Obstructions for the disk and the cylinder embedding extension problems*

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Abstract

Let S be a closed surface with boundary ∂S and let G be a graph. Let $K \subseteq G$ be a subgraph embedded in S such that $\partial S \subseteq K$. An *embedding extension* of K to G is an embedding of G in S which coincides on K with the given embedding of K. Minimal obstructions for the existence of embedding extensions are classified in cases when S is the disk or the cylinder. Linear time algorithms are presented that either find an embedding extension, or return an obstruction to the existence of extensions. These results are to be used as the basic building stones in the design of linear time algorithms for the embeddability of graphs in an arbitrary surface and for solving more general embedding extension problems.

1 Introduction

Let K be a subgraph of G. A K*–component* or a K*–bridge* in G is a subgraph of G which is either an edge $e \in E(G)\backslash E(K)$ (together with its endpoints) which has both endpoints in K, or it is a connected component of $G - V(K)$ together with all edges (and their endpoints) between this component and K. Each edge of a K-component R having an endpoint in K is a *foot* of R. The vertices of $R \cap K$ are the *vertices of attachment* of R. A vertex of K of degree in K different from 2 is a *main vertex* of K. For convenience, if a connected component of K is a cycle, then we choose an arbitrary vertex of it and declare it to be a main vertex of K as well. A *branch* of K is any path in K whose endpoints are main vertices and such that no internal vertex on this path is a main vertex. If a K-component is attached at a single branch of K, it is said to be *local*. The number of branches of K is called the *branch size* of K.

Let $K \subseteq G$, and suppose that we are given an embedding of K into a (closed) surface Σ. The *embedding extension problem* asks whether it is possible to extend the given embedding

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of K to an embedding of G, and any such embedding is said to be an *embedding extension* of K to G. Let Σ be the (closed) disk or the cylinder. Let K be embedded in Σ such that $\partial\Sigma \subseteq K$. An *obstruction* for embedding extensions is a subgraph Ω of $G - E(K)$ such that the embedding of K cannot be extended to $K \cup \Omega$. The obstruction is *small* if $K \cup \Omega$ has bounded branch size. If Ω is small, then it is easy to verify that no embedding extension to $K \cup \Omega$ exists, and hence Ω is a good verifier that there are no embedding extensions of K to G as well. In this paper, minimal obstructions for embedding extension problems in the disk and the cylinder are classified for several "canonical" choices of K. Although much work has been done on "embedding obstructions", our results seem to be new, apart from the case of the disk (cf. [20]; see also Section 3) or the case when $K = \emptyset$ and Σ is a closed surface [19]. It is interesting that minimal obstructions are not always small. They can be arbitrarily large but their structure is easily described. We also present linear time algorithms to either find an embedding extension, or return a (minimal) obstruction to the existence of extensions.

The basic results of this paper (Theorems 3.1, 4.3, 5.3, and 6.2) are to be used as the basic building stones in the design of linear time algorithms for embedding graphs in general surfaces [10, 12, 16, 11]. Moreover, we are able to solve even more general embedding extension problems in linear time.

Robertson and Seymour (cf. [19] and the graph minors papers preceding it) proved a Kuratowski theorem for general surfaces. In our further project [17], results of this paper are used to obtain a reasonably short proof of Robertson and Seymour's result. It is worth mentioning that all our results are direct and constructive, in the tradition of Archdeacon and Huneke [1]. (Recently, also Seymour [23] obtained a constructive proof by using graph minors and tree-width techniques.)

Embeddings in orientable surfaces can be described combinatorially [6] by specifying a *rotation system*: for each vertex v of the graph G we have the cyclic permutation π_v of its neighbors, representing their circular order around v on the surface. In order to make a clear presentation of our algorithm, we have decided to use this description only implicitly. Whenever we say that we have an embedding, we mean such a combinatorial description. Whenever used, it is easy to see how one can combine the embeddings of some parts of the graph described this way into the embedding of larger species.

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for basic operations. This model was introduced by Cook and Reckhow [4]. More precisely, our model is the *unit-cost* RAM where operations on integers, whose values are $O(n)$, need only constant time (n is the order of the given graph).

2 Basic definitions

Let G and K be graphs (both subgraphs of some graph H). Then we denote by $G - K$ the graph obtained from G by deleting all vertices of $G \cap K$ and all their incident edges. If $F \subseteq E(G)$, then $G - F$ denotes the graph obtained from G by deleting all edges in F. If K and L are subgraphs of G, then we say that a path P in G *joins* K and L if P is internally disjoint from $K \cup L$ and one of its ends is in K and the other end is in L. Moreover, if an end of P is in both K and L, then P is a trivial path.

A *block* or *2-connected component* of a graph G is either an isolated vertex, a loop, a bond of G, or a maximal 2-connected subgraph of G. One can also define the concept of 3-connected components. A graph G is said to be *k-separable* if it can be written as a union $G = H \cup K$ of (non-empty) edge-disjoint graphs H and K which have exactly k vertices in common, and each of them contains at least k edges. Such a pair $\{H, K\}$ is called a *k-separation* of G. A graph is *nonseparable* if it has no 0- or 1-separations. Let G be a nonseparable graph and let $\{H, K\}$ be a 2-separation of G. Let x, y denote the vertices of $V(H) \cap V(K)$. The 2-separation is *elementary* if either $H - \{x, y\}$ or $K - \{x, y\}$ is non-empty and connected, and either H or K is nonseparable. It turns out [26] that nonseparable graphs without elementary 2-separations are either 3-connected graphs, cycles C_n $(n \geq 3)$, $p \geq 1$ parallel edges, K_1, or a loop. Assume now that the 2-separation $\{H, K\}$ of G is elementary. Denote by H' and K' the graphs obtained from H and K, respectively, by adding to each of them a new edge between the vertices of $H \cap K$. The added edges are called *virtual edges*. It is easy to verify that H' and K' are both nonseparable, and we may repeat the process on their elementary 2-separations (if there are any) until no further elementary 2-separations are possible. As mentioned above, the obtained graphs are either 3-connected, cycles, edges in parallel, or rather small. Each of the graphs obtained this way is called a *3-connected component* of G. It was shown by MacLane [14] (cf. also [26]) that the set of 3-connected components of the graph is uniquely determined although different choices of the 2-separations may have been used during the process of constructing them. Every 3-connected component consists of several edges of G and several virtual edges. It is obvious by construction that each edge of G belongs to exactly one 3-connected component, and each virtual edge has a corresponding virtual edge in some other 3-connected component. The 3-connected components of G may be viewed as subgraphs of G, where each virtual edge corresponds to a path in G. These subgraphs are positioned in G in a tree-like way [26]. We also speak of *3-connected components* when the graph is separable. In that case we define them to be the 3-connected components of the blocks of the graph.

A linear time algorithm for obtaining the 3-connected components of a graph was devised by Hopcroft and Tarjan [7].

There are very efficient (linear time) algorithms which for a given graph determine whether the graph is planar or not. The first such algorithm was obtained by Hopcroft and Tarjan [8] back in 1974. There are several other linear time planarity algorithms (Booth and Lueker [2], Fraysseix and Rosenstiehl [5], Williamson [27, 28]). Extensions of the original algorithms produce also an embedding (rotation system) whenever the given graph is found to be planar [3], or find a small obstruction — a subgraph homeomorphic to K_5 or $K_{3,3}$ — if the graph is non-planar [27, 28] (see also [13]). The subgraph homeomorphic to K_5 or $K_{3,3}$ is called a *Kuratowski subgraph* of G.

Lemma 2.1 *There is a linear time algorithm that, given a graph G, either exhibits an embedding of G in the plane, or finds a Kuratowski subgraph of G.*

We will refer to the algorithm mentioned in Lemma 2.1 as *testing for planarity*. This procedure not only checks the planarity of the given graph but also takes care of exhibiting an embedding, or finding a Kuratowski subgraph.

Let C be a cycle of a graph G. Two C-components B_1 and B_2 *overlap* if either B_1 and B_2 have three vertices of attachment in common, or there are four distinct vertices a, b, c, d which appear in this order on C and such that a and c are vertices of attachment of B_1, and b, d are vertices of attachment of B_2. In the latter case, B_1 and B_2 contain disjoint paths P_1

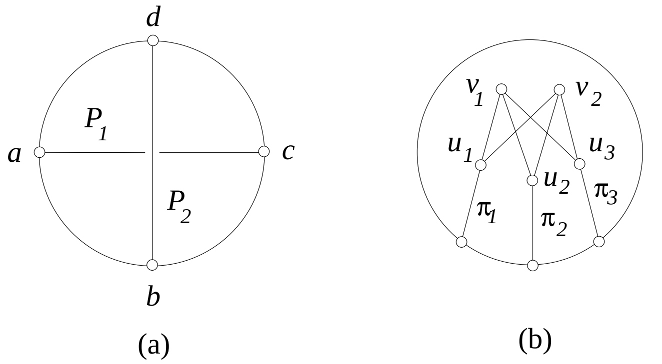

Figure 1: Disjoint crossing paths and a tripod

and P_2 whose ends a, c and b, d, respectively, interlace on C. Such paths will be referred to as *disjoint crossing paths*. See Figure 1(a). We will need another type of subgraphs of G that are attached to C. A *tripod* is a subgraph T of G that consists of two main vertices v_1, v_2 of degree 3, whose branches join them with the same triple of vertices u_1, u_2, u_3, together with three vertex disjoint paths π_1, π_2, π_3 joining u_1, u_2, and u_3 with C. Moreover, T intersects C only at the ends of π_1, π_2, and π_3. One or more of the paths π_i are allowed to be trivial, in which case $u_i \in C$. See Figure 1(b). If all three paths π_1, π_2, and π_3 are trivial (just vertices), then the tripod is said to be *degenerate*. We use the same name for attachments of the tripod in the case when the corresponding path is trivial.

3 The disk

Let D be the closed unit disk in the euclidean plane. Given a graph G and a cycle C in G, we would like to find an embedding of G in D so that C is embedded on ∂D. Of course, this is a case of the embedding extension problem for which an easy answer is at hand. First, we construct the *auxiliary graph* $\tilde{G} = \mathrm{Aux}(G, C)$ which is obtained from G by adding a new vertex v (called the *auxiliary vertex*) and joining it to all vertices on C. It is easy to see that an embedding extension of C on ∂D to G exists if and only if the auxiliary graph $\tilde{G}$ is planar. Its plane embedding also determines an embedding extension. In case of the non-planarity of $\tilde{G}$, a Kuratowski subgraph $\tilde{K}$ of $\tilde{G}$ determines the subgraph $K = \tilde{K} - v$ of G which is an obstruction for the embedding extension in the disk. Although $K \cup C$ can have arbitrarily large branch size, it can easily be modified to an obstruction Ω for which $\Omega \cup C$ has bounded branch size. Our answer seems to solve the question reasonably well. However, there is a better solution. Namely, it is known that in case when G is 3-connected, a pair (G, C) for which there is no disk embedding extension necessarily contains either a pair of disjoint crossing paths or a tripod. This simple but useful result was "in the air" for quite some time. It seems to have appeared for the first time in a paper by Jung [9] in a slightly weaker version. It also appeared in a paper by Seymour [21] (with the complete proof in [22]), Shiloach [24], Thomassen [25], all in relation to the non-existence of two disjoint paths between specified vertices. This result recently appeared in a more explicit form in Robertson and Seymour's work on graph minors [20]. In this section we will prove a slightly more specific result by also taking care of the case when G is not 3-connected. Moreover, we will show how to obtain such an obstruction in linear time.

Theorem 3.1 *Let G, C, D be as above. Let $\tilde{G} = \mathrm{Aux}(G, C)$. There is a linear time algorithm that either finds an embedding of G in D with C on ∂D, or returns a small obstruction Ω. In the latter case, Ω is one of the following types of subgraphs of $G - E(C)$:*

(a) a pair of disjoint crossing paths,

(b) a tripod, or

(c) a Kuratowski subgraph contained in a 3-connected component of $\tilde{G}$ distinct from the 3-connected component of $\tilde{G}$ containing C.

Before giving the proof of Theorem 3.1 we state a lemma whose easy proof is left to the reader.

Lemma 3.2 *Let H be a graph with a cycle C and let e be an edge of H which is not a chord of C. If the edge-contracted graph H/e contains a tripod or a pair of disjoint crossing paths with respect to C (or C/e if $e \in E(C)$), then H also contains a tripod or a pair of disjoint crossing paths.*

Proof. (of Theorem 3.1). By testing $\tilde{G}$ for planarity we can check if $\tilde{G}$ is planar. If yes, then we also get a required disk embedding of G.

Suppose now that $\tilde{G}$ is non-planar. Determine the 3-connected components of $\tilde{G}$, for example by using the linear time algorithm of Hopcroft and Tarjan [7]. Note that C (with some of its edges which possibly became virtual) and the auxiliary vertex are in the same 3-connected component. Denote this 3-connected component by R. If R is planar, then the obstruction to the planarity of $\tilde{Q}$ lies in one of the other 3-connected components. We get (c) in one of the planarity tests. From now on we may thus assume that R is non-planar. Let us show how to get disjoint crossing paths or a tripod.

Let K be a Kuratowski subgraph of R found by a planarity test on R. Denote by H the graph $(K \cup C) - w$ where w is the auxiliary vertex of $\tilde{G}$. Note that the branch size of H is not necessarily small. We will first try to find disjoint crossing paths or a tripod in H. Consider the C-components in H. First of all we check if two of them overlap. In order to perform efficient checking, we split the bridges into two classes: the bridges that contain main vertices of K (possibly as their vertices of attachment) are *main bridges* of C in H, and the remaining bridges are called *chords* since they are just paths joining two distinct vertices of C. There are at most 20 main bridges. To be more efficient in the sequel, we can temporarily replace every bridge by a single vertex joined to all of its vertices of attachment on C.

Step 1: *Are there two main bridges that overlap?* If yes, we either have a (degenerate) tripod or disjoint crossing paths. If not, proceed with the next step. Since only the main bridges have to be considered, this step can be carried over in constant time.

Step 2: *Is there a main bridge that overlaps with a chord?* If yes, we have disjoint crossing paths. Otherwise continue with Step 3. This question can easily be answered in linear time. Observe that the number of candidates for one of the disjoint crossing paths in the main bridges is small.

Step 3: *Are there two overlapping chords?* Note that no chord contains a main vertex. Thus, at most two chords are attached at the same vertex. A simple way to find overlapping chords is to start building a stack by traversing C once around starting at an arbitrary vertex of C. If there are two chords at the same point on C, we first consider chords that have already been met during the traversal. If both chords are not new, we give priority to the one that is on the top of the stack (if none is on the top, their order is not important). Then we process the new chords. If both are new, we give priority to the one whose other attachment is further away in the direction of the traversal. Every new chord met during the traversal is put to the top of the stack. Meeting the chord for the second time, we check if the top element in the stack is the same chord. If yes, the chord is removed from the stack and the traversal is continued. If not, then we have another chord at the top. It is easy to see that these two chords overlap and they give rise to disjoint crossing paths.

We may now assume that no two distinct bridges of C in H overlap. This means that the obstruction is in one of the main bridges. Such a bridge B can be discovered in $O(1)$ time since the number of main bridges is small, and each of them has small branch size. We will also assume that B is minimal in the sense that for every branch e of B, the graph $(B-e)\cup C$ is planar (if not, we can remove e and repeat the above procedure in order to get (a), (b), or a new bridge B with smaller number of branches). Note that $B \cup C$ is non-planar and has small branch size. Therefore B can be used as a legitimate obstruction in some applications. However, our goal is to show more: we want a tripod or disjoint crossing paths.

Since $B \cup C$ has constant branch size, it is easy to find a tripod or disjoint crossing paths in B whenever B contains one of them. Assume from now on that this is not the case. We will prove that under this assumption, B has at most two vertices of attachment on C. Let K' be a Kuratowski subgraph of $B \cup C$. By the minimality property of B, K' contains the whole B plus possibly some parts on C. If two main vertices of K' lie on C, then they are either non-adjacent in K', or connected by a branch which is contained in C. Therefore it is easy to see that at most three main vertices of K' lie on C (the case of four vertices of $K_{3,3}$ forming a cycle on C is the only possibility, but they give rise to disjoint crossing paths). Similarly, we can exclude three main vertices of K' being on C. (In the case analysis for the last claim, an application of Lemma 3.2, using the "contraction" argument as also used below, makes the number of cases much smaller.)

Now, if B has a vertex of attachment on C that is not a main vertex of K', we may contract the corresponding branch e of $B \cup C$ and obtain the non-planar graph $(B/e)\cup C = (B\cup C)/e$ with more main vertices of K'/e ($\approx K'$) on C. Inductively, we have a tripod or disjoint crossing paths. By Lemma 3.2, also $B \cup C$ contains a tripod or disjoint crossing paths.

Suppose now that B has $t \leq 2$ vertices of attachment on C, and recall that we know how to get in linear time a tripod or disjoint crossing paths in case of three or more vertices of attachment. Let $\overline{B}$ be the C-component in R that contains B as a subgraph. Since R is 3-connected, there are disjoint paths $e_1 \ldots, e_{3-t}$ in $\overline{B}$ starting at $C - B$ and terminating in $B - C$ whose only vertices in B are their endpoints. Such paths can be found in linear time by applying, for example, the appropriate modification of the augmented paths method used to test connectivity of graphs [18, Chapter 9]. The connectivity test should be applied on the graph $\overline{B} \cup C$ with the t attachments of B removed. Since $t \leq 2$, the graph $H = B\cup C\cup e_1\cup\cdots\cup e_{3-t}$ contains a copy of K' that does not contain the endpoints of $e_1 \ldots, e_{3-t}$ on C (this fact is really needed only in case $t = 2$). Therefore, the graph $H' = H/(e_1\cup\cdots\cup e_{3-t})$ also contains a copy of K'. Note that the only 3-separations in Kuratowski subgraphs intersect at the three vertices of the same color class in $K_{3,3}$. Therefore H' is equal to C plus a single bridge (plus possibly a branch between two vertices on C that can be replaced by a segment on C), except when the three vertices of $K' \subseteq H'$ that lie on C are the three vertices of the same color class of $K_{3,3}$. In the latter case we clearly have a tripod in H'. In the other cases we can apply the results from above since H' is of appropriate form and has three attachments on C: We can find a tripod in it. By Lemma 3.2, we have a tripod or disjoint crossing paths in H. □

4 The cylinder

In this section we will consider the embedding extension problems in the cylinder. Let C_1 and C_2 be disjoint cycles in the graph G, and for an integer $k \geq 0$, let $P_1, P_2, \ldots, P_k$ be vertex disjoint paths in G joining C_1 and C_2 (with no interior points on $C_1 \cup C_2$). Suppose, moreover, that the endpoints of the paths P_i appear on both cycles C_1, C_2 in the same (cyclic) order. The embedding extension problem in the cylinder with respect to the subgraph $K = C_1 \cup C_2 \cup P_1 \cup \cdots \cup P_k$, where K is embedded in such a way that C_1 and C_2 cover the boundary, will be referred to as the *k-prism embedding extension problem.* Note that in cases when $k \leq 2$, we have two essentially different problems depending on the embeddings of $C_1 \cup C_2$ on the boundary of the cylinder.

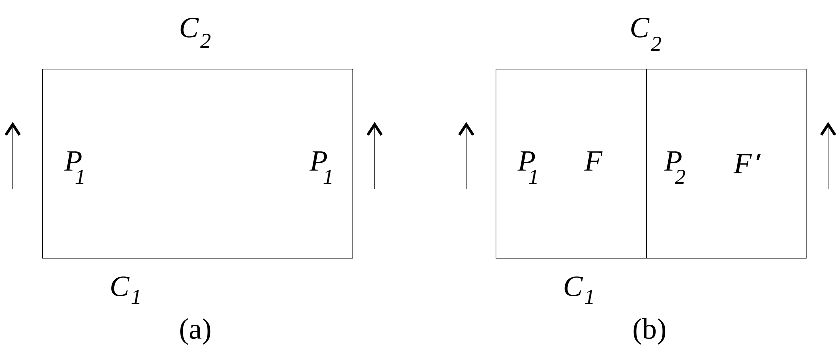

Figure 2: The 1- and 2-prism embedding extension problem

In testing for the k-prism embedding extension of K to G we make use of the *auxiliary graph* $\tilde{G}$ which is obtained from G by adding two new *auxiliary vertices* v_1 and v_2, and for $i = 1, 2$, joining v_i to all vertices of C_i. If $k \geq 3$, then an embedding extension of K to G exists if and only if $\tilde{G}$ is planar, and a planar embedding of $\tilde{G}$ determines a cylinder extension. Something similar holds also in cases when $k \leq 2$. More details will be provided later. Note that in the cylinder case, the auxiliary graph contains two auxiliary vertices while the auxiliary graph for the disk embeddability has just one. Although we are using the same name and notation, there will be no confusion since it will always be clear from the context which case is applied.

If there are local bridges attached to one or more of the paths P_i, we may get arbitrarily long chains of successively overlapping local bridges on P_i (see Figure 3). There are examples where after eliminating any of the branches, there exists an embedding extension. So we can have arbitrarily large minimal obstructions. On the other hand, in applications using the obstructions, certain connectivity conditions on the involved graphs can be achieved. In that case, the local bridges can be eliminated efficiently (in linear time; see [15] and [10] for more details). Since we are usually allowed to change the paths P_i during the pre-processing time, it makes sense to assume that there are no local bridges attached to any of the paths $P_1, \ldots, P_k$.

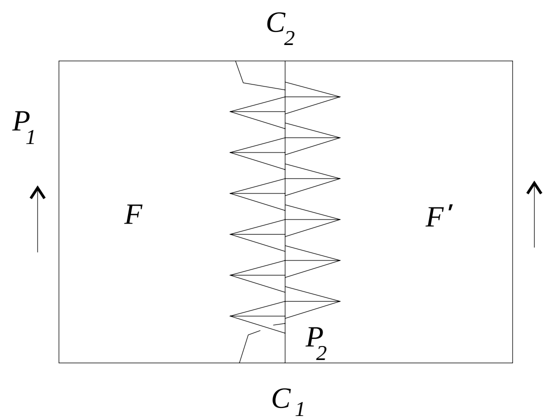

Figure 3: A large obstruction using local bridges

Obstructions for the k-prism embedding extension problem with $k \geq 3$ are easy to find. They are not much more complicated than the closed disk obstructions classified in Theorem 3.1. Besides the disjoint crossing paths and the tripods, we get another type of obstruction. A *dipod* (with respect to the cycle C) is a subgraph H of G consisting of distinct vertices $a, b, c, d \in V(C)$ that appear on C in that order, distinct vertices v, u where $v \notin V(C)$, and $u \notin V(C)$ unless $u = b$, and branches va, vc, vu, ub, and ud (Figure 4). The branches are internally disjoint from C. If $u = b$, the branch ub vanishes. See Figure 4(b). We also define a *triad* (with respect to a subgraph K of G) as a subgraph of G consisting of a vertex $x \notin V(K)$ and three paths joining x with K that are pairwise disjoint except at their common end x.

For $K = C_1 \cup C_2 \cup P_1 \cup \cdots \cup P_k$ embedded in the cylinder with C_1 and C_2 on its boundary, let $F_1, \ldots, F_k$ be the faces of K. We suppose that for $i = 1, 2, \ldots, k$, ∂F_i contains P_i and P_{i+1} (index modulo k).

Theorem 4.1 *Let $K \subseteq G$ be the subgraph of G for the k-prism embedding extension problem, where $k \geq 3$. Suppose that no K–component of G is attached to just one of the paths P_i of K, $1 \leq i \leq k$. Then there is no embedding extension of K to G if and only if $G - E(K)$ contains a subgraph Ω of one of the following types:*

(a) A path joining two vertices of K that do not lie on the boundary of a common face of K, or (with $k = 3$) a triad attached to P_1, P_2, and to P_3.

(b) A tripod attached to the boundary of one of the faces F_i. Not all three attachments of the tripod lie on just one of the paths P_i, P_{i+1} on ∂F_i.

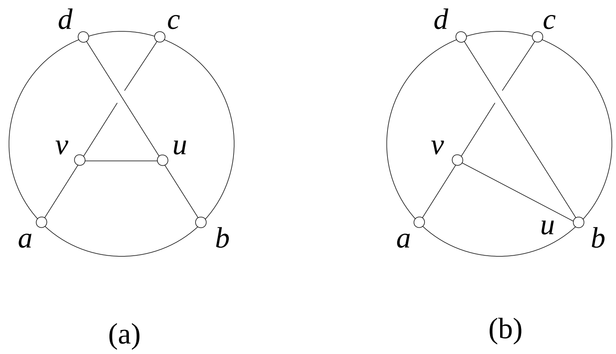

Figure 4: A dipod

(c) A pair of disjoint crossing paths with respect to the boundary of one of the faces F_i. None of the two paths is attached to just one of the paths P_i, P_{i+1} on ∂F_i.

(d) A dipod with respect to the boundary cycle of some F_i. In this case, the vertices a, c, and d from the definition of the dipods all lie on one of the paths P_i, or P_{i+1}, while $b \in \partial F_i$ does not lie on the same path.

(e) A Kuratowski subgraph contained in a 3-connected component L of the auxiliary graph $\tilde{G}$ of G, where L is such that it does not contain auxiliary vertices of $\tilde{G}$.

There is a linear time algorithm that either finds an embedding extension of K to G, or returns an obstruction Ω which fits one of the above cases.

Proof. We can find embedding extensions, if they exist at all, by testing the auxiliary graph $\tilde{G}$ for planarity. Suppose now that embedding extensions do not exist. Our goal is to show how to find the required obstruction Ω.

Since $k \geq 3$ and there are no local bridges at the paths P_j, every K–component is embeddable in at most one of the faces F_i. If one of the bridges contains a path whose ends do not belong to the boundary of the same face, then this path is clearly an obstruction for the embedding extendibility. If a bridge B of K does not have all of its vertices of attachment on the boundary of a single face F_i, then B either contains such a path, or it contains a triad attached to P_1, P_2, and P_3. (Note that the latter case is needed only in case when $k = 3$.) So, we have (a). Otherwise, every K–component is attached to ∂F_i for exactly one i, $1 \leq i \leq k$. Therefore, there is no embedding extension if and only if for some i, $1 \leq i \leq k$, we have a closed disk obstruction (cf. Theorem 3.1) in the subgraph G_i consisting of $C = \partial F_i$ and all the K–components attached to C. By Theorem 3.1, an obstruction to the (G_i, C) disk embeddability is either a pair of disjoint crossing paths, or a tripod, or a Kuratowski subgraph in a 3-connected component of $\tilde{G}_i$ not containing the auxiliary vertex. In the last case, $\tilde{G}_i$ is the auxiliary graph of G_i with respect to C for the disk embedding extension problem. Since there are no local bridges attached to the paths P_i and P_{i+1}, the 3-connected components of $\tilde{G}_i$ not containing the auxiliary vertex are also 3-connected components of $\tilde{G}$. Consequently, the Kuratowski subgraph obstruction in G_i gives (e).

Suppose now that in G_i we have a tripod T. If T is not local on P_i and not local on P_{i+1}, we have (b). Otherwise, assume all three attachments of T are on P_i. Denote by $v_1, v_2, u_1, u_2, u_3, \pi_1, \pi_2, \pi_3$ the elements of T as they are shown on Figure 1, and suppose that π_2 is attached at P_i between π_1 and π_3. Construct a path P, internally disjoint from C, that connects $C - P_i$ with an interior vertex x of T. The existence of P is guaranteed since the bridges containing T are not local on P_i. If x is on π_s for some $s \in \{1, 2, 3\}$, then we can replace the segment of π_s from x to P_i by P and get a tripod satisfying (b). If x is an interior vertex of the branch $u_2 v_1$, then $T \cup P$ contains a dipod satisfying (d). By the symmetries of T, the only essentially different remaining case is when x is on the branch $u_1 v_1$, where $x \neq u_1$ but possibly $x = v_1$. Let Q_1 be the path $P x v_1 u_2 \pi_2$ and let Q_2 be the path $\pi_1 u_1 v_2 u_3 \pi_3$ in $T \cup P$. If Q_1 and Q_2 are in the same K–component of G, then we can find a path P' from Q_1 to Q_2 that is disjoint from C, and $Q_1 \cup Q_2 \cup P'$ is a dipod satisfying (d). On the other hand, if Q_1 and Q_2 are in different bridges B_1, B_2 of K, respectively, let P' be a path from the interior of Q_2 to C that is disjoint from P_i. Such a path exists, again, because B_2 is not local on P_i. Now, $Q_1 \cup Q_2 \cup P'$ contains disjoint crossing paths satisfying (c), unless the endpoints of P and P' on C coincide. But in this case, $Q_1 \cup Q_2 \cup P'$ is a degenerate dipod with the attachment b (see Figure 4(b)) corresponding to the common point of P and P'.

It remains to consider the case of disjoint crossing paths, say Q_1 and Q_2, obtained as an obstruction in G_i. If both Q_1 and Q_2 are attached locally to P_i, we change one of them so that it has an attachment on $C - P_i$. For this purpose, the same method as above can be applied. If just one of the paths (possibly after the previous change) is local on P_i, the same procedure can be applied as it was undertaken above with the paths Q_1 and Q_2 in case of the tripods. We either get a dipod or disjoint crossing paths satisfying (d) or (c), respectively.

It is easy to perform the above construction in linear time. To find disk obstructions we use Theorem 3.1, and to find paths P, P', etc., we can use standard graph search. □

Once we know how the case $k \geq 3$ works, we can also solve the 0-prism embedding extension problem. If C_1, C_2 are cycles of G embedded on the boundary of the cylinder, an orientation of the cylinder yields consistent orientations of C_1 and C_2. If $P_1, \ldots, P_k$ are disjoint (C_1, C_2)–paths, they are said to be *attached consistently* if their ends on C_1 follow each other in the inverse cyclic order than on C_2, i.e., the embedding of $C_1 \cup C_2$ can be extended to $C_1 \cup C_2 \cup P_1 \cup \cdots \cup P_k$. Note that for $k \leq 2$, the paths are always attached consistently.

Before stating our next result on obstructions, let us formulate a lemma which will be used in its proof.

Lemma 4.2 *Let G be a 3-connected graph, C a cycle of G, and B a C-component in G. Let $G(B, C)$ be the graph obtained from $B \cup C$ by adding a new vertex adjacent to all vertices on C. Then $G(B, C)$ is 3-connected.*

Proof. The graph is clearly 2-connected. It is also easy to see that it has no 2-separations. □

Theorem 4.3 *Let C_1 and C_2 be disjoint cycles of a graph G that are embedded on the boundary of the cylinder. There is no embedding extension to G if and only if $G - E(C_1) - E(C_2)$ contains a subgraph Ω of one of the following types:*

(a) Three disjoint paths from C_1 to C_2 that are not attached consistently on C_1 and C_2.

(b) Disjoint paths P_1, P_2, P_3 where P_1, P_2 join C_1 with C_2, both endpoints of P_3 are on C_1 (respectively, on C_2) and the endpoints of P_3 interlace with the endpoints of P_1 and P_2 on C_1 (respectively, on C_2).

(c) A tripod or a pair of disjoint crossing paths with respect to C_1 (respectively, C_2). If G is not 3-connected, then this obstruction may have a vertex, two vertices, or a segment of one of its branches contained in C_2 (respectively, in C_1).

(d) A path P from C_1 to C_2 together with a tripod T with respect to $C_1 \cup C_2$ disjoint from P which has two attachments on C_1 and one on C_2, or vice versa.

(e) Disjoint paths P_1, P_2, P_3 from C_1 to C_2 attached consistently on C_1 and C_2 together with a triad attached to P_1, P_2, and to P_3.

(f) A Kuratowski subgraph contained in a 3-connected component L of the auxiliary graph $\tilde{G}$ of G, where L is such that it does not contain auxiliary vertices of $\tilde{G}$.

Moreover, there is a linear time algorithm that either finds an embedding extension of $C_1 \cup C_2$ to G, or returns an obstruction Ω for the embedding extendibility. In the latter case, Ω fits one of the above cases (a)–(f).

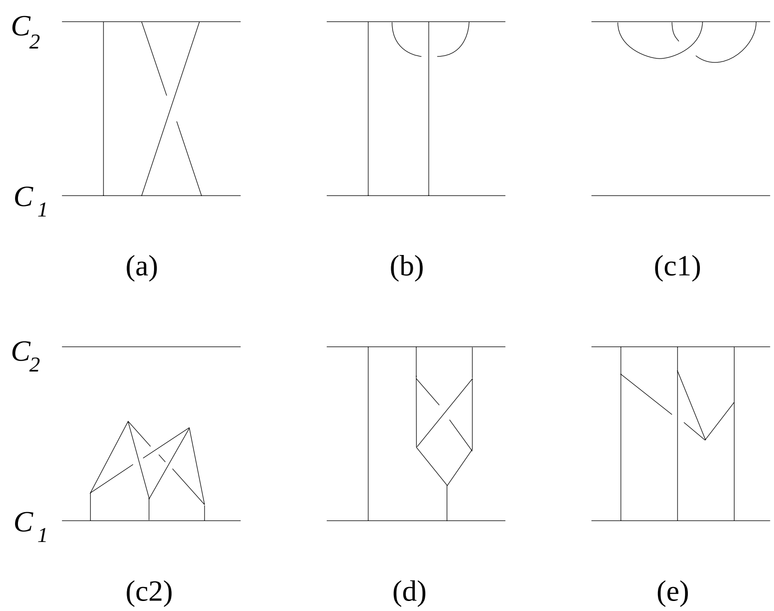

Figure 5: 0-prism obstructions

Proof. First of all we try to find three disjoint (C_1, C_2)–paths in G. If such paths exist, let $k = 3$, and let P_1, P_2, P_3 be the paths. Otherwise, let $k \leq 2$ be the maximal number of disjoint paths from C_1 to C_2. All these can be obtained in linear time by standard connectivity algorithms using flow techniques [18, Chapter 9].

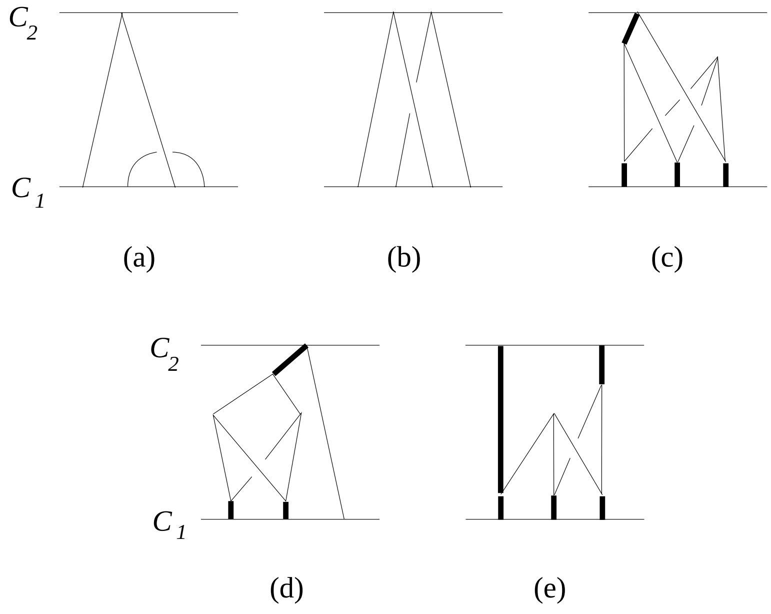

Figure 6: Some obstructions of type (c) meeting both cycles

Let us first consider the case when $k = 3$. If P_1, P_2, P_3 are not attached consistently at C_1 and C_2, then $\Omega = P_1 \cup P_2 \cup P_3$ is a small obstruction satisfying (a). Otherwise, we first reduce the problem to the 3-connected case. Without loss of generality we can remove the 3-connected components of the auxiliary graph that do not contain the auxiliary vertices (or we get (f)). So, we assume from now on that $\tilde{G}$ is 3-connected. Next we try to replace the paths P_1, P_2, and P_3 by disjoint paths joining the same pairs of endpoints so that no local bridge of $K' = C_1 \cup C_2 \cup P_1 \cup P_2 \cup P_3$ will be attached to some P_i only. It can be shown that this is always possible to do since $\tilde{G}$ is 3-connected, but it is not entirely obvious how to perform it in linear time. For $i = 1, 2, 3$, let G_i be the graph consisting of P_i together with all its local bridges and with an additional edge joining the ends of P_i. If G_i is planar, then an algorithm from [15] replaces P_i with a new path P_i' joining the same endvertices which is internally disjoint from $K' - P_i$, and such that no local bridge of $(K' - P_i) \cup P_i'$ is attached to P_i'. So, we either achieve our goal, or get one of G_i, say G_1, to be non-planar. Let us first deal with the latter possibility. Let C be the cycle composed of the paths P_2 and P_3 together with the segments on C_1 from P_3 to P_1 and from P_1 to P_2, and the segments on C_2 from P_2 to P_1 and from P_1 to P_3. Denote by B the $(C_1 \cup C_2 \cup P_2 \cup P_3)$–component in G that contains P_1. If B contains a vertex of $(C_1 \cup C_2) - C$, then a path in B from that vertex to an end of P_1, together with P_2 and P_3 determines three non-consistently attached paths from C_1 to C_2, and so we have case (a). Therefore we may assume that B is attached to C only. Let $H = B \cup C$. It is clear that H is 2-connected, and by Lemma 4.2, its auxiliary graph $\tilde{H}$ with respect to C is 3-connected. Moreover, $\tilde{H}$ is non-planar since G_1 is contained in H (with the edge joining the ends of P_1 replaced by a path in C). By Theorem 3.1, we can find in H a tripod T or disjoint crossing paths Q_1 and Q_2 with respect to C. Let us first consider the case when we have disjoint paths Q_1, Q_2. For $j = 1, 2$, denote by e_j the foot of P_1 on C_j, and let C_j° be the open segment of C_j obtained from $C_j \cap C$ by removing its endpoints. If $Q_1 \cup Q_2$ is not attached to C_1°, take a path P in $B - C$ from e_1 to $Q_1 \cup Q_2$. Such a path clearly exists since Q_1, Q_2 are both contained in the same bridge. Using this path, we can change Q_1 or Q_2 to get disjoint crossing paths that are attached to C_1°. We repeat the same procedure at C_2°. Up to symmetries, there are three possible outcomes:

(i) Q_1 *joins* C_1° *and* C_2°*:* If Q_2 is attached on P_2 and P_3, take a path P in $B - C$ joining Q_1 and Q_2. Now, the paths Q_1, P_2, P_3 and the triad $Q_2 \cup P$ satisfy (e). Otherwise, it is easy to see that we get a subgraph of type (a), or (b) contained in $Q_1 \cup Q_2 \cup P_2 \cup P_3$.

(ii) Q_1 *is attached to* C_1° *and* Q_2 *is attached to* C_2°*:* Excluding the above possibility (i), we may assume that the other attachment of Q_1 is on $P_2 - C_1$. Then $Q_1 \cup Q_2 \cup P_2$ contains disjoint crossing paths between C_1 and C_2. Together with P_3 they determine a subgraph of type (a).

(iii) Q_1 *is attached to* C_2° *and both endpoints of* Q_2 *are on* P_2*:* Let P be a path in $B - C$ joining Q_1 and Q_2. Then $Q_1 \cup Q_2 \cup P \cup P_2$ is a tripod on $C_1 \cup C_2$, and together with P_3 we have (d).

Suppose now that T is a tripod with respect to C that is contained in B. If T is not attached on C_1°, let P be a path in $B - C$ from e_1 to T. Then $T \cup P$ either contains a pair of disjoint crossing paths (which we already covered above), or a tripod T' with an attachment on C_1°. If T' is not attached to P_3, then $T' \cup P_2$ contains a tripod T'' with respect to $C_1 \cup C_2$ that is either attached to C_1 only (case (c)), or is attached to C_1 and C_2 (in this case $T'' \cup P_3$ satisfies (d)). Similarly, if T' is attached only to $C - P_2$. We are left with the case when T' is attached to C_1° and to P_2 and P_3. In this case we construct a path in $B - C$ from e_2 to T'. It gives rise to disjoint crossing paths, or to a tripod that are disjoint from P_2 or P_3, and both of these cases were already covered above.

From now on we may assume that we have P_1, P_2, P_3 without local bridges. Let $K' = C_1 \cup C_2 \cup P_1 \cup P_2 \cup P_3$. By Theorem 4.1 we either extend the embedding of K' to G, or find an obstruction. The first outcome is fine, while in the second case we get one of the obstructions (a)–(d) of Theorem 4.1. Obstruction (a) of Theorem 4.1 together with P_1, P_2, P_3 necessarily contains one of our cases (a), (b), or (e). Case (b) of Theorem 4.1 together with P_1, P_2, P_3 implies our cases (c) or (d). The possibility (c) of Theorem 4.1 yields either (a), (b), or (c). Finally, a dipod D of type (d) in Theorem 4.1 attached three times on P_1, say, gives rise to a tripod with respect to $C_1 \cup C_2$ contained in $D \cup P_1$ (plus a segment on P_2 (or P_3) if D is attached to P_2 (respectively, on P_3)). This tripod is disjoint from one of the paths, and fits our case (d).

Finally, we have reached the cases $k = 0, 1, 2$. The first two ($k = 0, 1$) are easy. We are faced with two disk embedding extension problems, and to solve each of them, we apply Theorem 3.1. The resulting obstruction fits (c). If a cutvertex v of G separating C_1 and C_2 is on C_2 (assuming that the block containing C_1 is non-planar), the obstruction may contain v. (The possible cases are shown in Figure 6(a), (c), and (d).) Note that this is the first time that disjoint crossing paths or a tripod with respect to C_1 have a vertex on C_2.

Suppose now that $k = 2$. Let Q be the block of G containing C_1 and C_2. If the embedding of $C_1 \cup C_2$ extends to Q, then we test the other blocks of G for planarity. We either get an embedding extension to G, or one of the blocks is non-planar. In the latter case we have (f). So we may assume from now on that G is 2-connected and that there is no embedding extension. Since $k = 2$, Menger's theorem guarantees that C_1 and C_2 are in distinct 3-connected components of the auxiliary graph $\tilde{G}$. If all 3-connected components of $\tilde{G}$ are planar, then $\tilde{G}$ is also planar. (This is easily seen by constructing a plane embedding of a graph by using plane embeddings of graphs forming its 2-separation.) Unfortunately, it may happen that the plane embedding of $\tilde{G}$ obtained this way will not determine an embedding extension since C_1 and C_2 are not oriented consistently. In this case, let $Q_1 \supset C_1$ and $Q_2 \supset C_2$ be the graphs used in merging at the time when C_1 and C_2 merge in the same part, and let e be the corresponding virtual edge. Fixing the embedding of Q_1, there are two possibilities for the embedding of Q_2 that differ from each other only by the choice of orientation. One of them gives the consistent orientation of C_1 and C_2.

We may assume now that one of the 3-connected components of $\tilde{G}$ is non-planar. If the two 3-connected components containing C_1 and C_2, respectively, are planar then we get (f). Suppose now that the 3-connected component $Q_1 \supset C_1$ of $\tilde{G}$ is non-planar. This is equivalent to the property that Q_1 minus the auxiliary vertex has no embedding into the disk having C_1 on its boundary. By Theorem 3.1, we know how to handle this case. Since Q_1 is 3-connected, we get disjoint crossing paths or a tripod in it. This almost always gives rise to a subgraph of G satisfying (c). The only trouble may arise if our obstruction in Q_1 contains the virtual edge e having its pair in the 3-connected component Q_2 of $\tilde{G}$ that contains C_2. In this case, the replacement of e by a path P in $Q_2 - e$ should be done carefully so that $P \cap C_2$ is either empty, a vertex, or a segment on C_2. Since this is easy to achieve, we are through with our case analysis. It is worth remarking that some of the possibilities when $P \cap C_2$ is non-empty lead to cases (b) or (d). Some of the really new cases are shown in Figure 6, where each of the bold segments can be contracted to a point. The cases shown in Figure 6 include all possibilities that arise when the obstruction in Q_1 is either a pair of disjoint crossing paths, or a tripod whose intersection with C_2 is a vertex, or a segment.

At the end we remark that all the steps of the algorithm that follows the above proof are easy to implement in linear time. □

It is worth remarking that all cases of Theorem 4.3 are indeed obstructions for the 0-prism embedding extension problem, and that they are minimal (except in some cases of (c) when the intersection with the other cycle is non-empty) in the sense that if any of the branches is removed from such an obstruction, there exists an embedding extension. Note that the branch size of minimal 0–prism obstructions is at most 12. The obstructions (a)–(e) (without showing their "degenerate" versions) are presented in Figure 5.

5 The 2-prism embedding extension problem

It may happen that minimal obstructions for the k-prism embedding extension problems are arbitrarily large. However, under the additional assumption that there are no local bridges attached to the paths P_i $(1 \le i \le k)$, large minimal obstructions are unavoidable only for the k-prism embedding extension problem with $k = 1$ or 2. An example of such an obstruction is shown in Figure 7. Since the general case of large minimal obstructions look like our example in Figure 7, we use the name millipede. More precisely, a *millipede* M for the 2-prism embedding extension problem is a subgraph of $G - E(K)$ which can be expressed as $M = B_1^\circ \cup B_2^\circ \cup \cdots \cup B_m^\circ$ $(m \ge 2)$ where $B_1^\circ, \ldots, B_m^\circ$ are subgraphs of distinct K–bridges $B_1, B_2, \ldots, B_m$ (respectively) and satisfy the following conditions:

(1) Each of B_1° and B_m° is embeddable in exactly one of the faces of K. If m is even, then B_1° and B_m° are embeddable in the same face of K. If m is odd, then B_1° and B_m° are embeddable in distinct faces of K.

(2) For $2 \le i \le m-1$, B_i° is embeddable in both faces of K.

(3) For each $i = 1, 2, \ldots, m-1$, B_i° and B_{i+1}° cannot be embedded simultaneously in the same face of K.

(4) No other pair B_i°, B_j° $(1 \le i < i+2 \le j \le m)$ interferes with each other, i.e., for any embedding of B_i°, there is an embedding of B_j° in the same face of K unless such an embedding is not possible by (1) (when $i = 1$ or $j = m$).

(5) B_i° $(1 \le i \le m)$ are minimal in the sense that the removal of any branch from B_i° destroys either (1), or (3).

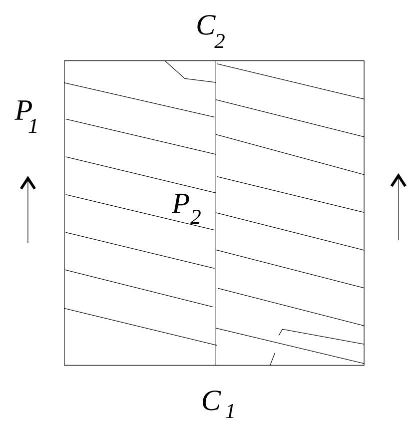

Figure 7: A millipede

It is easy to see that a millipede is a minimal obstruction for the embedding extendibility. It follows from the minimality property (5) that each B_i° $(1 \le i \le m)$ contains at most 6 feet (at most a triple for overlapping with B_{i-1}° and possibly another triple for overlapping with B_{i+1}°) and has at most 11 branches. (We will see that it suffices to consider only millipedes in which every B_i° contains at most 4 feet.) Let us remark that the millipedes constructed by our succeeding theorems will satisfy an even stronger "minimality" condition: Properties (1), (2) and (4) will hold not only for the subgraphs B_i° but also for their "master-bridges" B_i.

Given the 2-prism embedding extension problem with $C_1, C_2, P_1, P_2, F, F'$ as in Figure 2(b), and $K = C_1 \cup C_2 \cup P_1 \cup P_2$, we define the *overlap graph* $O(G, K)$ of K-bridges in G as follows. Its vertices are the K-bridges, and two of them are adjacent in $O(G, K)$ if they overlap in one of the faces, i.e., they can be embedded in the same face, say F, but their union cannot be embedded in F. The *extended overlap graph* $AO(G, K)$ is obtained from the overlap graph by adding two new vertices, w and w', which are adjacent to each other. Moreover, w is adjacent to all bridges of K that are not embeddable in F, and w' is adjacent to all bridges that are not embeddable in F'.

Lemma 5.1 *The embedding of K for the 2-prism embedding extension problem has an extension to G if and only if the extended overlap graph $AO(G,K)$ is bipartite.*

Proof. If a K-bridge B cannot be embedded in any of the faces F, F', then B together with w and w' determines a triangle, and the extended overlap graph is not bipartite. Therefore we may assume from now on that every bridge can be embedded in at least one of the faces, F, or F'.

Suppose now that we have an embedding extension. Color the bridges that are embedded in F using color 1, and color the bridges in F' using color 2. Moreover, let w be colored 1, and let w' be colored 2. It is easy to see that this determines a 2-coloring of $AO(G,K)$, so the extended overlap graph is bipartite.

Conversely, if the extended overlap graph is bipartite, choose one of its 2-colorings having w and w' colored 1 and 2, respectively. Consider the bridges colored 1. Each of them can be embedded in F since otherwise it would be adjacent in $AO(G,K)$ to w which also has color 1. Moreover, all these bridges can be embedded in F simultaneously, since no two of them overlap. Similarly, the bridges colored 2 have an embedding in F', and we get a required embedding extension. □

The above lemma provides a simple answer for the 2-prism embedding extension problem. It also yields an algorithm that is linear in the number of edges of $AO(G,K)$. Having a 2-coloring, we easily get an embedding extension. Otherwise, an odd cycle in $AO(G,K)$ determines an obstruction. Unfortunately, the usual 2-coloring algorithm can be of quadratic complexity in terms of the size of G since the number of edges of $AO(G,K)$ may be quadratic in terms of the number of bridges, and this number can be linear in terms of $|E(G)|$. Therefore we have to solve the biparticity problem of $AO(G,K)$ with some additional care in order to fulfil our linearity goal. One possible approach is explained in more detail in [12].

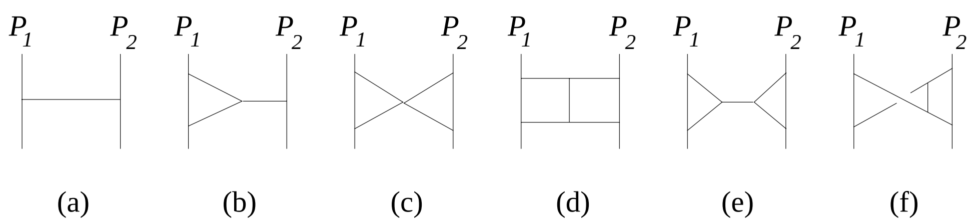

Figure 8: H–graphs

In the following results, we will use some special subgraphs of K-bridges. Let B be a K-bridge in G. For each branch e of K that B is attached to, let e_1 and e_2 be feet of B attached as close as possible on e to one and the other end of e (including the possibility of being attached to the end). Furthermore, let these feet be chosen in such a way that their total number is as small as possible, i.e., if there is just one attachment on e, we select $e_1 = e_2$, and similarly when different branches of K share an attachment of B. Let $H = H(B)$ be a minimal subtree of B that contains all chosen feet. The obtained graph H is said to be an H–*graph* of B. Suppose now that B is attached only to $P_1 \cup P_2$. Then H contains at most 4 feet. If there are three or just two distinct feet in H, then H is unique up to homeomorphisms. But in the case of four distinct edges, there are four homeomorphically distinct cases for H (see Figure 8). Let us remark that the last case of Figure 8 is excluded if B can be embedded in F since it contains disjoint crossing paths. Note that H–graphs can be constructed in linear time by standard graph search algorithms. The following simple result justifies the introduction of H–graphs.

Lemma 5.2 *Let G be a graph and K be a subgraph that is 2-cell embedded in some surface. Let B and B' be K-bridges in G that can be embedded in the same face F of K. If ∂F is a cycle in G and neither B nor B' is a local bridge, then B and B' overlap in F if and only if their H–graphs overlap in F.*

Theorem 5.3 *Let $K \subseteq G$ be the subgraph of G for the 2-prism embedding extension problem, and let F, F' be the faces of K. Suppose that no K–component of G is attached just to one of the paths P_1, P_2 of K. Then there is no embedding extension of K to G if and only if $G - E(K)$ contains a subgraph of one of the following types:*

(a) A path that is internally disjoint from K and connects a vertex of $\partial F - (P_1 \cup P_2)$ with a vertex of $\partial F' - (P_1 \cup P_2)$.

(b) A tripod T with respect to the boundary of one of the faces F, F'. Not all three attachments of T lie on just one of the paths P_1, P_2, and if all are in $P_1 \cup P_2$, then the tripod is non-degenerate.

(c) A pair of disjoint crossing paths with respect to the boundary of one of the faces F, F'. Each of the two paths is attached to a vertex of $K - (P_1 \cup P_2)$.

(d) A non-degenerate dipod with respect to the boundary cycle of F or F'. The vertices a, c from the definition of dipod lie on $P_1 \cup P_2$, and not all attachments of the dipod lie on just one of the paths P_1 or P_2.

(e) Internally disjoint triads T_1, T_2 attached to the same triple of vertices on $P_1 \cup P_2$ together with a path joining the main vertices of T_1 and T_2. Not all three attachments of $T_1 \cup T_2$ are on just one of the paths P_1, or P_2.

(f) Subgraphs H_1, H_2, H_3 that are pairwise overlapping in F or in F'. They are minimal pairwise overlapping subgraphs of H-graphs of distinct K-bridges. H_2 and H_3 are attached to $P_1 \cup P_2$ only.

(g) A millipede.

(h) A Kuratowski subgraph contained in a 3-connected component L of the auxiliary graph $\tilde{G}$ of G, where L is such that it does not contain auxiliary vertices of $\tilde{G}$.

Moreover, there is a linear time algorithm that either finds an embedding extension of K to G, or returns an obstruction Ω of one of the above types.

Proof. We may check the embedding extendibility by testing the planarity of the auxiliary graph $\tilde{G}$ (cf. case $k = 2$ in the proof of Theorem 4.3 for details). Moreover, if there is no embedding extension, then we can reduce the problem to the case when G is 2-connected (or we get (h)). Note that C_1 and C_2 are in the same block of G. If C_1 and C_2 are in the same

3-connected component of $\tilde{G}$, then we can also reduce the problem to the case when $\tilde{G}$ is 3-connected (or we get (h)). If this is not the case, let $\tilde{G}'$ be the graph obtained from $\tilde{G}$ by adding the edge joining the auxiliary vertices. By considering the 3-connected components of $\tilde{G}'$, we can also in this case either get (h) (since the 3-connected components of $\tilde{G}'$ are also 3-connected components od $\tilde{G}$ except for the one containing C_1 and C_2), or reduce the problem to the case when $\tilde{G}'$ is 3-connected. The latter case will be assumed henceforth.

Consider the K–components of G. Suppose that B is one of them, and that the embedding of K cannot be extended to $K \cup B$. We either have (a), or B is attached to the boundary of one of the faces of K, say to ∂F. In the latter case, let $L = \partial F \cup B$. Since $\tilde{G}'$ is 3-connected, the auxiliary graph $\tilde{L}$ of L (with respect to ∂F) is 3-connected by Lemma 4.2. Clearly, $\tilde{L}$ is non-planar, and by Theorem 3.1, B contains a tripod or a pair of disjoint crossing paths with respect to ∂F. Since B is not local on P_1 or P_2, we can use the same strategy as in the proof of Theorem 4.1 to change the obtained obstacle so that not all of its attachments are on just one of P_1 or P_2. Usually we will get case (b), or (c). But there are also two exceptions. The first possibility is when we get a degenerate tripod with all attachments on $P_1 \cup P_2$. Add a path P in B between the two triads in the tripod. If P joins the two main vertices of the tripod, then we have case (e). In all other cases, the union of P and the tripod contains a non-degenerate tripod, i.e., a subgraph satisfying (b). The other case is when we have disjoint crossing paths that do not satisfy (c). One of the paths is then attached only to $P_1 \cup P_2$. Hence, by adding a path in B that joins the two paths, we get a non-degenerate dipod satisfying (d).

From now on we may assume that every K–component is embeddable either in F, or in F' (or both). A linear time algorithm of [12] shows how to solve this problem. That algorithm finds an induced odd cycle Γ in the extended overlap graph $AO(G,K)$. There are 4 cases to be distinguished.

(i) *Both vertices w and w' lie on* Γ: In this case, the edge ww' is on Γ since Γ is an induced cycle. Let $B_1, \ldots, B_m$ be the K-bridges corresponding to the sequence of vertices of Γ from w to w' (but not including these two). By our assumptions, $m > 1$. By the definition of $AO(G,K)$, B_1 cannot be embedded in F, and B_m cannot be embedded in F'. Note that m is odd, so the conclusion here fits condition (1) of the definition of millipede. Next we describe how to get the subgraph B_i° of B_i, for $i = 1, 2, \ldots, m$. Since the cycle Γ of $AO(G,K)$ is induced, no bridge B_i, $2 \le i \le m-1$, is adjacent to w, or w'. This means that B_i itself is embeddable in F and in F'. Therefore also arbitrary subgraphs B_i° of B_i satisfy (2). The bridges B_i and B_{i+1} ($i < m$) cannot be simultaneously embedded in the same face, and at least one of them is embeddable in both faces of K. By Lemma 5.2, their H-subgraphs (which are easy to find) overlap in the same way as the bridges themselves. By taking such obstructions for all bridges B_i, we get small subgraphs of $B_1, \ldots, B_m$ satisfying (1) and (3). Since these subgraphs are small, we can check for each of them whether it satisfies the minimality requirement (5), and remove the superfluous branches whenever necessary. Finally, (4) is satisfied automatically since Γ is an induced cycle of $AO(G,K)$. Therefore we have a millipede.

(ii) $w \in V(\Gamma)$: We get a millipede in the same way as in Case (i), except that m is even.

(iii) $w' \in V(\Gamma)$: Same as Case (ii).

(iv) $w, w' \notin V(\Gamma)$: We will show that in this case the length m of Γ is rather small. Let $B_1, \ldots, B_m$ be the K-bridges corresponding to the successive vertices on Γ. Suppose first that $m = 3$. If two of the bridges, say B_1, B_2, are adjacent in $AO(G,K)$ to w (or w'), then we replace Γ by the triangle B_1, B_2, w (respectively, B_1, B_2, w'), and by (ii) (respectively, (iii)) we get a millipede. If one of them is adjacent to w, another to w', we get a millipede of length 3 as it was the case in (i). (This works even though the corresponding cycle in Γ obtained by replacing the edge B_1B_2 by the path $B_1ww'B_2$ is not induced.) We may therefore assume that B_2 and B_3 are embeddable in F and F' and that B_1 is embeddable at least in F. For $i = 1, 2, 3$, let H_i be an H-graph of B_i. By Lemma 5.2, the H-graphs overlap as much as the original bridges. We therefore have (f).

Suppose now that $m \geq 5$. As above, the cases when two of the bridges B_i are adjacent to w or w' (possibly one to w, another to w') can be reduced to the previously treated cases. We may thus assume that at most one of the bridges is not embeddable in both F and in F'. If there is a bridge that cannot be embedded in one of the faces, we will assume that this is B_1, and that this bridge can be embedded in F. Let us write $B_i \prec B_j$ if $B_i \cup B_j$ can be embedded in F and B_i is embedded closer to C_1 than B_j. Since none of the bridges is local on P_1 or P_2, the relation $\prec$ is well defined. The relation $\prec$ is transitive, and since $\tilde{G}'$ is 3-connected, it is also asymmetric. Therefore it has minimal elements. We may assume that B_1 is a minimal element for this relation. (If B_1 is only embeddable in F we thus assume that it is attached to $C_1 - (P_1 \cup P_2)$, and then B_1 is clearly minimal by the definition of the relation $\prec$.) We claim that for $i = 1, 2, \ldots, m-2$, $B_i \prec B_{i+2}$. Suppose that this is not true. Let i be the smallest index for which $B_{i+2} \prec B_i$. Since B_j and B_{j+2} do not overlap, they are $\prec$–comparable for every j and thus such an index i exists. By our choice of B_1, we have $i > 1$. Since B_{i+2} is attached closer to C_1 than B_i, and B_i overlaps with B_{i-1}, B_{i+2} has an attachment on P_1 or P_2 that is closer to C_1 than one of the attachments of B_{i-1} on the same path. Similarly, since B_{i+2} overlaps with B_{i+1} and $B_{i+1} \succ B_{i-1}$, the bridge B_{i+2} has an attachment that is further away from C_1 than an attachment of B_{i-1} on the same path P_1, or P_2. This implies that B_{i+2} overlaps with B_{i-1}. But this is not possible since $m \geq 5$. This proves the claim. In particular, we know that $B_{m-2} \prec B_m$. Since B_1 is $\prec$–minimal and $\prec$–comparable with B_j if $j \neq 1, 2, m$, we have $B_1 \prec B_{m-2}$. By transitivity we have $B_1 \prec B_m$. This contradicts the fact that B_1 and B_m overlap. The proof is thus complete. □

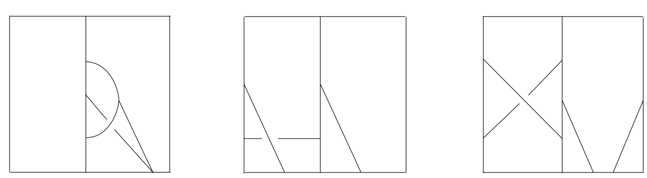

Figure 9: One-sided millipedes

In the last part of the above proof, we have learned even more than needed. A straightforward extension gives the following result. Let us call a millipede *two-sided* if it is attached to $C_1 - (P_1 \cup P_2)$ and to $C_2 - (P_1 \cup P_2)$. Otherwise it is *one-sided*. Some one-sided millipedes are shown in Figure 9.

Proposition 5.4 *If M is a one-sided millipede, then the number of K-bridges it includes is at most 4. In particular, M is a small obstruction.*

6 The 1-prism embedding extension problem

It remains to determine minimal obstructions for the 1-prism embedding extension problem. Let us first extend a few definitions used in previous sections for the purpose of this section. If F is a face of an embedded graph $K \subseteq G$, and P, P' are paths in G with endpoints on ∂F but otherwise disjoint from K, they are said to be *disjoint crossing paths* with respect to F if they cannot be simultaneously embedded in F. The essentially different cases of disjoint crossing paths with respect to the face F of a 1-prism embedding extension problem are shown in Figure 10 (up to symmetries). The only case where we have one of the paths attached to P_1 is (d) which also includes the possibility of the attachment at $C_1 \cap P_1$ or $C_2 \cap P_1$. *Tripods* with respect to the face F are another kind of obstructions for the 1-prism embedding extension problem. They are defined in the same way as in the case when the face is bounded by a cycle, with the additional requirement that it should be an obstruction. It turns out that tripods with respect to F can be divided into four classes:

1. Attached twice to $C_1 - P_1$ or twice to $C_2 - P_1$ with the third attachment anywhere else on ∂F and with no restrictions on non-degeneracy.

2. Attached to $C_1 - P_1$, to $C_2 - P_1$, and to P_1. The attachment on P_1 is non-degenerate.

3. Attached once to $C_1 - P_1$ (or to $C_2 - P_1$) and twice to P_1. The attachment on P_1 that is closer to C_1 (respectively, closer to C_2) is non-degenerate.

4. Attached only to P_1. The middle attachment on P_1 is non-degenerate.

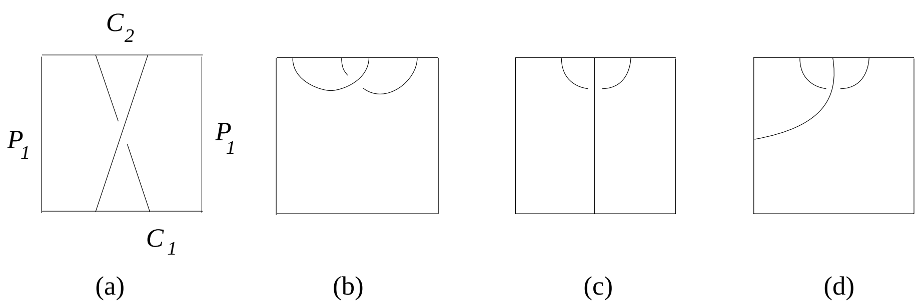

Figure 10: Disjoint crossing paths with respect to F

The *1-millipedes* are another type of obstructions for the 1-prism embedding extension problem. These obstructions are of the same type as the millipedes are for the 2-prism embedding extension problem, and they can be arbitrarily large, though minimal, as well. A 1-millipede is a subgraph consisting of a path P_2 joining C_1 and C_2 and disjoint from P_1, together with a millipede for the 2-prism problem with respect to $K \cup P_2$. Moreover, the following additional requirement is imposed on 1-millipedes:

(6) For $j = 3, 4, \ldots, m-2$, denote by l_i^- and r_i^- the vertices of attachment of $B_{i-1}^\circ \cup B_{i-2}^\circ$ on K closest to C_1 and C_2, respectively. Similarly, let l_i^+ and r_i^+ be the extreme vertices of attachment of $B_{i+1}^\circ \cup B_{i+2}^\circ$. Then r_i^+ is strictly closer to C_2 than r_i^- and l_i^- is strictly closer to C_1 than l_i^+.

Note that (6) is void if $m < 5$. It should also be pointed out that we assume that $l_3^- \in C_1 - P_1$ (an attachment of B_1°), and this is considered as being strictly closer to C_1 than any vertex on P_1. Similarly on the other side, where $r_{m-2}^+ \in C_2 - P_1$.

Yet another type of obstruction will be needed. Let $x_1, x_2 \in V(P_1)$ and suppose that x_2 is closer on P_1 to C_1 than x_1 is. A subgraph Ω of $G - E(K)$ is a *left side obstruction* with respect to x_1 and x_2 if it satisfies:

(i) Ω contains a path joining $C_1 - P_1$ with x_1 (respectively, a path joining $C_2 - P_1$ with x_2).

(ii) No attachment of Ω to K is closer to C_2 than x_1 (respectively, closer to C_1 than x_2) and no attachment of Ω is on $C_2 - P_1$ (respectively, $C_1 - P_1$).

(iii) Ω can be embedded in F in such a way that all feet of Ω attached to P_1 (strictly) between x_1 and x_2 are touching P_1 at the right side of F. (The left and the right are well defined with respect to Figure 2.)

(iv) Ω cannot be embedded in F in such a way that all feet of Ω attached to P_1 (strictly) between x_1 and x_2 are touching P_1 at the left side of F.

We define similarly the *right side obstructions*. Their attachments on P_1 between x_1 and x_2 can be embedded on the left side of F but cannot be embedded on the right side.

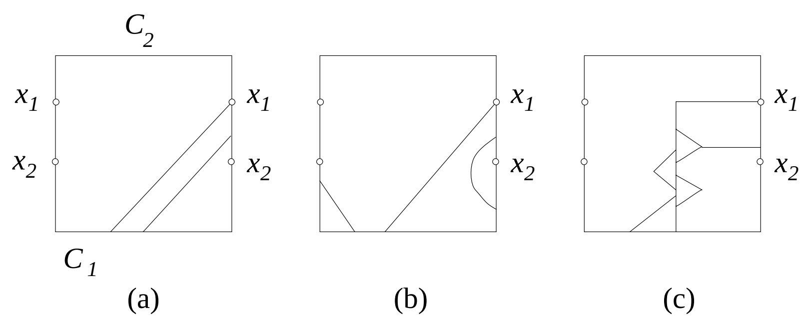

Figure 11: The minimal left side obstructions

Examples of left side obstructions are given in Figure 11. We will prove in Theorem 6.3 that Figure 11 contains all minimal left side obstructions attached to $C_1 - P_1$, where case (c) of Figure 11 represents arbitrary two-sided millipedes for the following 2-prism embedding extension problem. Add the edge x_1x_2 (x_1 and x_2 are formerly non-adjacent) and embed it across the face F so that it is attached to x_1 on the left and to x_2 on the right. Add also a path P_2 from $C_1 - P_1$ to x_1. Let P_1' be the segment of P_1 from C_1 till x_2 and let C_2' be the cycle consisting of the segment x_2x_1 on P_1 together with the new edge. Then we consider the 2-prism problem with respect to $K' = C_1 \cup C_2' \cup P_1' \cup P_2$ embedded in the cylinder as described above. Note that the first bridge in a two-sided millipede for this problem will be attached between P_1 and P_2 on C_1, while the last one will be attached to the segment of P_1 between x_2 and x_1. It is easy to see that such a millipede is a left side obstruction. Note that one-sided millipedes do not give rise to left side obstructions.

Having a left side obstruction Ω_1 attached to $C_1 - P_1$ and a left side obstruction Ω_2 attached to $C_2 - P_1$ (with respect to the same pair x_1, x_2), which do not intersect out of P_1, their union $\Omega_1 \cup \Omega_2$ cannot be embedded in F. This way we get a rich family of 1-prism embedding obstructions.

Before stating the main result of this section, we will prove a lemma about 3-connected subgraphs which will be needed in the proof. Let us recall that a graph H is *nodally 3-connected* if the graph obtained from H by replacing each branch with an edge between the corresponding main vertices is 3-connected.

Lemma 6.1 *Let G be a graph with disjoint nodally 3-connected subgraphs K and L. Let π_1, π_2, π_3 be disjoint paths in G joining three main vertices of K with a triple of main vertices of L. Let $J = K \cup L \cup \pi_1 \cup \pi_2 \cup \pi_3$. If for every branch e of $K \cup L$, no two consecutive (on e) connected components of $e \cap (\pi_1 \cup \pi_2 \cup \pi_3)$ belong to the same path π_i, then J is nodally 3-connected.*

Proof. J is clearly 2-connected. Suppose now that there is a 2-separation $J = J_1 \cup J_2$, $J_1 \cap J_2 = \{x, y\}$ where x and y are main vertices of J and each of J_1, J_2 contains two or more branches of J. One of the paths π_i is disjoint from x, y and is thus totally contained in J_2, say. Since K, L are nodally 3-connected, J_2 contains all main vertices of $K \cup L$. By our assumption, J has no parallel branches. Thus J_1 contains a main vertex z of J. Clearly, z is obtained as the intersection of one of the paths, say π_1, with a branch e in K, say. Since J_2 contains all main vertices of $K \cup L$, x, y both lie on e and both lie on π_1. Follow π_1 from z in a direction out of the branch e. The first intersection with $K \cup L$ must be on e. Otherwise we could reach a main vertex of $K \cup L$ different from x, y. By our assumption on J, there is a main vertex w between the two intersections of π_1 with e such that $w \notin \pi_1$. That vertex is neither x nor y and belongs to J_1. By repeating the above arguments with w, we see that x, y also belong to another path, π_2, or π_3. This is a contradiction. □

Our next result describes minimal obstructions for 1-prism embedding extension problems. In cases (f)–(h) of Theorem 6.2, obstructions (and, in particular, 1-millipedes) are defined with respect to the following 2-prism embedding extension problem. Suppose that in $G - P_1$, there is no path from $C_1 - P_1$ to $C_2 - P_1$. Let P_2 be a path from $C_1 - P_1$ to a vertex x on P_1 such that x is as close as possible to C_2. Let y be the neighbor of x on P_1 that is closer to C_1 than x. Then let P_1' be the segment of P_1 from C_1 to y, and let C_2' be a cycle yx_1xx_2y where x_1 and x_2 are new vertices. We consider the 2-prism problem for the subgraph $K' = C_1 \cup C_2' \cup P_1' \cup P_2$ of the graph G' obtained from K' by adding all K–bridges in G with an attachment on $C_1 - P_1$. (In particular, no attachment on P_1 of these K–bridges is closer to C_2 than x.) In case (g) (and (h)) an additional edge xz is added into G'. The vertex $z \in V(P_1')$ has the property that in G there is a path internally disjoint from P_1 joining $C_2 - P_1$ with z. The 2-prism problems for (f)–(h) in case when P_2 joins $C_2 - P_1$ with P_1 are defined similarly. It should also be mentioned that the millipedes appearing in (h) are defined with respect to above 2-prism embedding extension problems.

Theorem 6.2 *Let G and $K = C_1 \cup C_2 \cup P_1 \subseteq G$ be graphs for a 1-prism embedding extension problem. Suppose that no K–component of G is attached just to P_1. Then there is no embedding extension to G if and only if $G - E(K)$ contains a subgraph Ω of one of the following types:*

(a) *Disjoint crossing paths with respect to the face of K.*

(b) *A tripod with respect to the face of K. If $G - P_1$ contains a path joining C_1 and C_2, then at least one attachment of the tripod is not on P_1.*

(c) *A path P_2 joining C_1 and C_2 and disjoint from P_1 together with a dipod attached three times to P_1 and once to $(C_1 \cup C_2) - (P_1 \cup P_2)$. If the dipod is degenerate, its degenerate attachment is not on P_1.*

(d) *A path P_2 from C_1 to C_2 disjoint from P_1 together with a 2-prism embedding extension obstruction of type (b)–(f) of Theorem 5.3 with respect to $K \cup P_2$. (It may happen that such an obstruction is not minimal. In this case, a part of P_2 can be removed.)*

(e) *A 1-millipede.*

(f) *Same as (d) but with P_2 joining $C_1 - P_1$ (or $C_2 - P_1$) with a vertex $x \in P_1$. No other attachment of the obstruction is closer to C_2 (respectively, to C_1) than x.*

(g) *Same as (f) where P_2 joins $C_1 - P_1$ with $x \in P_1$ (respectively, $C_2 - P_1$ with $x \in P_1$) where one of the branches of the obstruction joins x with another vertex $z \in P_1$. In Ω, this branch is replaced by a branch joining $C_2 - P_1$ (respectively, $C_1 - P_1$) with z.*

(h) *A one-sided 1-millipede attached to C_1 (or C_2) and to a segment of P_1. The path P_2 of the 1-millipede joins $C_1 - P_1$ (respectively, $C_2 - P_1$) with the attachment x on P_1 closest to C_2 (respectively, C_1). If the 1-millipede contains a branch joining x with another vertex z on P_1, then this branch is possibly replaced in Ω by a branch joining $C_2 - P_1$ (respectively, $C_1 - P_1$) with z.*

(i) *Union $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 \cap \Omega_2 \subseteq P_1$. For $i = 1, 2$, Ω_i contains a path π_i joining $C_i - P_1$ with P_1. The end x_2 of π_2 on P_1 is closer to C_1 than the end x_1 of π_1. Moreover, Ω_1 and Ω_2 are both left side, or both right side obstructions with respect to x_1 and x_2.*

(j) *A Kuratowski subgraph contained in a 3-connected component of the auxiliary graph $\tilde{G}$. The 3-connected component does not contain auxiliary vertices.*

Cases (f), (g), (h), and (i) appear only when in $G - P_1$ there is no path from C_1 to C_2. Moreover, there is a linear time algorithm that either finds an embedding extension of K to G, or returns an obstruction Ω for the embedding extendibility. In the latter case, Ω fits one of the above cases (a)–(j).

Proof. First of all, we can check the embedding extendibility by applying Theorem 4.3. Suppose now that there is no embedding extension of K to G. Consider the block(s) of G containing C_1 and C_2. If there is an embedding extension of K to this (these) block(s), then there is also an embedding extension to G, unless we have (j). If C_1 and C_2 are in different blocks of G, then there is no embedding extension if and only if one of them, say the one containing C_1, has no embedding extension in the disk with C_1 on the boundary. We leave this case to the end of the proof since we will reduce it to the 2-connected case. Suppose now that C_1 and C_2 are in the same block of G. To simplify notation, we will assume that this block is G itself, i.e., G is 2-connected. Then there is no embedding extension if and only if one of the 3-connected components of the cylinder auxiliary graph $\tilde{G}$ is non-planar. (See the details in the proof of Theorem 4.3, case $k = 2$.) If the 3-connected component(s) of $\tilde{G}$ containing the auxiliary vertices is (are) planar, then we have (j). Otherwise, let $\overline{G}$ be the graph obtained from $\tilde{G}$ by adding an edge between the auxiliary vertices. Since G is 2-connected, the cycles C_1 and C_2 and the auxiliary vertices will be in the same 3-connected component of $\overline{G}$. The other 3-connected components of this graph are also 3-connected components of $\tilde{G}$, and can thus be eliminated.

We assume from now on that $\overline{G}$ is 3-connected. First of all, we try to find two disjoint paths in $G - P_1$ joining C_1 and C_2. The search for such paths can be performed in linear time by standard flow techniques which were also used in our previous results. Suppose first that we have found such paths P_2 and P_3. If P_1, P_2, P_3 are not attached consistently on C_1 and C_2, then we have (a). Suppose now that this is not the case. Then we try to change P_2 and P_3 in such a way that there are no local bridges of $K \cup P_2 \cup P_3$ attached only to P_2 or only to P_3. To achieve this goal, we use the same technique as in the proof of Theorem 4.3. Let $K' = K \cup P_2 \cup P_3$. For $i = 2, 3$, let G_i be the graph consisting of P_i together with all its local K'–bridges and with an additional edge joining the ends of P_i. If G_i is planar, then an algorithm from [15] replaces P_i with a new path which has no local bridges attached to it. So, we either achieve our goal, or get one of G_i, say G_3, to be non-planar. Let C be the cycle composed of the paths P_1 and P_2 together with the segments on C_1 from P_1 to P_3 and from P_3 to P_2, and the segments on C_2 from P_1 to P_3 and from P_3 to P_2. Denote by B the $(K \cup P_2)$–component in G that contains P_3. If B contains a vertex of $(C_1 \cup C_2) - C$, then a path in B from that vertex to an end of P_3 together with P_2 determines an obstruction of type (a). Therefore we may assume that B is attached only to C. Since G_3 is non-planar, $C \cup B$ is non-planar. By Lemma 4.2, the auxiliary graph of $C \cup B$ with respect to C is 3-connected. By Theorem 3.1, we can find in B a pair of disjoint crossing paths or a tripod with respect to C. We can change the obtained obstruction as in the proof of Theorem 5.3 to get a 2-prism obstruction in the graph $H = K \cup P_2 \cup B$ with respect to its subgraph $K \cup P_2$. It follows from the proof of Theorem 5.3 that the only obstructions that appear in this case are types (b)–(e) of Theorem 5.3. Thus we have our case (d).

Suppose now that P_2 and P_3 do not have local K'–bridges. By Theorem 4.1, the obstructions to the non-extendibility of the embedding of K' to G are rather simple. Since $\overline{G}$ is 3-connected and in G there are three disjoint paths from C_1 to C_2, also the auxiliary graph $\tilde{G}$ of G is 3-connected. Therefore we need to consider only cases (a)–(d) of Theorem 4.1. Case (a) of Theorem 4.1 gives our case (a) or (d) (the latter one being case (d) of Theorem 5.3). Case (b) yields case (b) of Theorem 5.3, thus our case (d). (Here we need to be careful in selecting two paths among P_1, P_2, P_3 for which we get the 2-prism obstruction. We need to take P_1. The other path is P_3 if the tripod obstruction of case (b) is attached to P_2. Otherwise, we take P_2.) Case (c) of Theorem 4.1 yields case (c) of Theorem 5.3, thus our case (d). Finally, in case (d) of Theorem 4.1, we either have cases (b) or (d) of Theorem 5.3 (thus our case (d)), or we have a degenerate dipod attached three times to P_1 and disjoint from P_2, say. In the latter case, we get our possibility (c). (Note that (c) is contained in (d) if the dipod is non-degenerate.)

Next we suppose that there are no two paths P_2, P_3 as asked for above. Suppose that there is a path P_2 disjoint from P_1 joining C_1 and C_2. Let B be the K-bridge containing P_2. We will show that B either contains a subgraph of type (a), or (b), or else P_2 can be changed so that the local bridges on P_2 will disappear. In the latter case, we will be able to get an obstruction of type (a), (d), or (e).

For $i = 1, 2$, let S_i be the segment on C_i between the "leftmost" and the "rightmost" attachment of B to $C_i - P_1$. Let $B' = (B - P_1) \cup S_1 \cup S_2$. For $i = 1, 2$, if S_i is not just a vertex, let x_i be the end of P_2 on S_i. Otherwise, we add to B' a pendant edge attached to the vertex S_i, and we let x_i be the new vertex on this edge. Let $Q_0, Q_1, \ldots, Q_t$ be the shortest sequence of blocks of B' satisfying the following conditions:

(i) $x_1 \in V(Q_0)$, $x_2 \in V(Q_t)$, and

(ii) for $i = 1, 2, \ldots, t$, Q_{i-1} and Q_i intersect at a cutvertex w_i of B'.

By the minimality requirement for t, the blocks Q_i and the cutvertices w_i are all distinct and uniquely determined. We also define $w_0 = x_1$ and $w_{t+1} = x_2$. Let us mention that the sequence $w_0, Q_0, w_1, Q_1, \ldots, w_t, Q_t, w_{t+1}$ can be determined in linear time by standard biconnectivity algorithms. We also note that $w_1, w_2, \ldots, w_t$ are vertices of P_2. By our assumption, B' contains no two disjoint paths from C_1 to C_2. Thus, by Menger's Theorem, we have $t \geq 1$.

Suppose that one of the blocks Q_i $(1 \leq i < t)$ cannot be embedded in the plane with w_i and w_{i+1} on the boundary of the outer face. Let K_i be its subgraph obtained from the Kuratowski subgraph of $Q_i + w_i w_{i+1}$ by deleting its edge $w_i w_{i+1}$ if necessary. Next, find in G three disjoint paths π_1, π_2, π_3 from K to the main vertices of K_i (in linear time by using flow techniques). This is possible since $\overline{G}$ is 3-connected. We will show next that we can change the paths π_j in such a way that one of them will be attached to the end of P_2 on C_1 and one of them to the end of P_2 on C_2. Since Q_i is 2-connected, there are two disjoint paths in Q_i from $\{w_i, w_{i+1}\}$ to main vertices of K_i. Let π_1' and π_2' be obtained from these paths by adding a segment of P_2 from w_i to C_1 and from w_{i+1} to C_2, respectively. If π_1' is disjoint from π_1, π_2, π_3, it can replace π_1. Otherwise, suppose that π_1' first meets π_1 (in direction from C_1). Do the same with π_2': if it does not intersect any of the paths, it can replace π_2. If its first intersection from C_2 with $\pi_1 \cup \pi_2 \cup \pi_3$ is on π_2 (or similarly π_3), then we replace π_1 with the segment of π_1' up to its intersection with π_1 followed by the remaining segment of π_1, and we replace π_2 with a path consisting of the initial segment of π_2' and the terminal segment of π_2. Our goal for the paths to attach to C_1 and to C_2 is then satisfied. The remaining possibility is when π_2' first intersects π_1 as well as π_1' does. In this case, let $y_1, y_2, \ldots$ be the consecutive intersections of π_1' with $\pi_1 \cup \pi_2 \cup \pi_3$. Similarly, let $z_1, z_2, \ldots$ be the intersections of π_2' with the union of the paths. By our assumption, $y_1, z_1 \in V(\pi_1)$. Suppose that p, q are the largest indices so that all $y_1, \ldots, y_p$ and $z_1, \ldots, z_q$ belong to π_1. Let y be the vertex among $y_1, \ldots, y_p, z_1, \ldots, z_q$ which is closest to the end of π_1 in K'. Suppose that $y \in \pi_1'$. Replace now π_1 by the segment of π_1' till y and the segment of π_1 from y to its end at a main vertex of K_i. Now, π_2' either does not intersect the paths π_j at all, or intersects first a path distinct from π_1, and we can apply the above procedure to fulfil our task.

Now we have three disjoint paths π_1, π_2, π_3 joining K with three of the main vertices of K_i, where one of the paths starts at $C_1 \cap P_2$ and another one starts at $C_2 \cap P_2$. Note that these two paths pass through w_i and w_{i+1}, respectively. Let H be the graph obtained from K_i by adding the three paths and the cycle C obtained as follows. Let e_1 and e_2 be edges of C_1 and C_2, respectively, that are adjacent to P_1 (both at the same side of P_1 with respect to the given embedding of K in the cylinder). Then let C be the cycle obtained from $K - e_1 - e_2$ by adding a new edge between the two vertices of degree one. We can change π_1, π_2, π_3 so that the graph $H = C \cup K_i \cup \pi_1 \cup \pi_2 \cup \pi_3$ has no parallel branches. By Lemma 6.1, the auxiliary graph of H with respect to the cycle C is nodally 3-connected. By Theorem 3.1, H contains a tripod T (since there are only three attachments on C). By construction of H, the tripod T is also a tripod in G with respect to the face of K since it is attached twice to

$K - P_1$, and if the third attachment lies on P_1, it is non-degenerate. Thus we have our case (b).

Consider now Q_0 and suppose that it is non-trivial, i.e., S_1 is not just a vertex. Let p and q be the endpoints of the segment S_1, and let Q_0' be the graph obtained from Q_0 by adding the edges pw_1 and qw_1 to it. If Q_0' has no embedding in the plane so that the cycle $S_1 + pw_1 + w_1q$ bounds the outer face, then we can get an obstruction Ω_0 by using Theorem 3.1. If Ω_0 is attached to the cycle at w_1, then we add to it the segment on P_2 from w_1 to C_2. Now Ω_0 is either a pair of disjoint crossing paths with respect to the face of K (case (a)) or a tripod attached to $K - P_1$ (case (b)).

We perform the similar procedure with Q_t. Not having obtained an obstruction, we know that the graph $Q = Q_0 \cup Q_1 \cup \cdots \cup Q_t$ can be embedded in the face F of K. Since $P_2 \subseteq Q$, every $(K \cup P_2)$–bridge in G that is locally attached to P_2 is totally contained in Q. Now, since Q can be embedded in F, the algorithm of [15] enables us to remove the local bridges at P_2 in linear time. Note that also after a possible change of P_2 by another path in Q with the same endpoints, P_2 still passes through $w_1, \ldots, w_t$ and that for every $(K \cup P_2)$–bridge B, there is some i, $0 \le i \le t$, such that B is attached to P_2 only between w_i and w_{i+1}.

Now we can apply Theorem 5.3 for the subgraph $K' = K \cup P_2$ of G. We note that P_2 plus a 2-prism embedding obstruction of Theorem 5.3 is not necessarily a minimal obstruction for our embedding extension problem. Clearly, the deletion of any branches not in P_2 is not possible since we have a minimal obstruction for the embedding extension of K'. However, P_2 or a part of it may be superfluous in the obstruction and may then be omitted. Note that case (a) of Theorem 5.3 gives our case (a), and cases (b)–(f) give (d). Case (h) of Theorem 5.3 can be excluded because of our initial connectivity reductions. In the remaining case (g) of millipedes, we claim that we really get a 1-millipede. We need to show that the corresponding millipede for K' satisfies (6). In order to achieve this property, we change P_2 before applying Theorem 5.3 as explained in the next paragraph.

For $i = 0, 1, \ldots, t$, let Q_i' be K' together with all K'–bridges attached to the segment w_iw_{i+1} of P_2, except for those K'–bridges whose only attachment on P_2 is one of w_i, w_{i+1}. If the embedding of K' cannot be extended to the obtained subgraph Q_i' of G, we get an obstruction of type (a), (d), or a millipede. Having a 1-sided millipede, its length m is at most 4 (Proposition 5.4), and thus it is clear that it satisfies the required property (6). Two-sided millipedes are excluded since in Q_i', bridges of K' are either not attached to $C_1 - P_1$ (if $i \ne 0$), or are not attached to $C_2 - P_1$ (if $i \ne t$) since $t \ge 1$. Thus we may assume that we have an extension of the embedding of K' to Q_i'. Suppose first that $i \ne 0, t$. Consider the induced embedding of $Q_i \subseteq Q_i'$ and change the segment w_iw_{i+1} of P_2 to be the leftmost path in Q_i from w_i to w_{i+1}. After this change of P_2, there is just one bridge R_i attached to the right side of the segment (with respect to the embedding of Q_i') that has an attachment on P_1 since Q_i is 2-connected. Unfortunately, some local bridges attached at the right side of the segment may arise. In such a case, replace the obtained segment of P_2 by the rightmost path through such local bridges. It is easy to see that, because of the 3-connectivity, local bridges disappear after this change. On the right side of the new segment, the same bridge R_i remains as the only bridge on the right of it, while on the left, we can get more than one bridge. No two of the left bridges overlap. Moreover, every left bridge overlaps with R_i since R_i is attached to w_i and to w_{i+1}. We perform similar change for $i = 0, t$ (and possibly change the ends of P_2 on C_1 and C_2).

Suppose that, after the above change of P_2, we get a millipede M when applying Theorem 5.3. If M is one-sided, it satisfies (6) since $m \le 4$. If M is two-sided, then it can contain at most three bridges from Q_i'. If it contains two or three such bridges, R_i is among them. Suppose now that for some j, $3 \le j \le m-4$, $r_j^+ \preceq r_j^-$ (i.e., (6) is violated). Notation $\preceq$ means being closer to C_1 than to C_2 on P_1. Then $r_j^+ = r_j^-$. By (4), $r = r_j^+$ is the only attachment of $B_{j+1}^\circ \cup B_{j+2}^\circ$ on P_1. Since $B_j^\circ \preceq B_{j+2}^\circ$, B_{j+1}° overlaps with B_j° and with B_{j+2}° on P_2. This implies that $B_j^\circ, B_{j+1}^\circ, B_{j+2}^\circ$ are the three bridges of some Q_i' and that $B_{j+1}^\circ = R_i$. Since B_{j+2}° is attached only to $P_1 \cup P_2$, we have $j+2 < m$. Thus, there is a bridge B_{j+3}° overlapping with B_{j+2}° and not overlapping with B_{j+1}°. But this is, clearly, not possible. We have a contradiction, so $r_j^- \prec r_j^+$. The proof that $l_j^- \prec l_j^+$ is similar. Hence, (6) holds.

We have covered the case when, in $G - P_1$, there is a path from C_1 to C_2. Suppose now that this is not the case. Then the K-bridges can be partitioned into classes $\mathcal{B}_1, \mathcal{B}_2$ such that $\mathcal{B}_i$ ($i = 1, 2$) contains exactly those bridges that are attached to $C_i - P_1$. Let x_1 be the vertex of P_1 as close to C_2 as possible such that there is a bridge in $\mathcal{B}_1$ that is attached to x_1. (If none of the bridges in $\mathcal{B}_1$ is attached to P_1, we let x_1 be the end of P_1 on C_1.) Define similarly x_2 for the bridges in $\mathcal{B}_2$. For $i = 1, 2$, we let G_i be the graph consisting of C_i, the segment of P_1 from C_i to x_i, and the bridges in $\mathcal{B}_i$. We will use the same notation later when providing details for the case when C_1 and C_2 are in distinct blocks of G. Clearly, this is not the case if and only if on P_1, x_1 is strictly closer to C_2 than x_2.

Let π_1 be a path in G_1 joining C_1 with x_1 such that $\pi_1 \cap P_1 = \{x_1\}$. Define similarly π_2 in G_2. Suppose that we have an embedding extension of K to G. If π_1 is attached to x_1 at the left side of F (with the obvious meaning of the "left" with respect to Figure 2), then π_2 is attached at the right side. Then all the attachments of G_1 to P_1 at x_1 and between x_1 and x_2 are also on the left. Thus we say that G_1 has the *left side embedding* and, similarly, G_2 has the *right side embedding* with respect to x_1 and x_2. There are two possibilities for the non-existence of embedding extensions: either G_1 (or G_2) admits neither the left nor the right side embedding, or each of G_1 and G_2 does not admit the left side embedding (respectively, the right side embedding), but each of them admits the right (left) side embedding.

Suppose first that G_1 admits neither side embeddings. What are the possible obstructions? Define the graph G_1' as follows. Let $y \in V(P_1)$ be the neighbor of x_1 that is closer to C_1 than x_1. Replace the edge yx_1 of G_1 by a pair of paths of length two between these two vertices and denote by C_2' the obtained cycle of length 4. In addition to this, add the edge x_1x_2. It is easy to see that G_1 has neither side embeddings if and only if the obtained graph G_1' has no embedding extending the 1-prism embedding of $K' = C_1 \cup C_2' \cup (P_1 \cap G_1')$ with C_1 and C_2' on the boundary. (Note: the two embeddings of K' are really equivalent to each other.) This type of 1-prism embedding extension problem has been covered above — the case when there is a path π_1 disjoint from P_1 joining the two cycles. Since C_2' is joined to C_1 only through y and x_1, we have just one such path. It is easy to see that G_1' is 2-connected and that the auxiliary graph G_1' with the auxiliary vertices joined to each other is 3-connected (since there are no local bridges on P_1). As shown above, this gives rise to obstructions of types (a), (b), (d), or (e) for the extension problem of G_1'. The obtained obstruction possibly contains the new edge x_1x_2 which is not present in G. By replacing this edge by π_2, we get an obstruction Ω for our original 1-prism embedding extension problem. If Ω is of type (a) (with respect to G_1'), then it is easy to see that $\pi_2 \not\subseteq \Omega$ and that Ω fits our case (a) as well. If Ω is of type (b) then, again, π_2 is not a part of Ω. Thus Ω fits case (b). Note that in this case Ω can be a tripod attached only to P_1. In the case of type (d), the path appearing in (d) is π_1 (which may have been changed when constructing the obstruction). We have our case (f) or (g) with $P_2 = \pi_1$. In case (e) in G_1', note that Ω cannot be attached to $C_2' - (P_1 \cup P_2)$. Thus the corresponding 1-millipede is one-sided, and we have (h).

Up to symmetries, the only remaining case is when neither G_1 nor G_2 admits the left side embedding. We may as well assume that G_1 and G_2 admit the right side embeddings. Then we are looking for minimal left side obstructions. It is clear that we get case (i).

It remains to consider the case when C_1 and C_2 are in distinct blocks of G. Let G_1 and G_2 be the corresponding blocks. We suppose that G_1 has no embedding in the plane with C_1 bounding a face. An obstruction for this will obstruct the original embedding extension problem. Let us remark that using Theorem 3.1 is not a straightforward success since an obstruction obtained by using that result could intersect P_1 too many times. However, the application of Theorem 3.1 is possible if $G_1 \cap P_1$ is just the end of P_1 on C_1. Then the obstruction is either our case (a), or (b) (or (j)). Thus we assume that this is not the case.

Note that $G_1 \cap P_1$ is a segment of P_1 from C_1 to x_1. Let y be the neighbor of x_1 that is closer to P_1 than x_1. By the assumption made above, y is well-defined. Define G_1' to be the graph obtained from G_1 by replacing the edge yx_1 with two paths of length two between x_1 and y. Denote by C_2' the obtained cycle of length four consisting of these two paths. Clearly, G_1' has an embedding in the plane with C_1 and C_2' bounding faces if and only if G_1 has an embedding with C_1 bounding a face. By our assumption, this is not the case. Thus G_1' has no 1-prism embedding extension with respect to $K' = C_1 \cup C_2' \cup (P_1 \cap G_1)$. Possible obstructions have been classified above since in this problem we have a path disjoint from the first one. We get obstructions of types (a), (b), (f), or (h). □

Case (i) of the previous theorem is well-described if we know what are the minimal left side obstructions and how we get them in linear time. Their discovery was covered in the theorem. The proof of the last theorem was also detailed enough to yield a simple classification of these obstructions.

Theorem 6.3 *Let Ω be a minimal left side obstruction with respect to x_1 and x_2. Then Ω is one of the graphs shown in Figure 11 where case (c) represents an arbitrary two-sided millipede for the 2-prism embedding extension problem described before Lemma 6.1.*

Proof. We will use all the notation and assumptions introduced in the preceding proof up to the point where we encountered the case (i). We suppose that there is a left side obstruction in G_1.

Let P_1' be the segment of P_1 from C_1 to x_2, and let C_2' be the segment of P_1 from x_2 to x_1 together with an additional edge x_1x_2. (If x_2 is just preceding x_1 on P_1, then we add a path of length two in order not to get parallel edges.) Define G_1' to be the graph obtained from $C_1 \cup P_1' \cup C_2'$ by adding all bridges from $\mathcal{B}_1$. Then G_1 has no left side embedding if and only if G_1' has no embedding in the cylinder with C_1 and C_2' on the boundary (with C_1 at the bottom and C_2' with its new edge on the right side). Thus we are looking for 1-prism embedding extension obstructions in G_1' with respect to $K' = C_1 \cup C_2' \cup P_1'$. Since in $\mathcal{B}_1$ there is a bridge joining $C_1 - P_1'$ with $x_1 \in C_2' - P_1'$, we get the case with two or three disjoint paths from C_1 to C_2' (counting also the path P_1'). If there are three disjoint paths, they obstruct the right side embeddings of G_1. By applying Theorem 6.2, we get an obstruction Ω_1 of type (a), (b), (d), or (e) with respect to G_1' since these are the possible cases that arise when in addition to P_1', there is only one path. In cases (d) and (e), we may suppose that the corresponding path P_2 is π_1 (which has possibly been changed during the procedure of constructing the obstruction, but its end x_1 has reminded unchanged). We know, moreover, that Ω_1 has right side embeddings since G_1 has such an embedding in F. Let us now consider particular cases for the obtained left side obstruction.

If Ω_1 is of type (a) (in G_1') then we claim that the disjoint crossing paths Q_1, Q_2 can be changed in such a way that one of them is attached to x_1. If this is not the case already in Ω_1, add the path π_1. Note that each of Q_1, Q_2 is attached to $C_1 - P_1$ and to the segment of P_1 between x_2 and x_1 since otherwise they also obstruct the right side embeddings. If π_1 intersects Q_1 or Q_2 in an internal vertex, then we can change Q_1 or Q_2 so that one of them is attached to x_1. Otherwise, we can either replace one of the paths by π_1, or $\Omega_1 \cup \pi_1$ obstructs the right side embeddings. Hence the claim. Consequently, we have a left side obstruction represented in Figure 11(a).

If Ω_1 is of type (b), then it is also the right side obstruction. If it is of type (d), corresponding to one of the cases (b), (d), (e), or (f) of Theorem 5.3, it is a right side obstruction as well. Type (d), case (c) is a right side obstruction except in two cases. One of them is represented in Figure 11(b), while the other one contains case (a) of Figure 11 after removing the middle part of π_1. Finally, if Ω_1 is a millipede, it does not obstruct the right side embeddings if and only if it is two-sided. So, the last type of minimal left side obstruction is as claimed. □

7 Conclusion

There is an additional property that the millipedes may be assumed to have. This property is essential for our further applications of the results of this paper and will be stated in our last results.

An *extended millipede* (or an *extended 1-millipede*) is defined by the same conditions (1)–(4) (and (6)) as the millipede, but (5) is replaced by the requirement that B_i° $(1 \le i \le m)$ is an H–graph of a K–bridge in G. This assures, in particular, that B_i° $(2 \le i \le m-1)$ are attached to P_1 and to P_2, but we lose the minimality property of millipedes as obstructions.

For $\Omega \subseteq G - E(K)$ we define $b(\Omega)$ to be the number of branches of Ω where all vertices of attachment of Ω (including those of degree 2 in Ω) are considered to be main vertices of Ω.

Theorem 7.1 *Let $K \subseteq G$ be a subgraph for a 2-prism embedding extension problem. There is a linear time algorithm that either finds an embedding extension of K to G, or returns an obstruction $\Omega \subseteq G - E(K)$ for such extensions. In the latter case, the obstruction Ω satisfies one of the following conditions:*

(a) Ω is small, $b(\Omega) \le 20$, $K \cup \Omega$ has at most four K–bridges, and at most 8 vertices of Ω are on P_2.

(b) $\Omega = B_1^\circ \cup B_2^\circ \cup \cdots \cup B_m^\circ$ is an extended millipede of length $m \ge 5$. Then $b(\Omega) \le 5m$ and at most $2m$ vertices of Ω are on P_2. Moreover, if D $(\supseteq B_2^\circ \cup \cdots \cup B_{m-1}^\circ)$ is the union of all K–bridges in G that are attached only to $P_1 \cup P_2$, then there is an embedding extension of K to $K \cup D$.

Proof. Consider first the 2-prism problem for $K \cup D$. By applying Theorem 5.3, we either get an embedding extension to $K \cup D$ or a small obstruction since millipedes have attachments out of $P_1 \cup P_2$. In the latter case we have (a). Otherwise, we apply Theorem 5.3 again, this time for the original 2-prism problem. What we have gained, is that in the case of millipedes, we can guarantee the property stated in (b). The stated bounds follow from Theorem 5.3. Note that the case of millipedes having $m < 5$ is hidden in our case (a). □

Theorem 7.2 *Let $K = C_1 \cup C_2 \cup P_1 \subseteq G$ be a subgraph for a 1-prism embedding extension problem. Suppose, moreover, that there is a path in G disjoint from P_1 that joins C_1 and C_2. There is a linear time algorithm that either finds an embedding extension of K to G, or returns an obstruction $\Omega \subseteq G - E(K)$ for such extensions. In the latter case, the obstruction Ω satisfies one of the following conditions:*

(a) Ω is small, and $b(\Omega) \leq 29$.

(b) $\Omega = P_2 \cup B_1^\circ \cup B_2^\circ \cup \cdots \cup B_m^\circ$ is an extended 1-millipede of length $m \geq 5$. Then $b(\Omega) \leq 7m$. Moreover, if D ($\supseteq B_2^\circ \cup \cdots \cup B_{m-1}^\circ$) is the union of all $(K \cup P_2)$–bridges in G that are attached only to $P_1 \cup P_2$, then there is an embedding extension of K to $K \cup P_2 \cup D$.

Proof. Apply the algorithm described in the proof of Theorem 6.2 with the only difference being that instead of using Theorem 5.3 within that proof, we use Theorem 7.1 instead. The stated bounds on the branch size also follow from Theorem 7.1. □

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