The isomorphism classes of d-fold covering graphs over the graph G and the isomorphism classes of d-fold branched coverings over $K=K(G,g)$ are in a bijective correspondence with the conjugacy classes of homomorphisms $\pi_1(G,*)\to S_d$.*

Similar to the description of Proposition 5.3, there are two classical Hurwitz theorems on the existence and classification of branched coverings of surfaces (see [2] and [12]). A very similar description is also known for 3-manifolds and branched coverings with a collection of simple closed curves as the branched set [11]. In the above-mentioned results, d-fold branched coverings are described by homomorphisms $\pi_1(K-B,*)\to S_d$ where B is the branched set. The advantage of this approach is that $\pi_1(K-B,*)$ is independent of the triangulation of K.

Proposition 5.4. *Let K be a relative n-cycle and let B be a subpseudocomplex of K. The isomorphism classes of d-fold branched coverings over K with the branched set contained in B are in bijective correspondence with the conjugacy classes of homomorphisms $\pi_1(K-B,*)\to S_d$.*

Proof. Taking into account that a branched covering to K with the branched set in B is determined by the corresponding unbranched covering to $K-B$ [7], the result follows by well-known representation properties of covering spaces, see [17]. □

Corollary 5.5. *If $\pi_1(K-B)=\pi_1(K)$ then every branched covering with the branched set contained in B is unbranched.*

If $K=K(G,g)$ is locally simply connected then the branched set is pure $(n-2)$-dimensional by Theorem 4.1, and is thus determined by its $(n-2)$-simplices. In this case, the fundamental group of the pseudocomplex can also be easily calculated. It is equal to the fundamental group of the 2-skeleton of the dual cone complex, and $\pi_1(K-B)$ can be obtained as follows. Take the generators of $\pi_1(G)$ (the edges of a cotree). Then add relations corresponding to links of the $(n-2)$-simplices which are not in B. That's all.

We note that the representation theorems can be useful in the theory of crystallizations. Notice that every n-crystallization is a branched covering over $K(H_n,h_n)$, where H_n is the graph consisting of two vertices and $n+1$ parallel edges between them, and h_n sends each arc of H_n to its inverse.

6. Regular Branched Coverings

A branched covering $f\colon M\to N$ is *regular* if the corresponding unbranched covering is regular. This means that the local groups act transitively on fibers, and this is true if and only if the dual graph covering f^*: $M^* \to N^*$ is a regular covering projection. It is known [9], [10] that every regu.ar graph covering projection over the graph G can be represented by an *ordinary voltage assignment* on G. This is a function φ: $D(G) \to \Gamma$ (a 1-cocycle with values in Γ) where Γ is some group. The triple (G, Γ, φ), called the *ordinary voltage graph*, gives rise to the *derived graph* G^φ, which has vertex set $V(G) \times \Gamma$, and two arbitrary vertices, (u, g) and (v, h), adjacent if and only if $(uv) \in D(G)$ and $h = \varphi(uv)g$. The derived graph is a $|\Gamma|$-fold regular covering graph over G.

Remark. There is the obvious generalization of both ordinary and permutation voltage graphs. If Γ is a group acting on the set X, then a given 1-cocycle of G with values in Γ gives rise to the derived graph with vertex set $V(G) \times X$, and two vertices adjacent if the same condition as above is satisfied.

Proposition 6.1. *Each regular branched covering over the relative n-cycle K can be represented by a 1-cocycle on the dual graph with values in some group.*

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