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Some relations between analytic and geometric properties of infinite graphs

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Abstract

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For locally finite infinite graphs the following analytic invariants and properties are considered: the spectrum of the transition and difference Laplacian matrix, amenability and the Kazhdan property (T). They are related to several geometric invariants, such as the isoperimetric number, growth, the structure of the space of ends, etc. Usually, only the global behaviour of invariants is important. It is shown how each of the above properties has its 'essential' counterpart, e.g. the essential isoperimetric number, the essential spectrum, the essential maximum degree, etc. These invariants do not change if we add or delete finitely many edges in the graph.

1. Introduction

The main aim of this paper is to bring to attention some properties and invariants of infinite locally finite graphs (with possible loops and multiple edges). Several relations between some of these invariants are derived. We also introduce, for each invariant considered here, its essential counterpart, which is a similar quantity but with the property that it does not change if we add, or delete finitely many edges from the graph. These 'essential' invariants describe the global properties of graphs. If a finite piece of a graph looks arbitrarily ugly, it does not affect the essential properties.

The considered invariants are classified as analytic (algebraic), or combinatorial (geometric). In the first class we count the spectrum of several matrices associated to graphs, the amenability, and the Kazhdan property (T). The last two of these were originally introduced for locally compact topological groups, and it is

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suggested here how to define them for arbitrary graphs, so that they coincide on Cayley graphs of finitely generated discrete groups with the classical parameters. The related combinatorial invariants are the isoperimetric constants of graphs, some numbers based on vertex degrees and the growth of graphs, and the structure of the space of ends of graphs.

The paper intends to be something between introductory, expository, and research. We believe that some ideas presented here are worthwhile deeper investigation not only because of their applications in graph theory but also because of their close relationship with some other areas of mathematics, e.g. differential geometry, representation theory, random walks, stochastic processes, etc.

2. Some geometric invariants

Graphs in this paper are undirected and locally finite.

Graphs G and H are said to be *equivalent* (*by finite*), or *almost isomorphic*, if there is a bijection $\psi : V(G) \to V(H)$ which maps adjacent, or non-adjacent pairs of vertices of G to adjacent and non-adjacent pairs, respectively, in H, with finitely many exceptions. Equivalently, H can be obtained from G by first deleting and then adding finitely many edges, up to an isomorphism.

We shall denote by $\Delta(G)$ and $\delta(G)$ the maximal and minimal degrees of vertices in G, respectively. They have their 'essential' counterparts, the *essential maximal degree* $\Delta'(G)$ and the *essential minimal degree* $\delta'(G)$, which are defined by

$$\Delta'(G) := \inf\{\Delta(H) \mid H \text{ equivalent to } G\}$$

and

$$\delta'(G) := \sup\{\delta(H) \mid H \text{ equivalent to } G\}.$$

It may happen that $\Delta'(G) = \infty$ and $\delta'(G) = \infty$. It is clear that $\Delta'(G)$ is equal to the minimal value d for which there are only finitely many vertices of G with degree $> d$. Similarly, $\delta'(G)$ is the minimal d with infinitely many vertices of degree d. It is obvious that $\delta'(G) \leq \Delta'(G)$. These two quantities are defined in such a way that no finite perturbation of G changes their values, i.e., if G and H are equivalent then $\Delta'(G) = \Delta'(H)$ and $\delta'(G) = \delta'(H)$. Graph invariants with this property are said to be *essential*. So, $\Delta'(G)$ and $\delta'(G)$ are first examples of essential graph invariants we met.

Let $v \in V(G)$. Denote by $B_n(v)$ the set of all vertices of G at distance at most n from v (the *ball* of radius n centered at v), and let $b_n(v) := |B_n(v)|$. If necessary to expose the graph, we write $b_n(G, v)$, or $B_n(G, v)$. The graph G has *exponential growth* (from the vertex v) if $b_n(v) \geq Cq^n$ for some constants $C > 0$, $q > 1$, and each $n \geq 0$. It has *polynomial growth* if $b_n(v) \leq p(n)$ where $p(\ )$ is a polynomial. G has *subexponential growth* if it does not grow exponentially.

Note that the type of the growth is independent of v if G is connected. Let

$$\varepsilon(G, v) := \limsup_{n\to\infty} (b_n(G, v))^{1/n}$$

and

$$\tau(G, v) := \liminf_{n\to\infty} (b_n(G, v))^{1/n}.$$

If v and u are vertices in the same component of G then $b_n(u) \leqslant b_{n+d}(v) \leqslant b_{n+2d}(u)$ where $d = \mathrm{dist}(u, v)$. It follows that $\epsilon(G, v)$ and $\tau(G, v)$ are constant on each component of the graph. Define

$$\varepsilon(G) := \sup\{\varepsilon(G, v) \mid v \in V(G)\}$$

and

$$\tau(G) := \sup\{\tau(G, v) \mid v \in V(G)\}.$$

By the above conclusion, the suprema may be taken on representatives of each of the connected components only.

Lemma 2.1. *If e is an edge of the graph G then $\varepsilon(G-e) = \varepsilon(G)$ and $\tau(G-e) \leqslant \tau(G)$. If e is not a bridge then $\tau(G-e) = \tau(G)$.*

Proof. The proof of $\varepsilon(G-e) = \varepsilon(G)$ can be found in [13]. To verify the relations for τ, let $e = uv$. Clearly, $\tau(G, v) \geqslant \tau(G-e, v)$ and $\tau(G, v) = \tau(G, u) \geqslant \tau(G-e, u)$. Consequently, $\tau(G) \geqslant \tau(G-e)$. Assume now that e is not a bridge. Then u and v belong to the same component of $G-e$. Let d be the distance between u and v in $G-e$. Clearly, $b_n(G, v) \leqslant b_{n+d}(G-e, v)$. Therefore $\tau(G, v) \leqslant \tau(G-e, v)$. The same holds for u instead of v. This implies that $\tau(G) \leqslant \tau(G-e)$, so by the converse inequality proved above, $\tau(G) = \tau(G-e)$. □

Note. It may happen that $\tau(G-e) < \tau(G)$. There are also cases where $\tau(G) > 1$ and $\tau(G-e) = 1$. Specific examples are not hard to construct.

Corollary 2.2. *The numbers $\varepsilon(G)$ and $\tau'(G) := \inf\{\tau(H) \mid H$ equivalent to $G\}$ are essential numerical graph invariants.*

Proof. ϵ is essential by Lemma 2.1, since addition or deletion of edges does not affect its value. On the other hand, τ' is essential since for G and G' equivalent, the set of graphs which are equivalent to G is equal to the equivalence class of G'. □

Proposition 2.3. *Let G be a graph and $v \in V(G)$. Then $\tau(G, v) > 1$ if and only if G has exponential growth at v.*

Proof. First, if $b_n(v) \geq Cq^n$, $q > 1$, $C > 0$, then clearly $\tau(G) \geq \tau(G, v) \geq q > 1$. Conversely, if $\tau(G) > 1$, pick $v \in V(G)$ such that $\tau(G, v) > 1$. This imples that for an arbitrarily small $\alpha > 0$,

$$b_n(v)^{1/n} \geq \tau(G, v) - \alpha \tag{2.1}$$

for all but finitely many values of n. Take $\alpha := \frac{1}{2}(\tau(G, v) - 1)$ and $q := \tau(G, v) - \alpha > 1$. By (2.1), $b_n(v) \geq q^n$ for all but finitely many n, thus $b_n(v) \geq Cq^n$ for some small enough $C > 0$ and each n. □

The *isoperimetric number* $h(G)$ of G is the number

$$h(G) := \inf\left\{ \frac{|\delta X|}{|X|} \;\middle|\; |X| < \infty,\ X \neq \emptyset,\ X \subset V(G) \right\}$$

where $\delta X = \{e \in E(G) \mid e = vu,\ v \in X,\ u \notin X\}$. Sometimes we will have various graphs on the same vertex set. To specify in which graph G the coboundary δX is taken we use the subscript, e.g. $\delta_G X$. The *essential isoperimetric number* of G is

$$h'(G) := \sup\{h(G') \mid G' \text{ equivalent to } G\}.$$

It is clear that $h'(G) \geq h(G)$. It is also immediate that $h(G) \leq \delta(G)$ and $h'(G) \leq \delta'(G)$. However, these inequalities need not be very tight since there are easily constructed examples where $h'(G) = 0$ and $\delta'(G) = \infty$. On the other hand also

$$h'(G) \leq \Delta'(G) - 2. \tag{2.2}$$

This can be shown as follows: Let G' be equivalent to G. Then it has only finitely many vertices of degree greater than $\Delta'(G)$. Therefore one can find an arbitrarily large finite $X \subset V(G')$ such that all vertices in X have degree at most $\Delta'(G)$ in G', and the subgraph $\langle X \rangle$ of G' induced on X is connected. Then, in G',

$$|\delta X| \leq \Delta'(G) \cdot |X| - 2\,|E(\langle X \rangle)| \leq \Delta'(G)\,|X| - 2(|X| - 1),$$

which easily implies (2.2). Notice that this bound is best possible since for the Δ-regular tree T_Δ, $h(T_\Delta) = h'(T_\Delta) = \Delta - 2$.

Theorem 2.4. *Let G be a connected locally finite graph with $\Delta'(G) < \infty$. Then*

$$h(G) \leq h'(G) \leq \Delta'(G)\,\frac{\tau(G) - 1}{\tau(G) + 1}. \tag{2.3}$$

Proof. The first inequality is obvious. By Lemma 2.1, $\tau(G)$ does not change by adding finitely many new edges to G since G is connected. Thus we may assume that $h(G)$ is arbitrarily close to $h'(G)$, and so it is sufficient to show that

$$h(G) \leq \Delta'(G) \cdot \frac{\tau(G) - 1}{\tau(G) + 1}. \tag{2.4}$$

Choose a vertex $v \in V(G)$ and denote by $s_n := b_n(v) - b_{n-1}(v)$. For n large enough all vertices at distance n from v have degree at most $\Delta'(G)$, thus

$$\Delta'(G)s_{n+1} \geq |\delta B_{n+1}(v)| + |\delta B_n(v)| \geq h(G)(b_{n+1} + b_n) = h(G)(2b_n + s_{n+1}).$$

It follows that $(\Delta'(G) - h(G))s_{n+1} \geq 2h(G)b_n$ and hence

$$b_{n+1} = b_n + s_{n+1} \geq b_n \frac{\Delta'(G) + h(G)}{\Delta'(G) - h(G)}$$

for each n large enough. Therefore

$$b_n \geq C \cdot \left(\frac{\Delta'(G) + h(G)}{\Delta'(G) - h(G)}\right)^n,$$

for some $C > 0$ and each $n \geq 1$. So $\tau(G) \geq (\Delta'(G) + h(G))/(\Delta'(G) - h(G))$. This is equivalent to (2.4), so we are done. $\square$

Corollary 2.5. *If $\Delta'(G) < \infty$ and $h'(G) > 0$ then $\tau(G) > 1$, so G has exponential growth in some of its components.*

Note 2.6. There are graphs with exponential growth and with $h'(G) = 0$. For example, Cayley graphs of some soluble groups which are not nilpotent by finite are known to grow exponentially but having $h'(G) = 0$, see [18]. On the other hand, there are graphs with $\Delta'(G) = \infty$ and $h'(G) > 0$ and having polynomial growth. Take, for example, the graphs having as vertices all integers and between i and $i + 1$ having $|i| + 1$ parallel edges, $-\infty < i < +\infty$. This graph is easily seen to have $h(G) = h'(G) = 1$, but it grows linearly. If we want simple graphs, we may add a vertex of degree two on each edge, or replace each vertex i by a clique of order $|i| + 1$, and between any two consecutive cliques put a complete join. All of this shows that Corollary 2.5 is best possible.

There are other possibilities to define isoperimetric constants of graphs, by taking other distances in graphs. Let us introduce only the *transition isoperimetric number* $s(G)$ of a graph G. If G is a graph with no edges then $s(G) := 0$, and otherwise

$$s(G) := \inf\left\{\frac{|\delta X|}{S(X)} \;\middle|\; X \subset V(G),\ X \neq \emptyset,\ |X| < \infty\right\}$$

where $S(X) := \sum_{v \in X} \deg(v)$ is the sum of the degrees of the vertices in X. Clearly, $0 \leq s(G) < 1$. In contrast to $h(G)$, the transition isoperimetric number does not properly measure the connectivity of the graph since it may happen that adding an edge to a graph strictly decreases its value. It is easily seen that $\delta(G)s(G) \leq h(G) \leq \Delta(G)s(G)$. Therefore $s(G)$ has advantage over $h(G)$ in case $\Delta(G) = \infty$ only. The *essential transition isoperimetric number* $s'(G)$ is equal to the supremum of all $s(G')$, where G' is equivalent by finite to G. A similar result as Theorem 2.4 holds for $s(G)$.

Theorem 2.7. *Let G be a connected locally finite graph. Then:*

(a) *Regardless of the degrees in G*

$$\tau(G) \geq \sqrt{\frac{1+s'(G)}{1-s'(G)}}, \quad \text{i.e.} \quad s(G) \leq s'(G) \leq \frac{\tau^2(G)-1}{\tau^2(G)+1}. \tag{2.5}$$

(b) *For a vertex $v \in V(G)$ let D_n be the maximal degree of vertices at distance n from v. If* $\limsup_{n\to\infty} D_n^{1/n} = 1$ *then*

$$\tau(G) \geq \frac{1+s'(G)}{1-s'(G)}, \quad \text{i.e.} \quad s(G) \leq s'(G) \leq \frac{\tau(G)-1}{\tau(G)+1}. \tag{2.6}$$

(c) *If there is a constant $M < \infty$ such that each vertex of G is contained in at most M edge-disjoint cycles, then* (2.6) *holds.*

Proof. If $s'(G) = 0$ then all the statements of the theorem are clear. Thus we assume that $s'(G) > 0$. Choose an arbitrarily small $\eta > 0$, $\eta < s'(G)$. There is a graph G_1 equivalent to G such that $s(G_1) \geq s'(G) - \eta > 0$. G_1 has finitely many components and all of them are infinite. It is easily seen that one of the components, say G_2, has $s(G_2) = s(G_1)$. We shall prove that the claimed inequalities (2.5) and (2.6) hold in G_2, for $s(G_2)$ in place of $s'(G)$. Since $\tau(G) \geq \tau(G_2)$ and $s(G_2)$ is arbitrarily close to $s'(G)$, this will be enough. To make the notation easier we will do all the calculations with the graph G (instead of G_2), in particular we assume henceforth that G is connected.

Choose a vertex $v \in V(G)$. Let $S_n := B_n(v) \setminus B_{n-1}(v)$. Then

$$S(B_n) - S(B_{n-1}) = S(S_n) \geq |\delta B_n| + |\delta B_{n-1}| \geq s(G)(S(B_n) + S(B_{n-1}))$$

and so $S(B_n) \geq S(B_{n-1})(1+s(G))/(1-s(G))$. It follows that for some constant $C > 0$

$$S(B_n) \geq C \cdot \left(\frac{1+s(G)}{1-s(G)}\right)^n. \tag{2.7}$$

We now consider cases (a), (b), and (c).

(a) Since each edge with an end in B_n has its other end in B_{n+1},

$$S(B_n) \leq |B_n| \cdot |B_{n+1}| \leq |B_{n+1}|^2.$$

Combining this with (2.7) gives the lower bound (2.5) on $\tau(G)$.

(b) Choose any $\varepsilon > 0$. By the assumption on D_n, there is n_0 such that for each $n \geq n_0$, $D_n < (1+\varepsilon)^n$. Let $n_1 \geq n_0$ be such that $\varepsilon(1+\varepsilon)^{n_1} \geq S(B_{n_0})$. Then for each $n > n_1$

$$\begin{aligned} S(B_n) &< (1+\varepsilon)^n |S_n| + \cdots + (1+\varepsilon)^{n_0} |S_{n_0}| + S(B_{n_0}) \\ &\leq (1+\varepsilon)^n |S_n| + \cdots + (1+\varepsilon) \cdot (1+\varepsilon)^{n_1} |S_{n_1}| + \cdots + (1+\varepsilon)^{n_0} |S_{n_0}| \\ &\leq (1+\varepsilon)^n (|S_n| + \cdots + |S_{n_0}|) \leq (1+\varepsilon)^n b_n. \end{aligned}$$

Now (2.7) implies that

$$(1+\varepsilon)^n b_n \geq C \cdot \left(\frac{1+s(G)}{1-s(G)}\right)^n.$$

Since ϵ was arbitrary, this gives the required lower bound on $\tau(G)$.

(c) Let G_n be the subgraph of G induced on $B_n(v)$. Take a cycle C_1 in G_n and remove its edges. Repeat this by choosing another, edge-disjoint cycle C_2, remove its edges, etc. until an acyclic graph is obtained. By our condition, the total number of removed edges is at most $M\,|B_n|$, and therefore

$$|E(G_n)| \leq M\,|B_n| + (|B_n| - 1) < (M+1)\,|B_n|. \tag{2.8}$$

We also have

$$S(B_n) \leq 2\,|E(G_{n+1})| \tag{2.9}$$

and now a combination of (2.7), (2.8), and (2.9) gives

$$|B_{n+1}| \geq \frac{C}{2(M+1)} \left(\frac{1+s(G)}{1-s(G)}\right)^n$$

which implies (2.6). □

Corollary 2.8. *If* $s'(G) > 0$ *then* $\tau(G) > 1$, *so G has exponential growth in some of its components.*

3. The spectrum

With a graph we may associate several matrices. For our purpose, only the adjacency, the transition and the Laplacian matrix will be introduced. These matrices have their rows and columns indexed by the vertices $V(G)$ of the graph. No order of $V(G)$ is assumed, so a matrix in fact means a whole class of permutation similar matrices with ordered rows and columns. Let $\ell^2(V)$ denote the Hilbert space of complex vectors $x = (x_v)_{v \in V(G)}$ with coordinates indexed by $V = V(G)$, and such that

$$\|x\|^2 = \sum_{v \in V} |x_v|^2 < \infty.$$

The inner product in $\ell^2(V)$ is defined $\langle x, y \rangle = \sum_{v \in V} x_v \bar{y}_v$. Our matrices act naturally on $\ell^2(V)$ as linear operators.

The *adjacency matrix* $A(G) = [a_{uv}]$ of the graph G has entries a_{uv} equal to the number of edges between vertices u and v. If $D(G)$ denotes the diagonal matrix with diagonal entries containing vertex degrees, $d_{uu} = \deg(u)$, $u \in V(G)$, then the matrix $L(G) := D(G) - A(G)$ is called the *difference Laplacian matrix* of G. The *transition matrix*, $P(G) = [p_{uv}]$ has entries $p_{uv} = a_{uv}/\deg(u)$. This one is assumed to act on the Hilbert space $\ell_d^2(V)$ where the inner product is given by

$$\langle x, y\rangle_d = \sum_{v\in V} x_v \bar{y}_v \deg(v)$$

while the adjacency and the Laplacian matrix act on $\ell^2(V)$. With this convention all three matrices define symmetric linear operators on the corresponding Hilbert spaces. It will be assumed, whenever speaking about the Laplacian or the adjacency matrix, that the graphs have bounded degrees, i.e., $\Delta(G) < \infty$. This will not be required in case of the transition operator. Under this assumption, $A(G)$, $L(G)$ and $P(G)$ are everywhere defined and self-adjoint linear operators on the corresponding Hilbert space. Their spectra are real. If we denote the spectrum of $A(G)$, $L(G)$ and $P(G)$ by $\sigma_A(G)$, $\sigma_L(G)$, and $\sigma_P(G)$, respectively, then it is well known (cf., e.g. [15]) that

$$\sigma_A(G) \subseteq [-\Delta, \Delta], \qquad \sigma_L(G) \subseteq [0, 2\Delta], \quad \text{and} \quad \sigma_P(G) \subseteq [-1, 1],$$

where $\Delta = \Delta(G)$. The reader is referred to [15] for more details about the spectra of infinite graphs. Let us add that in each case, the spectrum is the *approximate point spectrum* in the corresponding Hilbert space; for example, $\lambda \in \sigma_L(G)$ if and only if there is a sequence of unit vectors $x^{(n)}$ such that

$$\|L(G)x^{(n)} - \lambda x^{(n)}\| \to 0 \quad \text{as } n \to \infty.$$

The vectors $x^{(n)}$ may be assumed to have only real entries $x_v^{(n)}$.

The introduced spectra are closely related if G is regular. It is easily seen that

$$k \cdot \sigma_P(G) = \sigma_A(G) = k - \sigma_L(G)$$

where k is the valency and the equalities hold as equalities between sets.

For us there are two particularly important numerical invariants based on the spectrum. Let us denote by

$$\lambda_1(G) := \inf \sigma_L(G)$$

and

$$\rho_1(G) := \sup \sigma_P(G).$$

They can also be expressed as

$$\lambda_1(G) = \inf\left\{ \frac{\langle L(G)x, x\rangle}{\langle x, x\rangle} \,\middle|\, x \in \ell^2(V),\, x \neq 0 \right\} \tag{3.1}$$

and

$$\rho_1(G) = \sup\left\{ \frac{\langle P(G)x, x\rangle_d}{\langle x, x\rangle_d} \,\middle|\, x \in \ell_d^2(V),\, x \neq 0 \right\}. \tag{3.2}$$

The 'essential' invariant corresponding to the spectrum is the *essential spectrum* of $A(G)$, $L(G)$, or $P(G)$, respectively. The essential spectrum of a self-adjoint linear operator B on a Hilbert space consists of all those elements from the spectrum of B which are not isolated eigenvalues of finite multiplicity. It is well known (cf., e.g., [9, Section X.2]) that these are exactly those $\lambda \in \sigma(B)$ which remain in the spectrum if B is changed by any compact perturbation, i.e. $\lambda \in \sigma(B+K)$ for every compact K. It follows that the essential spectrum is an essential graph invariant since equivalence by finite perturbes the graphs matrices in finitely many entries only and thus presents only compact perturbations. The elements of the essential spectrum are also characterized as the approximate eigenvalues of infinite multiplicity [5] (cf. also [9]). This means that λ is in the essential spectrum of B if and only if there are *pairwise orthogonal* unit vectors $x^{(n)}$, $n = 1, 2, 3, \dots$, such that

$$\|Bx^{(n)} - \lambda x^{(n)}\| \to 0 \quad \text{as } n \to \infty.$$

We shall denote by $\lambda_1'(G)$ the infimum of the essential spectrum of $L(G)$, and by $\rho_1'(G)$ the supremum of the essential spectrum of $P(G)$. Clearly, $\lambda_1(G) \leqslant \lambda_1'(G)$ and $\rho_1(G) \geqslant \rho_1'(G)$. Later we shall need the following inequality:

$$\lambda_1'(G) \leqslant \frac{\Delta'(G) + \delta'(G)}{2} - 1 < \Delta'(G) \tag{3.3}$$

if $\delta'(G) \geqslant 1$. The proof goes as follows: There are infinitely many pairs u_i, v_i of adjacent vertices, such that $\deg(u_i) \leqslant \delta'(G)$ and $\deg(v_i) \leqslant \Delta'(G)$. Let $y^{(1)}, y^{(2)}, \dots$ be their characteristic vectors (i.e., $y^{(i)}$ has the coordinate of v_i and u_i equal to $1/\sqrt{2}$, all others 0). Then

$$\langle L(G)y^{(i)}, y^{(i)} \rangle \leqslant \frac{\deg(u_i) + \deg(v_i) - 2}{2} \leqslant \frac{\Delta'(G) + \delta'(G)}{2} - 1.$$

Since this happens for infinitely many pairwise orthogonal unit vectors $y^{(i)}$, an element of the essential spectrum must be $\leqslant (\Delta'(G) + \delta'(G))/2 - 1$. Since we were not able to find this last statement mentioned in the available literature, we give a short proof of it. We have to show that if $\langle Lx^{(i)}, x^{(i)} \rangle \leqslant t$ for infinitely many pairwise orthogonal unit vectors $x^{(i)}$ then there is some $\lambda \leqslant t$ in the essential spectrum of L. It is known [22] that for some compact symmetric operator K, the spectrum of $L + K$ is equal to the essential spectrum of L. Notice that since K is compact, $\langle Kx^{(i)}, x^{(i)} \rangle \to 0$. Consequently

$$\begin{aligned} \lambda_1'(L) &= \lambda_1(L + K) = \inf\{\langle (L+K)x, x \rangle \mid \|x\| = 1\} \\ &\leqslant \inf\{\langle Lx^{(i)}, x^{(i)} \rangle + \langle Kx^{(i)}, x^{(i)} \rangle\} \leqslant t. \end{aligned}$$

Theorem 3.1. *Let G be a locally finite graph, and let $\Delta = \Delta(G) < \infty$, $\Delta' = \Delta'(G)$. Then*

$$h(G) \geqslant \lambda_1(G) \frac{\Delta^2}{\Delta^2 - \Delta - \lambda_1(G)} \geqslant \lambda_1(G) \tag{3.4}$$

and

$$h'(G) \geq \lambda_1'(G) \frac{\Delta'^2}{\Delta'^2 - \Delta' - \lambda_1'(G)} \geq \lambda_1'(G). \tag{3.5}$$

Proof. Let U be an arbitrary finite subset of $V = V(G)$. Define $x \in \ell^2(V)$ by setting $x_v = 1$ if $v \in U$, and otherwise $x_v = (1/\Delta)n_v$, where n_v is the number of edges from v to vertices in U. Denote by W the set of those vertices of $V \setminus U$ which have a neighbour in U. By (3.1),

$$\lambda_1(G)\,\|x\|^2 \leq \langle L(G)x, x\rangle. \tag{3.6}$$

It is easily seen that

$$\langle L(G)x, x\rangle = \sum_{uv \in E(G)} (x_u - x_v)^2 \tag{3.7}$$

and since $x_u - x_v$ is nonzero only if v or u lies in W, we get from (3.6) and (3.7):

$$\begin{aligned} \lambda_1\,\|x\|^2 &= \lambda_1\Big(|U| + \sum_{v \in W} \frac{n_v^2}{\Delta^2}\Big) \leq \sum_{vu \in E(G)} (x_u - x_v)^2 \\ &\leq \sum_{v \in W} \Big(n_v\Big(1 - \frac{n_v}{\Delta}\Big)^2 + (\Delta - n_v)\Big(\frac{n_v}{\Delta}\Big)^2\Big) \\ &= \sum_{v \in W} n_v - \sum_{v \in W} \frac{n_v^2}{\Delta} = |\delta U| - \frac{1}{\Delta} \sum_{v \in W} n_v^2. \end{aligned} \tag{3.8}$$

By rearranging this inequality and using the fact $n_v^2 \geq n_v$, one immediately gets

$$|\delta U| - \lambda_1\,|U| \geq \frac{\Delta + \lambda_1}{\Delta^2} \sum_{v \in W} n_v \geq \frac{\Delta + \lambda_1}{\Delta^2}\,|\delta U|$$

and it follows that

$$\frac{|\delta U|}{|U|} \geq \lambda_1 \frac{\Delta^2}{\Delta^2 - \Delta - \lambda_1}$$

which implies (3.4) since U was arbitrary.

The proof of (3.5) needs some more care. Fix an arbitrarily small $\varepsilon > 0$. First we shall prove that there exists a graph G_1, which can be obtained from G by adding finitely many edges, such that $\lambda_1(G_1) \geq \lambda_1'(G) - \varepsilon$.

The only elements in the Laplacian spectrum of G which are smaller than $\lambda_1' = \lambda_1'(G)$ are isolated eigenvalues of finite multiplicity. Let N be the span of the eigenspaces of all those eigenvalues of $L(G)$ which are smaller than $\lambda_1' - \varepsilon/2$, and let $n := \dim N < \infty$. Fix an orthonormal basis $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ of N. It should be mentioned at once that for any $x \in \ell^2(V)$ which is orthogonal to N

$$\langle L(G)x, x\rangle \geq \Big(\lambda_1' - \frac{\varepsilon}{2}\Big)\,\|x\|^2. \tag{3.9}$$

Let U be a finite subset of $V(G)$ such that for each $i = 1, 2, \ldots, n$,

$$\sum_{u\in U} (x_u^{(i)})^2 > 1-\delta, \text{ i.e., } \sum_{u\notin U} (x_u^{(i)})^2 < \delta, \tag{3.10}$$

where $\delta > 0$ is some very small number to be fixed later. It will depend on ε, n, and G.

Let G_1 be the graph which is obtained from G by adding, for each vertex $u \in U$, m edges joining u to m new distinct neighbours which lie out of U. Denote by $L_1 := L(G_1)$ its Laplacian matrix. We shall prove that $\lambda_1(G_1) \geqslant \lambda_1' - \varepsilon$, assuming m is large enough. Notice that m may depend on ε, n, and G.

Take an arbitrary $x \in \ell^2(V)$, $\|x\| = 1$. We shall prove that $\langle L_1 x, x\rangle \geqslant \lambda_1' - \varepsilon$ which will imply, by (3.1) that $\lambda_1(G_1) \geqslant \lambda_1' - \varepsilon$. It is easy to see that it may be assumed that $x_v \geqslant 0$ for each $v \in V(G)$ (apply, for example, (3.7)). Write now $x = y + z$ where y agrees with x on U, and z agrees with x at other coordinates.

If $\|y\|^2 \leqslant \delta$ then we do the following calculation: Let $x = p + q$ where $p \in N$ and $q \perp N$. By (3.9), $\langle Lq, q\rangle \geqslant (\lambda_1' - \varepsilon/2)\|q\|^2$ while $\langle Lp, p\rangle$ can be estimated as follows: If $p = \sum_{i=1}^n \alpha_i x^{(i)}$ then $\|p\|^2 = \sum_{i=1}^n |\alpha_i|^2$ and

$$|\alpha_i| = |\langle x, x^{(i)}\rangle| = \sum_{u\in U} x_u x_u^{(i)} + \sum_{v\notin U} x_v x_v^{(i)} < \|y\| + \|x\|\sqrt{\delta} \leqslant 2\sqrt{\delta}. \tag{3.11}$$

In the last inequality we applied the Cauchy–Schwartz inequality and (3.10). Consequently, $\|p\|^2 < 4n\delta = \varepsilon/2\Delta(G)$ if we choose $\delta := \varepsilon/8n\Delta(G)$. Therefore $\|q\|^2 = \|x\|^2 - \|p\|^2 > 1 - \varepsilon/2\Delta(G)$. Using this estimate, the fact that $\langle Lp, q\rangle = \langle Lq, p\rangle = 0$, and (3.3) we get

$$\begin{aligned}\langle L_1 x, x\rangle &\geqslant \langle Lx, x\rangle = \langle Lp, p\rangle + \langle Lq, q\rangle \\ &\geqslant \left(\lambda_1' - \frac{\varepsilon}{2}\right)\|q\|^2 > \left(\lambda_1' - \frac{\varepsilon}{2}\right)\left(1 - \frac{\varepsilon}{2\Delta(G)}\right) \geqslant \lambda_1' - \varepsilon.\end{aligned}$$

The other possibility to consider is the case when $\|y\|^2 > \delta = \varepsilon/8n\Delta(G)$. In this case let $B := L_1 - L(G)$ and let $F := E(G_1)\setminus E(G)$. If $z = p + q$ where $p = \sum_{i=1}^n \alpha_i x^{(i)} \in N$, $q \perp N$, then we see as in (3.11) that

$$|\alpha_i| = |\langle z, x^{(i)}\rangle| < \sqrt{\delta}\,\|z\|$$

and hence $\|p\|^2 < n\delta\,\|z\|^2$, so $\|q\|^2 > (1 - n\delta)\,\|z\|^2$. Therefore

$$\langle L_1 z, z\rangle \geqslant \langle Lz, z\rangle = \langle Lp, p\rangle + \langle Lq, q\rangle \geqslant \left(\lambda_1' - \frac{\varepsilon}{2}\right)\|q\|^2 > (\lambda_1' - \varepsilon)\,\|z\|^2. \tag{3.12}$$

Finally,

$$\begin{aligned}\langle L_1x, x\rangle &= \langle L_1y, y\rangle + 2\langle L_1y, z\rangle + \langle L_1z, z\rangle\\ &\geq (\lambda_1' - \varepsilon)\,\|z\|^2 + \langle By, y\rangle + 2\langle L_1y, z\rangle\\ &= (\lambda_1' - \varepsilon)\,\|z\|^2 + \sum_{uv\in F}(y_u - y_v)^2 + 2\sum_{uv\in E(G_1)}(y_u - y_v)(z_u - z_v)\\ &\geq (\lambda_1' - \varepsilon)\,\|z\|^2 + m\sum_{u\in U} y_u^2 - 2\sum_{uv\in E(G_1)} y_u z_v\\ &\geq (\lambda_1' - \varepsilon)\,\|x\|^2 + (m - \lambda_1')\,\|y\|^2 - 2\sum_{uv\in F_1} y_u z_v\end{aligned}$$

where $F_1 = F \cup \delta_G U \subseteq E(G_1)$, and it is assumed that $u \in U$, $v \notin U$. It remains to show that

$$(m - \lambda_1')\,\|y\|^2 - 2\sum_{uv\in F_1} y_u z_v \geq 0. \tag{3.13}$$

By the Cauchy–Schwartz inequality and assuming that each edge $uv \in F_1$, $u \in U$, has different endvertex $v \notin U$, we estimate

$$\Big(\sum_{uv\in F_1} y_u z_v\Big)^2 \leq \sum_{uv\in F_1} |y_u|^2 \sum_{uv\in F_1} |z_v|^2 \leq (m + \Delta(G))\,\|y\|^2 \cdot (\Delta(G) + 1)\,\|z\|^2.$$

This inequality and the assumption $\|y\|^2 > \delta$ imply that

$$\begin{aligned}&(m - \lambda_1')\,\|y\|^2 - 2\sum_{uv\in F_1} y_u z_v\\ &\quad\geq (m - \lambda_1')\sqrt{\delta} - 2\sqrt{m + \Delta(G)}\,\sqrt{\Delta(G) + 1}\,\sqrt{1 - \delta}\,.\end{aligned} \tag{3.14}$$

Obviously, the last expression is ≥ 0 if m is large enough. It should be added that m depends on λ_1', $\Delta(G)$ and $\delta = \delta(\varepsilon, n, \Delta(G))$ but not on x. This finally establishes (3.13), and thus we succeeded to show $\langle L_1x, x\rangle \geq \lambda_1' - \varepsilon$, and consequently that $\lambda_1(G_1) \geq \lambda_1' - \varepsilon$.

It is clear that adding edges to a graph can not decrease the isoperimetric number. Therefore there exists a graph G_2 which can be obtained from G_1 by adding finitely many edges so that $h(G_2) \geq h'(G) - \varepsilon$. Clearly, also $\lambda_1(G_2) \geq \lambda_1(G_1) \geq \lambda_1' - \varepsilon$.

Let $X \subset V(G_2)$ be the set containing all vertices of degree greater than $\Delta'(G)$ and all their neighbours. Let G_3 be the graph obtained from G_2 by adding m parallel edges between any two vertices of X, where $m = \Delta'(G)\,|X|$.

Choose a finite $U_1 \subset V(G_3)$ such that $|\delta_{G_3} U_1|/|U_1| \leq h(G_3) + \varepsilon$. If $U_1 \cap X = \emptyset$ then let $U := U_1$, otherwise let $U := U_1 \cup X$. So U either contains X, or it is disjoint from X. Therefore no vertex adjacent to U in G_2 has degree greater than

$\Delta'(G)$. It is easily seen that in each case

$$\frac{|\delta_{G_2}U|}{|U|} = \frac{|\delta_{G_3}U|}{|U|} \leqslant \frac{|\delta_{G_3}U_1|}{|U_1|} \leqslant h(G_3) + \varepsilon \leqslant h'(G) + \varepsilon \tag{3.15}$$

since $m = \Delta'(G)\,|X|$.

Now we may repeat the first part of the proof for the graph G_3, using $\Delta'(G)$ instead of Δ. Notice that in (3.8) we need the fact that $n_v \leqslant \Delta'(G)$ for each vertex v adjacent to U. This completes the proof since $h(G_3)$ and $\lambda(G_3)$ are arbitrarily close to their essential values. □

It is interesting that $h(G)$ is also bounded above by a function of $\lambda_1(G)$. This inequality, given in Theorem 3.2, is called the *strong discrete Cheeger inequality*, from its close relation to a well-known Cheeger inequality [4] (cf. also [3]) from the theory of differential operators on Riemannian manifolds. A weaker version of (3.16) is proved in [13].

Theorem 3.2. *For an arbitrary locally finite graph G with bounded degrees,*

$$h(G) \leqslant \sqrt{\lambda_1(G)(2\Delta(G) - \lambda_1(G))} \tag{3.16}$$

and

$$h'(G) \leqslant \sqrt{\lambda'_1(G)(2\Delta'(G) - \lambda'_1(G))}\,. \tag{3.17}$$

Proof. The spectrum is closed, so $\lambda_1 = \lambda_1(G) \in \sigma_L(G)$. For each small enough $\varepsilon > 0$ there is a vector $x \in \ell^2(V(G))$ with finite support (only finitely many x_v are nonzero) and with $\|x\| = 1$ such that

$$\langle Lx, x\rangle \leqslant \lambda_1 + \varepsilon \leqslant \Delta(G). \tag{3.18}$$

If

$$\eta := \sum_{uv \in E(G)} |x_u^2 - x_v^2|$$

then it can be shown by the summation per partes (see, e.g., [13]) that

$$\eta \geqslant h(G). \tag{3.19}$$

On the other hand

$$\begin{aligned} \eta^2 &= \Big(\sum_{uv\in E} |x_u^2 - x_v^2|\Big)^2 = \Big(\sum_{uv\in E} |x_u + x_v|\,|x_u - x_v|\Big)^2 \\ &\leqslant \sum_{uv\in E} (x_u + x_v)^2 \cdot \sum_{uv\in E} (x_u - x_v)^2 \\ &= \sum_{uv\in E} (2x_u^2 + 2x_v^2 - (x_u - x_v)^2) \cdot \langle Lx, x\rangle \\ &= \Big(2\sum_{v\in V} \deg(v)x_v^2 - \langle Lx, x\rangle\Big) \cdot \langle Lx, x\rangle \\ &\leqslant (2\Delta(G)\,\|x\|^2 - \langle Lx, x\rangle)\langle Lx, x\rangle \\ &\leqslant 2\Delta(G)(\lambda_1 + \varepsilon) - (\lambda_1 + \varepsilon)^2. \end{aligned} \tag{3.20}$$

In the first inequality we used the Cauchy–Schwartz inequality, while in the last one we needed (3.18). Since ε was arbitrarily small, it follows that $h^2(G) \leqslant \eta^2 \leqslant 2\Delta(G)\lambda_1 - \lambda_1^2$ and the proof of (3.16) is done.

To prove (3.17), let G_1 be a graph obtained from G by adding finitely many edges and such that $h(G_1) \geqslant h'(G) - \varepsilon$ where $\varepsilon > 0$ is arbitrarily small again. As we know from the proof of Theorem 3.1, there is a graph G_2 obtainable from G_1 by addition of finitely many edges such that $\lambda_1(G_2) \geqslant \lambda_1'(G) - \varepsilon$. Since $h(G_2) \geqslant h(G_1)$, we have $h(G_2) \geqslant h'(G) - \varepsilon$.

Let X be the set of vertices of G_2 having degree $> \Delta'(G)$. The set X is clearly finite. Since $\lambda_1' = \lambda_1'(G) = \lambda_1'(G_2)$ is in the essential spectrum, there are arbitrarily many unit vectors $x^{(i)}$, $i = 1, 2, \ldots$ such that $\langle L(G_2)x^{(i)}, x^{(i)} \rangle \leqslant \lambda_1' + \varepsilon$, and such that $\langle x^{(i)}, x^{(j)} \rangle = 0$ for $i \neq j$. Let us take $N = |X| + 1$ such vectors $x^{(i)}$. They are linearly independent, so any nontrivial linear combination is nonzero. On the other hand, there exist $\alpha_1, \alpha_2, \ldots, \alpha_N$ with some $\alpha_i \neq 0$ such that $\sum_{i=1}^N \alpha_i x^{(i)} =: x$ has coordinate $x_v = 0$ for each $v \in X$. Clearly, if we demand that $\sum \alpha_i^2 = 1$, then

$$\|x\|^2 = \left\langle \sum_{i=1}^N \alpha_i x^{(i)}, \sum_{i=1}^N \alpha_i x^{(i)} \right\rangle = \sum_{i=1}^N \alpha_i^2 = 1.$$

The vectors $x^{(i)}$ may also be assumed to satisfy $\|L(G_2)x^{(i)} - \lambda_1' x^{(i)}\| \leqslant \varepsilon / N$. Then

$$\begin{aligned} \langle L(G_2)x, x \rangle &= \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \langle L(G_2)x^{(i)}, x^{(j)} \rangle \\ &= \sum_{i=1}^N \alpha_i^2 \langle \lambda_1' x^{(i)}, x^{(i)} \rangle + \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \langle L(G_2)x^{(i)} - \lambda_1' x^{(i)}, x^{(j)} \rangle \\ &\leqslant \lambda_1' + \sum_{i=1}^N \sum_{j=1}^N |\alpha_i \alpha_j| \, \|L(G_2)x^{(i)} - \lambda_1' x^{(i)}\| \cdot \|x^{(j)}\| \\ &\leqslant \lambda_1' + \frac{\varepsilon}{N} \sum_{i=1}^N \sum_{j=1}^N |\alpha_i \alpha_j| \leqslant \lambda_1' + \varepsilon. \end{aligned}$$

Now we may carry out the same calculation as to obtain (3.19) and (3.20), this time for G_2. The inequality $\deg(v)x_v^2 \leqslant \Delta'(G)x_v^2$ is clearly satisfied, and the only remaning fact to verify is $\lambda_1' + \varepsilon \leqslant \Delta'(G)$. But we have shown this by (3.3) for any graph with $\delta'(G) \geqslant 1$. The remaining case $\delta'(G) = 0$ is unimportant since in this case $h'(G) = 0$ and $\lambda_1' = 0$, so (3.17) is trivial. □

There are similar relations between $s(G)$ and $\rho_1(G)$.

Theorem 3.3. *If G is a locally finite graph then*

$$s(G) \geqslant \frac{1 - \rho_1(G)}{1 - \nabla/\rho_1(G)} \tag{3.21}$$

where $\nabla = 1/\Delta(G) \geqslant 0$, and

$$s'(G) \geqslant 1 - \rho_1'(G). \tag{3.22}$$

Proof. Set $\rho_1 = \rho_1(G)$. Let A be a finite subset of $V(G)$. Define $x \in \ell^2(V)$ as follows: $x_v := 1$ if $v \in A$, and otherwise let $x_v := n_v/\rho_1 \deg(v)$, where n_v is the number of neighbours of v in the set A. Then a straightforward calculation gives

$$\langle x, x\rangle_d = S(A) + \sum_{v\notin A} \frac{n_v^2}{\rho_1^2 \deg(v)}$$

and

$$\begin{aligned}\langle P(G)x, x\rangle_d &\geq S(A) - |\delta A| + \sum_{u\in A}\sum_{v\notin A} a_{uv}x_v \\ &\geq S(A) - |\delta A| + 2\sum_{v\notin A} x_v n_v .\end{aligned}$$

Combining these inequalities with (3.2) gives

$$\begin{aligned}\rho_1\langle x, x\rangle_d &= \rho_1 S(A) + \sum_{v\notin A} \frac{n_v^2}{\rho_1 \deg(v)} \\ &\geq \langle P(G)x, x\rangle_d \geq S(A) - |\delta A| + 2\sum_{v\notin A} \frac{n_v^2}{\rho_1 \deg(v)}\end{aligned}$$

and after rearranging

$$|\delta A| - S(A)(1-\rho_1) \geq \sum_{v\notin A} \frac{n_v^2}{\rho_1 \deg(v)} \geq \frac{\nabla}{\rho_1}\sum_{v\notin A} n_v = \frac{\nabla}{\rho_1}|\delta A| .$$

Since A was arbitrary, this implies (3.21).

To prove (3.22), take G_1 to be a graph equivalent by finite to G and with $\rho_1(G_1) \leq \rho_1'(G) + \varepsilon$. Then, by (3.21)

$$s'(G) \geq s(G_1) \geq 1 - \rho_1(G_1) \geq 1 - \rho_1'(G) - \varepsilon .$$

This implies (3.22) since ε was arbitrarily small. □

Theorem 3.4. *If G is a locally finite graph then*

$$s(G) \leq \sqrt{1-\rho_1(G)} \tag{3.23}$$

and

$$s'(G) < \sqrt{1-\rho_1'(G)} . \tag{3.24}$$

Proof. The first inequality can be found in [13]. To prove (3.24), take any $\varepsilon > 0$, and let G_1 be a graph equivalent by finite to G with $s(G_1) \geq s'(G) - \varepsilon$. Then by (3.23)

$$s'(G) - \varepsilon \leq s(G_1) \leq \sqrt{1-\rho_1(G_1)} \leq \sqrt{1-\rho_1'(G)} .$$

The inequality (3.24) is now obvious. □

4. Amenability

In this section we shall shortly exhibit the notion of amenability of infinite graphs. This concept was classically introduced as a property of locally compact groups. We refer to [16, 17] for more information.

Let Γ be a locally compact group. By $L^\infty(\Gamma)$ we denote the set of all functions $\Gamma \to \mathbb{R}$ which are bounded a.e. with respect to the Haar measure $\lambda(.)$ on Γ. The group Γ is *amenable if* $L^\infty(\Gamma)$ has an *invariant mean*, i.e. a linear functional $m: L^\infty(\Gamma) \to \mathbb{R}$ which satisfies the following conditions:

(i) For $f \in L^\infty(\Gamma)$, if $f \geqslant 0$ (a.e.) then $m(f) \geqslant 0$.

(ii) $m(\chi_\Gamma) = 1$, where χ_Γ is the constant 1 function on Γ.

(iii) It is Γ-invariant, i.e., $m(g \cdot f) = m(f)$ for every $g \in \Gamma$. (Note that $(g \cdot f)(x) = f(g^{-1}x)$.)

By a theorem of Følner [6], the amenability is equivalent to the following condition:

(F) Given $\varepsilon > 0$ and a compact set $K \subseteq \Gamma$ there is a Borel set $U \subseteq \Gamma$ of positive finite measure $\lambda(U) < \infty$ such that

$$\lambda(kU \,\Delta\, U) < \varepsilon\lambda(U)$$

for all $k \in K$, where Δ denotes the symmetric difference of sets.

For a discrete group the Haar measure counts the number of elements of the set. Then it is easy to see that (F) is equivalent to

(F') For every $\varepsilon > 0$ and finite $K \subseteq \Gamma$ one can find a finite set $U \subseteq \Gamma$ such that

$$|KU \,\Delta\, U| < \varepsilon\, |U|.$$

Lemma 4.1. *Let Γ be a group generated by a finite set of generators $S = \{g_1, \ldots, g_k\}$. Then Γ is amenable if and only if for every $\varepsilon > 0$ there is a finite set $U \subseteq \Gamma$ such that $|SU \setminus U| < \varepsilon |U|$.*

Proof. ($\Rightarrow$) Let $K := S \cup \{e\}$, where e is the identity in Γ. Then $KU \,\Delta\, U = SU \setminus U$. We are done by the amenability of Γ.

($\Leftarrow$) Choose $\varepsilon > 0$ and a finite $K \subseteq \Gamma$, $K = \{k_1, k_2, \ldots, k_t\}$. It may be assumed that $e \notin K$. If k_i can be written as a word of length $l(i)$ in terms of $S \cup S^{-1}$, let

$$\varepsilon_i := \frac{\varepsilon}{2t \cdot l(i)}$$

and let $U \subseteq \Gamma$ be a set for which $|SU \setminus U| < \varepsilon' \,|U|$, where $\varepsilon' = \min \varepsilon_i$. For each generator $g_j \in S$, $|g_j U \setminus U| \leqslant |SU \setminus U| < \varepsilon' \,|U|$. From this we also see that $|g_j U \cap U| > (1 - \varepsilon')\,|U|$, therefore also $|g_j^{-1} U \cap U| > (1 - \varepsilon')\,|U|$, so $|g_j^{-1} U \setminus U| < \varepsilon' |U|$. It follows that $|k_i U \setminus U| < l(i)\varepsilon' |U| \leqslant (\varepsilon/2t)\,|U|$. This implies that $|KU \setminus U| < (\varepsilon/2)\,|U|$. Since $|KU| \geqslant |U|$, we get $|KU \,\Delta\, U| < \varepsilon\,|U|$. □

Corollary 4.2. *A finitely generated infinite group Γ is amenable if and only if the Cayley graph* $\mathrm{Cay}(\Gamma, S)$ *with respect to some (and hence for every) finite generating set S has the isoperimetric number* $h(\mathrm{Cay}(\Gamma, S)) = 0$.

By Corollary 4.2 it follows trivially that every finitely generated group with polynomial or subexponential growth is amenable. The converse is not true. For example, all soluble groups are amenable but some of them have exponential growth. Although finite graphs are out of our main concern, let us mention that every finite group is amenable (take $U = \Gamma$).

Corollary 4.2 was observed before by several authors. Its importance lies in the fact that it gives a possibility to extend the notion of amenability to graphs in such a way that a finitely generated group Γ is amenable if and only if it has an amenable Cayley graph. So, let us call a graph G *amenable* if $s(G) = 0$. We use $s(G) = 0$ instead of $h(G) = 0$ to cover the case of graphs with unbounded degrees.

The case of graphs with bounded degrees was investigated in detail by Gerl and we refer to his works [7–8] where several conditions equivalent to the amenability of graphs are derived.

The following result is a direct consequence of Corollary 2.8.

Proposition 4.3. *A graph with subexponential growth is amenable.*

At the end of this section we shall consider amenability of infinite covers of finite graphs. Let B be a given finite graph, and let $p: G \to B$ be a graph covering projection. Is there any relation between $h(G)$ and $h(B)$, or $\lambda_1(G)$ and $\lambda_1(B)$? It turns our that there are some obvious relations, but their converse depends on the fundamental groups of G and B. Let us mention that results of the same type as following here are known in the setting of Riemannian manifolds, cf. [1, 2].

Recall that a graph map $p: G \to B$ is a *covering projection* if it is a local isomorphism, i.e., the edges incident to any vertex $v \in V(G)$ are mapped bijectively onto the edges incident with $p(v)$. For each $b \in V(B)$, the set $p^{-1}(b) \subseteq V(G)$ is called the *fibre* of b. It can be shown that all fibres have equal cardinalities if B is connected (which we shall assume henceforth).

We need some more definitions concerning finite graphs. If K is a finite graph then we define its Laplacian and the transition matrix in the same way as we do in the infinite case. The Laplacian $L(K)$ is positive semidefinite and its smallest eigenvalue is $\lambda_0 = 0$ with a corresponding eigenvector $\mathbf{1}$ having all entries equal to 1. The second smallest eigenvalue of $L(K)$ plays the role of λ_1 in the infinite case, so we denote it by $\lambda_1(K)$. It is known that $\lambda_1(K) > 0$ if and only if K is connected. Similarly, let $\rho_1(K)$ be the second largest eigenvalue of $P(K)$, while the largest is always equal to 1 and has $\mathbf{1}$ as an eigenvector. Similarly, the isoperimetric numbers $h(G)$ and $s(G)$ of finite graphs are defined as for infinite graphs, as the minimum of the isoperimetric quotients $|\delta X|/|X|$ and $|\delta X|/S(X)$, respectively, where in the first case X is limited by the condition that $|X| \leq \frac{1}{2}|V(G)|$, and for $s(G)$ we take only all those sets for which $|S(X)| \leq \frac{1}{2}|S(V(G))| = |E(G)|$.

Assume that B is a fixed connected finite graph. If G is a finite covering of B then the Laplacian spectrum $\sigma_L(G)$ contains $\sigma_L(B)$. See e.g. [14] for details. In particular, $\lambda_1(G) \leq \lambda_1(B)$. If G is infinite this is not necessarily true any more. It might happen that $\lambda_1(G) > \lambda_1(B)$. For example, the k-regular tree T_k covers any connected k-regular graph. It is well known that $\lambda_1(T_k) = k - 2\sqrt{k-1}$ but (finite) k-regular graphs may have their second smallest eigenvalue arbitrarily close to zero. Similarly, $h(G) \leq h(B)$ for finite covers G of B, while for G infinite it may happen that $h(G) > h(B)$. It is also possible that $h(G)$ is arbitrarily close to zero even for finite covers G of B. The next results make the situation clear.

Proposition 4.4. *Let $p: G \to B$ be a graph covering, where B is a finite connected graph, and G is an infinite graph. Pick a spanning tree T in B, and let H be the graph obtained from G by contracting each component of $p^{-1}(T)$ to a single point. Then*

$$h(G) \leq \frac{1}{|V(B)|} h(H) \leq 2(|E(B)| - |V(B)| + 1) \cdot h(G). \tag{4.1}$$

Proof. Choose an $\varepsilon > 0$. Let U be a finite subset of $V(H)$ such that $|\delta_H U| \leq (h(H) + \varepsilon)|U|$. For each $u \in V(H)$ let T_u be the set of vertices of G which are contracted to u, and let $\tilde{U} := \bigcup \{T_u \mid u \in U\}$. Then $|\tilde{U}| = |V(B)| \cdot |U|$ and $|\delta_G \tilde{U}| = |\delta_H U|$. Therefore

$$h(G)|\tilde{U}| \leq |\delta_G \tilde{U}| = |\delta_H U| \leq (h(H) + \varepsilon)|U| = \frac{|\tilde{U}|}{|V(B)|}(h(H) + \varepsilon)$$

which implies the first inequality of (4.1) since ε can be arbitrarily small.

Take now a finite $X \subset V(G)$ such that $|\delta_G X| \leq (h(G) + \varepsilon)|X|$. Then let $X_1 \subset V(H)$ be the set of all $u \in V(H)$ for which $T_u \cap X \neq \emptyset$. Clearly, $|X_1| \geq |X|/|V(B)|$. For each edge in $\delta_H X_1$ with a vertex $u \in X_1$ and the other end outside X_1, there is at least one edge in $\delta_G X$ which has one of its ends in T_u. This is clear if $T_u \subseteq X$, and if $T_u \nsubseteq X$ then such an edge lies in T_u. Since $\deg_H(U) \leq 2|E(B) \setminus E(T)|$, it follows that $|\delta_H X_1| \leq 2|\delta_G X| \cdot |E(B) \setminus E(T)|$. Consequently,

$$\begin{aligned} h(H) &\leq \frac{|\delta_H X_1|}{|X_1|} \leq \frac{2|V(B)| \cdot |\delta_G X| \cdot |E(B) \setminus E(T)|}{|X|} \\ &\leq 2(h(G) + \varepsilon)|V(B)| \cdot |E(B) \setminus E(T)|. \end{aligned}$$

which implies the second inequality of (4.1). □

Proposition 4.4 by itself is not a very surprising result, but it turns into a very useful result if we give another interpretation to the graph H. Denote by $\pi_1(.)$ the fundamental group of a graph. Fixing the spanning tree T of B, and a base vertex

$b \in V(B)$, $\pi_1(B)$ is generated by the *fundamental cycles* (with respect to T). These are defined as follows: For each edge $e \in E(B)\setminus E(T)$ there is a unique cycle in $T+e$. Let γ_e be the closed walk in $T+e$ which starts at our chosen base vertex b, leads to the cycle, goes once around it (in one of the possible directions), and returns to b. The corresponding elements in $\pi_1(B)$ are denoted by the same symbol γ_e and called fundamental cycles. Let $S_T := \{\gamma_e \mid e \in E(B)\setminus E(T)\}$.

The fundamental group $\pi_1(G)$ embeds naturally as a subgroup in $\pi_1(B)$. The embedding is just the induced monomorphism $p_*: \pi_1(G) \to \pi_1(B)$. Let $\Gamma := \mathrm{Cay}(\pi_1(B), S_T)$ be the Cayley graph of $\pi_1(B)$ with respect to the fundamental cycles as the generators. It is isomorphic to the infinite $2|S_T|$-regular tree. Let $\Gamma_1 := p_*(\pi_1(G)) \leqslant \pi_1(B)$, and denote by Γ/Γ_1 the quotient graph of Γ with respect to the action of Γ_1 on $V(\Gamma)$. Such quotients of Cayley graphs are called *Schreier coset graphs*. The vertices of Γ/Γ_1 are the right cosets of $\pi_1(B)/\Gamma_1$ and $\Gamma_1\alpha$ and $\Gamma_1\beta$ are joined by an edge if $\Gamma_1\alpha\gamma = \Gamma_1\beta$ (or vice versa) for some $\gamma \in S_T$. Notice that Γ/Γ_1 is finite if and only if Γ_1 has finite index in $\pi_1(B)$. If Γ_1 is a normal subgroup of $\pi_1(B)$ then Γ/Γ_1 is just the Cayley graph of $\pi_1(B)/\Gamma_1$ with respect to S_T.

Corollary 4.5. *Let G be a connected infinite graph and B a connected finite graph. If $p: G \to B$ is a graph covering projection then G is amenable if and only if the quotient graph $\Gamma/\Gamma_1 = \mathrm{Cay}(\pi_1(B), S_T)/p_*(\pi_1(G))$ is amenable.*

Proof. In view of Proposition 4.4 it suffices to show that the quotient graph $G_1 := \Gamma/\Gamma_1$ is isomorphic to the graph H. Notice that $\Delta(G) = \Delta(B) < \infty$, so $s(G) = 0$ is equivalent to $h(G) = 0$.

Let $\pi_1(G) = \pi_1(G, \bar{b})$ be the fundamental group of G with respect to the base vertex $\bar{b} \in p^{-1}(b)$. Take a closed walk W in B with base point b, and let $\tilde{W}$ denote its lift to G such that the initial point of $\tilde{W}$ is $\bar{b}$. The terminal point of $\tilde{W}$, say b_1, is by the homotopy lifting property of covering spaces equal for all walks W' which are homotopic to W in B. The mapping which assigns with each W the corresponding point $b_1 \in p^{-1}(b) \equiv V(H)$ therefore determines a map $\phi: \pi_1(B) \to V(H)$. It is easy to see that under this mapping closed walks W_1, W_2 based at b have the same image if and only if $[W_1][W_2]^{-1} = [W_1 W_2^{-1}] \in \Gamma_1$. Therefore ϕ induces a bijection $\bar{\phi}: \pi_1(B)/\Gamma_1 \to V(H)$. To prove that $\bar{\phi}$ determines a graph isomorphism between G_1 and H it suffices to see that the fundamental cycles $\gamma_e \in S_T$ determine, via $\bar{\phi}$, the edges in H.

The graphs H and G_1 are both $2|S_T|$-regular (loops are counted twice). Choose a vertex $\Gamma_1\alpha \in V(G_1)$ and an edge η in G_1 incident to $\Gamma_1\alpha$, which corresponds to the generator $\gamma_e \in S_T$. The other end of η is $\Gamma_1\alpha\gamma_e$. Let W be a closed walk in B in the homotopy class α, and $\tilde{W}$ its lift to G with initial point $\bar{b}$. If its terminal point is b_1 then $\bar{\phi}(\Gamma_1\alpha) = b_1$. The lift of $W\gamma_e$ to G leads through b_1 and after that it uses the lift of γ_e which has all its edges, except the edge $\bar{e} \in p^{-1}(e)$, in copies of T. It is clear that the ends of $\bar{e}$ are in copies of T labelled $b_1 = \bar{\phi}(\Gamma_1\alpha)$ and

$b_2 = \bar{\phi}(\Gamma_1 \alpha \gamma_e)$. If we extend now $\bar{\phi}$ to the edge map $\bar{\phi}(\eta) := \tilde{e}$, this obviously determines a graph isomorphism $G_1 \to H$. □

Let us notice that most of the arguments in the above proof can easily be deduced from the general theory of covering spaces [12, pp. 161–164].

It should be mentioned that Proposition 4.4 yields more than just the result stated in Corollary 4.5. It gives explicit lower bounds on $h(G)$ in terms of $h(\Gamma/\Gamma_1)$. This is important if we want to have some applications of it. For example, the result of the next section (Theorem 5.1) show that for certain groups all their quotients have the isoperimetric number uniformly bounded away from zero. Using these to play the role of $\mathrm{Cay}(\pi_1(B), S_T)/\Gamma_1$, this might give a method to produce many covers G over B with the isoperimetric number bounded below by a positive constant. We shall leave the detailed discussion of this phenomenon for further works. Cf. also [2].

5. The Kazhdan property (T)

Group representations are used not only to make abstract groups 'visible' but also to make analytic methods accessible to do the analysis on groups. Let Γ be a group. If H is a Hilbert space and $\rho : \Gamma \to \mathrm{Aut}(H)$ is a homomorphism (where $\mathrm{Aut}(H)$ are all invertible linear mappings $H \to H$), then the pair (H, ρ) is called a *representation* of Γ. A *trivial* representation sends each group element $\gamma \in \Gamma$ to the identity of $\mathrm{Aut}(H)$. The representation is *unitary* if $\rho(\gamma)$ is a unitary transformation for each $\gamma \in \Gamma$. It is *irreducible* if $\rho(\Gamma)$ has no nontrivial invariant subspaces.

A locally compact group Γ has the *Kazhdan property* (T) (or is said to be a *Kazhdan group*) if there exist an $\varepsilon > 0$ and a compact subset K of Γ such that for every nontrivial irreducible unitary representation (h, ρ) of Γ and every $v \in H \setminus \{0\}$, $\|\rho(k)v - v\| > \varepsilon \|v\|$ for some $k \in K$. To be more specific we also say that Γ is (K, ε)-*Kazhdan*, or *Kazhdan for* K and ε.

An equivalent definition of property (T) is: Γ is Kazhdan if and only if there are $\varepsilon > 0$ and a compact $K \subseteq \Gamma$ such that for every unitary representation (H, ρ) of Γ which has some (K, ε)-invariant vectors ($\exists v \in H$: $\|\rho(k)v - v\| < \varepsilon \|v\|$, $\forall k \in K$) has in fact some nonzero Γ-invariant vectors ($\exists v \in H$, $v \neq 0$ and $\rho(g)v = v$ for every $g \in \Gamma$).

The property (T) was introduced by Kazhdan in [10] as a tool for studying discrete subgroups of Lie groups of finite co-volume. Examples of infinite Kazhdan groups are all semi-simple Lie groups with all factors having $\mathbb{R}$-rank $\geqslant 2$, and all their cofinite-volume discrete subgroups, e.g. $\mathrm{SL}(n, \mathbb{Z})$, $n \geqslant 3$, but not $\mathrm{SL}(2, \mathbb{Z})$. Let us mention also that free groups are not Kazhdan groups.

We restrict ourselves to finitely generated (discrete) groups. (It can also be shown that if a countable discrete group has property (T), then it is finitely generated.)

(i) A finite group is Kazhdan. To see this, let (H, ρ) be a unitary representation such that there exists a $v \in H$, $\|v\| = 1$, and $\|\rho(k)v - v\| < 1$ for each $k \in K := \Gamma$. Let

$$\tilde{v} := \frac{1}{|\Gamma|} \sum_{k \in \Gamma} \rho(k)v.$$

Then

$$\|\tilde{v} - v\| = \frac{1}{|\Gamma|} \left\| \sum_{k \in \Gamma} (\rho(k)v - v) \right\| < 1,$$

so $\tilde{v} \neq 0$. But

$$\rho(g)\tilde{v} = \frac{1}{|\Gamma|} \sum_{k \in \Gamma} \rho(g)\rho(k)v = \frac{1}{|\Gamma|} \sum_{k \in \Gamma} \rho(gk)v = \tilde{v},$$

for every $g \in \Gamma$. So Γ is Kazhdan (for $K = \Gamma$ and $\varepsilon = 1$).

(ii) Let Γ have property (T) for ε, K. If $K' \supseteq K$ is finite (compact) and $0 < \varepsilon' \leq \varepsilon$ then Γ has property (T) for ε', K'.

(iii) A countable discrete group with property (T) is finitely generated. Therefore we may assume by (ii) that K generates Γ.

(iv) If Γ has property (T) for ε, K, where K generates Γ, then Γ has property (T) for ε', K' where K' is any other generating set and $\varepsilon' = \varepsilon'(K') > 0$ some constant depending on K', K and ε. To prove this, express each $k \in K$ as a word in the alphabet $K' \cup (K')^{-1}$ and denote by $l(k)$ its length. Let $L = \max\{l(k) \mid k \in K\}$. Let (H, ρ) be a unitary representation of Γ and assume that there exists a vector $v \in H$, $\|v\| = 1$, such that $\|\rho(k')v - v\| < \varepsilon/L$ for all $k' \in K'$. Take now an arbitrary $k \in K$ and let $k = k'_n k'_{n-1} \cdots k'_1$ where $k'_i \in K' \cup (K')^{-1}$ and $n \leq L$. It will be assumed for the sake of easier notation that $K' = (K')^{-1}$. This may be done since the representation is unitary and hence $\|\rho(k)v - v\| = \|\rho(k^{-1})v - v\|$. We will use induction on n to show that $\|\rho(k)v - v\| = \|\rho(k'_n \cdots k'_1)v - v\| < (\varepsilon/L)n$. This is true by our assumption if $n = 1$. Otherwise

$$\begin{aligned} \|\rho(k'_n \cdots k'_1)v - v\| &= \|\rho(k'_{n-1} \cdots k'_1)v - \rho(k'^{-1}_n)v\| \\ &\leq \|\rho(k'_{n-1} \cdots k'_1)v - v\| + \|\rho(k'^{-1}_n)v - v\| \\ &\leq \frac{\varepsilon}{L}(n-1) + \frac{\varepsilon}{L} = \frac{\varepsilon n}{L}. \end{aligned}$$

It follows that any (K', ε')-invariant vector v is (K, ε)-invariant, and hence Γ has property (T) with respect to K', ε' where $\varepsilon' := \varepsilon/L$.

One of the fundamental properties of Kazhdan groups determines that such groups are something opposite to amenable groups, in a very strong sense. Let us recall that λ_1, ρ_1, and the isoperimetric numbers h and s for finite graphs were introduced in Section 4.

Theorem 5.1. *Let Γ be an infinite group generated by a finite set S, and let $G = \mathrm{Cay}(\Gamma, S)$ be its Cayley graph with respect to S. If Γ is (S, ε)-Kazhdan and Γ_1 is a proper subgroup of Γ then the quotient graph G/Γ_1 has*

$$\rho_1(G/\Gamma_1) \leq 1 - \frac{\varepsilon^2}{2k}, \quad \text{and} \quad \lambda_1(G/\Gamma_1) \geq \frac{1}{2}\varepsilon^2, \tag{5.1}$$

where $k := |S \cup S^{-1}|$. If no generator in S has order 2 then (5.1) can be improved to $\rho_1(G/\Gamma_1) \leq 1 - \varepsilon^2/k$ and $\lambda_1(G/\Gamma_1) \geq \varepsilon^2$.

Proof. Denote by $G_1 := G/\Gamma_1$, and assume first that G_1 is infinite. Let ρ be the right regular representation of Γ in $H := \ell^2(\Gamma/\Gamma_1) = \ell^2(V(G_1))$, i.e., for $\gamma \in \Gamma$, $\rho(\gamma)$ is a permutation matrix indexed by $V(G_1) = \Gamma/\Gamma_1$ and having entries

$$\rho(\gamma)_{\Gamma_1\alpha, \Gamma_1\beta} = \begin{cases} 1 & \text{if } \Gamma_1\alpha\gamma = \Gamma_1\beta, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that (H, ρ) is a unitary representation of Γ. If a vector $x \in H$ is fixed by each $\rho(\gamma)$, $\gamma \in \Gamma$, it must have all entries equal since Γ is transitive on $V(G_1)$, so $x = 0$. Since Γ is (S, ε)-Kazhdan we conclude that for each $x \neq 0$ there is some $s \in S$ such that

$$\|\rho(s)x - x\| \geq \varepsilon \, \|x\|. \tag{5.2}$$

The graph G_1 is regular of valency k. Its Laplacian matrix is easily seen to be equal to

$$L = L(G_1) = \sum_{t \in S \cup S^{-1}} (I - \rho(t)). \tag{5.3}$$

Let $x \in \ell^2(V(G_1))$ be a unit vector and s a corresponding element from S such that (5.2) holds. Since $\rho(s)$ is unitary, it follows by using (5.2) that

$$\begin{aligned} &\langle x - \rho(s)x, x\rangle + \langle x - \rho(s^{-1})x, x\rangle \\ &\quad = \langle x - \rho(s)x, x\rangle + \langle \rho(s)x - x, \rho(s)x\rangle \\ &\quad = \|x - \rho(s)x\|^2 \geq \varepsilon^2. \end{aligned} \tag{5.4}$$

Consequently, by (5.3)

$$\begin{aligned} \langle Lx, x\rangle &= \sum_{t \in S \cup S^{-1}} \langle (I - \rho(t))x, x\rangle \\ &\geq \tfrac{1}{2}(\langle x - \rho(s)x, x\rangle + \langle x - \rho(s^{-1})x, x\rangle) \geq \tfrac{1}{2}\varepsilon^2. \end{aligned} \tag{5.5}$$

Note that we need the factor $\frac{1}{2}$ to cover the case $s = s^{-1}$. Now (3.1) says that $\lambda_1(G_1) \geq \varepsilon^2/2$, and $\rho_1(G_1) = (k - \lambda_1(G_1))/k \leq 1 - \varepsilon^2/(2k)$.

If G_1 is finite we undertake a similar way. Matrices $\rho(\gamma)$ are the same as before but they are assumed to act as automorphisms of the subspace H^0 of $\ell^2(V(G_1))$, where

$$H^0 = \{x \in \ell^2(V(G_1)) \mid x \perp \mathbf{1}\},$$

i.e., the orthogonal complement of the eigensubspace of $\lambda_0 = 0$ as the eigenvalue of $L(G_1)$. Then (H^0, ρ) is a unitary representation. Relations (5.2)–(5.5) also hold if $x \in H^0$, $\|x\| = 1$, and we are done by the fact that $\lambda_1(G_1) = \inf\{\langle Lx, x\rangle \mid x \in H^0, \|x\| = 1\}$. □

Let us mention an important example of a non-Kazhdan group which has a similar property as stated in Theorem 5.1 for Kazhdan groups. The modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ is a quotient of a non-Kahdan group $\mathrm{SL}(2, \mathbb{Z})$, and thus it is not Kazhdan. Its congruence subgroups

$$\Gamma_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) \mid a, d \equiv 1 \pmod n,\ b, c \equiv 0 \pmod n \right\}$$

have the property that for each $n \geq 2$, $\lambda_1(\mathrm{Cay}(\Gamma, S)/\Gamma_n) \geq \varepsilon > 0$ for some explicitly known ε, which follows from the Selberg's Theorem [19], cf. also [11]. This is an advantage in various applications over the examples of Kazhdan groups for which no explicit values of ε are known.

Corollary 5.2. *If $G = \mathrm{Cay}(\Gamma, S)$ is the Cayley graph of a Kazhdan group then G has exponential growth, and so do all the quotients G/Γ_1 where Γ_1 is any subgroup of Γ of infinite index. Moreover, $\tau(G/\Gamma_1)$ is uniformly bounded away from* 1.

Proof. The Kazhdan group Γ is (S, ε)-Kazhdan for some $\varepsilon > 0$. By the preceding theorem, $\lambda_1(G/\Gamma_1) \geq \frac{1}{2}\varepsilon^2$, thus by Theorem 3.1, $h(G/\Gamma_1) \geq \frac{1}{2}\varepsilon^2$. Theorem 2.4 now completes the proof. □

It seems that the property of G stated in Theorem 5.1 is characteristic for Cayley graphs of Kazhdan groups. We use it to define what a graph with Kazhdan property (T) will be.

Let G be a graph and $\Gamma \leq \mathrm{Aut}(G)$. The pair (G, Γ) is said to have *Kazhdan property* (T) if there exists an $\varepsilon > 0$ such that for every subgroup Γ_1 of Γ, which is not transitive on $V(G)$, the graph G/Γ_1 has $s(G/\Gamma_1) \geq \varepsilon$. Here G/Γ_1 is the *directed* graph whose vertices are the orbits $\Gamma_1 v$, $v \in V(G)$, of the action of Γ_1 on $V(G)$, and in the direction from the orbit $\Gamma_1 v$ to $\Gamma_1 u$ there are as many directed edges as the number of neighbours of the vertex v in $\Gamma_1 u$. Although $\lambda_1(G_1)$ or $\rho_1(G_1)$ make sense for directed quotients $G_1 := G/\Gamma_1$ of undirected graphs (but this is not so obvious), we use in our definition the requirement that $s(G_1)$ is uniformly bounded away from zero. By the results of Section 3 this is equivalent to the condition that $\rho_1(G_1)$ is uniformly bounded away from 1. It should be mentioned that for a directed graph G_1, the transition isoperimetric number $s(G_1)$ is defined in the same way as for undirected graphs. However, $|\delta X|$ counts the number of edges with the initial point in X and the terminal point outside X, and $S(X)$ is the sum of the out-degrees of vertices in X. We must require Γ_1 not to be transitive on $V(G)$ to exclude the quotients with one vertex, for which the isoperimetric number is undefined.

If Γ is a Kazhdan group for (S, ε) where S generates Γ, then the pair $(\mathrm{Cay}(\Gamma, S), \Gamma)$ has the Kazhdan property (T).

Note that the quotient graphs G/Γ_1 are regular quotients, i.e. the natural projection $p: G \to G/\Gamma_1$, $p(v) = \Gamma_1 v$, is a regular covering projection. This means that p is a covering projection (i.e. a local isomorphism between directed graphs), and it is *regular* (the group of covering transformations, which contains Γ_1, is transitive on each fibre $p^{-1}(\Gamma_1 v) = \Gamma_1 v \subset V(G)$).

A graph G is said to be *ε-Kazhdan* if for every regular covering $p: G \to G_1$ where G_1 is a directed graph with at least two vertices, $s(G_1) \geqslant \varepsilon$. Let us stop with some open problems.

(1) Suppose that we know that the pair (G, Γ) is Kazhdan. Give any results to estimate the corresponding ε from below. Notice that upper bounds on ε can be given in terms of $\lambda_1(G)$, or $\tau(G)$. The importance of this question lies in the fact that there are several known families of Kazhdan groups but no explicit estimates on ε are known.

(2) How can one 'see' the Kazhdan property? Infinite Kazhdan groups (and their quotients) have exponential growth. But the converse is far from being true. What other geometric properties are shared by Kazhdan pairs (G, Γ)?

(3) It would be extremely important to get nontrivial examples of Kazhdan graphs. At the moment it seems that any infinite Kazhdan graph with rich automorphism group should be classified as a 'nontrivial' example.

6. Almost symmetric graphs and ends of graphs

In this section we present a short proof of a result relating the amenability and the ends of graphs (Proposition 6.2). Such a result does not hold without further restrictions on graphs since there are easily constructed examples of amenable, or non-amenable graphs (even trees) having arbitrary space of ends. Let us mention that Proposition 6.1 below is somehow implicit already in the classical work of Stallings [21] (for Cayley graphs), but we have not found it stated in the available literature. It was proved by Soardi and Woess [20] that a locally finite vertex transitive graph G with infinitely many ends has $h(G) > 0$. Our Proposition 6.2 is an obvious generalization of this result to 'almost symmetric' graphs.

A locally finite graph G is said to be *almost (vertex) symmetric* if there is a constant $D < \infty$ and a vertex $v \in V(G)$ such that for each $u \in V(G)$ there is an automorphism $\sigma \in \mathrm{Aut}(G)$ such that $\mathrm{dist}(u, \sigma(v)) \leqslant D$. For a connected graph G this is equivalent to saying that $\mathrm{Aut}(G)$ has only finitely many orbits on $V(G)$. It is clear that an almost symmetric graph G has $\Delta(G) = \Delta'(G) < \infty$. Similarly, one can prove that $h(G) = h'(G)$, $\lambda_1(G) = \lambda_1'(G)$, and $\rho_1(G) = \rho_1'(G)$.

For an arbitrary graph G one may define the *space of ends* of G, $\mathrm{Ends}(G)$, as follows: Let $\Omega(G)$ be the set of all one-way-infinite paths in G. Call two such paths A and B *equivalent*, $A \sim B$, if for each finite subgraph C of G, the infinite parts of $A\setminus C$ and $B\setminus C$ belong to the same connected component of $G\setminus C$. I.e., no finite subgraph can separate A and B. The equivalence classes $\Omega(G)/\sim$ are the *ends* of G, denoted by $\mathrm{Ends}(G)$. There is a natural topology in $\mathrm{Ends}(G)$. Pick a vertex in each of the components of G, and let C_n ($n \geq 0$) be the induced subgraph of G containing all those vertices which are at distance at most n from some of the chosen vertices. For an end $\omega \in \mathrm{Ends}(G)$, let U_n be the set of all those ends which have representative paths in the same component of $G\setminus C_n$ as ω. The sets U_n define then a decreasing sequence of basis neighbourhoods of ω. With this topology $\mathrm{Ends}(G)$ becomes a compact metrizable space. It can be shown that for a connected graph G the space $\mathrm{Ends}(G)$ is homeomorphic to a closed subset of the Cantor set, and conversely—for an arbitrary closed subset A of the Cantor set there are graphs with $\mathrm{Ends}(G)$ homeomorphic to A.

If $\omega_1, \omega_2, \ldots, \omega_k$ are distinct ends of G, then, since the Cantor set is totally disconnected, there is a finite set $C \subset V(G)$ such that $G\setminus C$ pairwise separates the ends ω_i. In particular, $G\setminus C$ has at least k infinite components. Let us denote the number of infinite components of $G\setminus C$ by $c_\infty(C)$. Notice that if $C' \supseteq C$ then $c_\infty(C') \geq c_\infty(C)$.

Proposition 6.1. *Let G be a locally finite connected almost vertex symmetric graph. Then* $\mathrm{Ends}(G)$ *contains either* 0, 1, 2, *or infinitely many ends. In the latter case,* $\mathrm{Ends}(G)$ *is homeomorphic to the Cantor set, and is thus uncountable.*

Proof. Assume that G has more than two ends. Fix $v \in V(G)$, and let $B_n := B_n(v)$ be the ball of radius n centered at v. Let D be the constant corresponding to G from the definition of almost symmetric graphs. For some $N > D$, $c_\infty(B_{N-D}) \geq 3$ since G has more than two ends. If u is a vertex at distance $2N - D + 1$ from v and lies in one of the infinite components of $G\setminus B_{N-D}$ then there is a vertex u' at distance d from v, $2N - 2D < d \leq 2N + 1$, and an automorphism σ of G such that $\sigma(v) = u'$. Clearly,

$$c_\infty(B_{N-D}(u')) = c_\infty(B_{N-D}(v)) \geq 3.$$

One of the components of $G\setminus B_{N-D}(u')$ contains $B_{N-D}(v)$. Therefore

$$c_\infty(B_{N-D}(u') \cup B_{N-D}(v)) \geq 4.$$

We repeat this for some u in each of the infinite components, and it follows that $c_\infty(B_{3N-D+1}) \geq 6$. By the same method we see that $c_\infty(B_{5N-D+2}) \geq 12$, etc. In general,

$$c_\infty(B_{(2t+1)N-D+t}) \geq 3 \cdot 2^t.$$

Thus G has infinitely many ends. Moreover, it follows that the space of ends of G is homeomorphic to the ends space of some locally finite infinite tree without vertices of degree two, and this is known to be homeomorphic to the Cantor set. □.

Proposition 6.2. *Let G be a locally finite, connected almost vertex symmetric graph.*

(a) *If G has infinitely many ends then $h(G) > 0$.*

(b) *If G has exactly two ends then $h(G) = 0$.*

Proof. (a) Let $\beta_n := \max\{|B_n(v)| \mid v \in V(G)\}$, and let N, D be as in the above proof. Let X be an arbitrary finite subset of $V(G)$. If $|X| < 2\beta_N$ then

$$\frac{|\delta X|}{|X|} \geq \frac{1}{4\beta_{2N}}.$$

Otherwise let $X_1 := \{u \in X \mid B_N(u) \nsubseteq X\}$ and $X_2 := \{u \in X \mid B_N(u) \subseteq X\}$. If $|X_1| \geq |X|/2$ then we estimate $|\delta X|$ as follows: For each $u \in X_1$ there is an edge $e \in \delta X$ at distance at most N form u. Any edge $e \in \delta X$ corresponds in this way to at most β_N distinct vertices of X_1. Therefore

$$\frac{|\delta X|}{|X|} \geq \frac{|X_1|}{\beta_N |X|} \geq \frac{1}{2\beta_N}.$$

In the remaining case $|X_2| \geq |X|/2$. In this case we first make sure that X_2 contains at least $t := \lfloor |X_2|/\beta_{2N} \rfloor$ disjoint N-balls, i.e. there are vertices $u_1, u_2, \ldots, u_t$ in X_2 which are pairwise at distance at least $2N + 1$. Next we see, by the same method as in the proof of Proposition 6.1, that $c_\infty(\bigcup_{i=1}^t B_N(u_i)) \geq t$. Consequently, $c_\infty(X) \geq c_\infty(X_2) \geq t$, and so

$$|\delta X| \geq t \geq \left\lfloor \frac{|X_2|}{\beta_{2N}} \right\rfloor \geq \left\lfloor \frac{|X|}{2\beta_{2N}} \right\rfloor \geq \frac{|X|}{4\beta_{2N}}.$$

In the last inequality we use the assumption that $|X| \geq 4\beta_{2N}$. It follows that in each case $|\delta X|/|X|$ is bounded away from 0, so $h(G) > 0$.

(b) Let U be a finite subset of $V(G)$ such that $c_\infty(U) = 2$. If σ is an automorphism of G which sends a vertex $u \in U$ (arbitrarily) far away from U, then a finite component of $G \backslash (U \cup \sigma(U))$ contains (arbitrary) many vertices but only constantly many edges on its boundary. Thus obviously $h(G) = 0$. □

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