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Coloring Graphs without Short Non-bounding Cycles

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It is shown that there is a constant c such that if G is a graph embedded in a surface of genus g (either orientable or non-orientable) and the length of a shortest non-bounding cycle of G is at least $c \log(g+1)$, then G is six-colorable. A similar result holds for three- and four-colorings under additional assumptions on the girth of G. © 1994 Academic Press, Inc.

1. INTRODUCTION

Graphs in this paper are finite, simple, and undirected. A *k-coloring* of a graph G is an assignment of "colors" 1, 2, ..., k to the vertices of G in such a way that adjacent vertices receive different colors. A graph is *k-colorable* if it admits a k-coloring. The *chromatic number* $c(G)$ of a graph G is the least integer k for which G is k-colorable. A *cycle* of a graph G is a connected two-regular subgraph of G.

Let S be a surface (closed, without boundary). It is well known [AH, He, RY] that graphs which can be embedded in S have bounded chromatic number. More precisely, if G is embedded in S, then

$$c(G) \leqslant \left\lfloor \frac{7+\sqrt{49-24\chi(S)}}{2} \right\rfloor, \tag{1}$$

where $\chi(S)$ is the Euler characteristic of S. If S is not the Klein bottle, then there are graphs (e.g., complete graphs) for which the equality holds in (1).

Albertson and Stromquist [AS] proved that if G is a graph embedded in the torus, C a shortest non-contractible cycle in G, and ω^* is the length of a shortest non-contractible cycle that is not homotopic to C, then if $\omega^* \geqslant 8$, G has a five-coloring. Their result was further extended by Hutchinson [H] who showed that if G has a two-cell embedding on an

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orientable surface of genus k, and G has a representation on the standard $4k$-gon representing the surface (with each side of unit length) such that every edge has length less than $\frac{1}{5}$, then G has a five-coloring. In each case the proof consists of applying the four-color theorem to suitable pieces of the surface. Their assumptions guarantee that the four-colorings can be patched together to form a five-coloring. Our results do not use the four-color theorem, nor explicitly construct a coloring. We show in particular that there is a constant c such that every graph embedded in a surface of genus g (either orientable, or non-orientable) and with the shortest non-bounding cycle of length at least $c \log(g+1)$ is six-colorable. The same condition, together with the additional assumption that the girth of G is at least four (respectively, six) guarantees that G can be four-colored (respectively, three-colored). It is surprising that the only "non-elementary" fact used in the proofs is the Euler's formula.

Our results improve and generalize results obtained by Cook [C], Kronk [K], Kronk and White [KW], and Woodburn [W]. They also generalize the result of Grötzsch and some of its improvements [Gro, Gru, SY] on the three-colorability of triangle-free planar graphs. The gap between the Grötzsch requirement of the girth being at least four, against our value of six, indicates the possibility of improvements of our Theorem 3.3. It should be noted, however, that our requirement on the length of a shortest non-bounding cycle being larger than $c \log(g+1)$ cannot be dropped (or replaced by a constant) since there are graphs with arbitrarily large girth and chromatic number [E].

After completing the manuscript of this paper it came to our attention that Thomassen [T] proved a five color theorem with similar hypotheses. Although his result is superior to ours, we point our that our results accomplish the Thomassen's five color theorem in various ways. For example, the result of [T] is proved only for orientable surfaces and needs the length of a shortest non-contractible cycle (which may be shorter than the length of a shortest non-bounding cycle) to be bounded below by an exponential function of the genus g. Moreover, our proofs need quite elementary techniques, and this fact sheds new light on the map coloring problems.

2. Short Non-bounding Cycles

Let S be a surface and let G be a graph embedded in S. A cycle C of G is *bounding* (resp. *non-bounding*) if it is bounding (resp. non-bounding) as a closed curve in S; i.e., $S - C$ is disconnected (resp. connected). Denote by $\text{nbd}(G)$ the length of a shortest non-bounding cycle in G. By elementary results of algebraic topology the bounding cycles of a two-cell embedded graph G in S lie in a subspace of the cycle space of G with codimension $2-\chi(S)$. It follows that two-cell embedded graphs always have non-bounding cycles (and so $0<\mathrm{nbd}(G)<\infty$) if S is not the two-sphere. More generally, an embedded graph (not necessarily two-cell) contains a non-bounding cycle if the embedding obtained from the given one by replacing each non-simply connected face by a union of discs (one for each boundary component of the face) is not a spherical embedding. In particular, a non-planar graph always contains non-bounding cycles.

LEMMA 2.1. *Let G be a graph embedded in a surface S of genus g. Denote by δ the minimal vertex degree of G. Suppose that one of the following conditions is satisfied*:

(a) $\delta \geqslant 6$ *and each vertex of degree six is either contained in a non-triangular face, or has a neighbor of degree seven or more.*

(b) $\delta \geqslant 4$, *the girth of G is at least four, and each vertex of degree four is either contained in a non-quadrangular face, or there is a vertex at distance at most two from this vertex whose degree is at least five.*

(c) $\delta \geqslant 3$, *the girth of G is at least six, and each vertex of degree three is either contained in a non-hexagonal face, or there is a vertex at distance at most three from this vertex whose degree is at least four.*

Then $\mathrm{nbd}(G) \leqslant c\log(g+1)$, *where c is a constant (independent of G and g).*

Proof. It is easy to see that a graph G satisfying (a), (b), or (c) is non-planar. Therefore it contains a non-bounding cycle.

For a vertex $v \in V(G)$ denote by

$$B_i = B_i(v) = \{u \in V(G) \mid \mathrm{dist}_G(u, v) \leqslant i\}, \qquad i = 0, 1, 2, ...,$$

and let G_i be the subgraph of G induced on B_i. Suppose that G_i contains a non-bounding cycle. Let C be a shortest one. Then C is isometric in G_i; i.e., the distance in G_i between any two vertices of C is equal to their distance on C. If, for example, vertices $x, y \in V(C)$ are joined by a path P whose length is smaller than the distance between x and y on C (and $E(P) \cap E(C) = \emptyset$, which we may assume), then each of the two cycles C_1, C_2 of $C \cup P$ different from C is shorter than C, and at least one of them is non-bounding since their sum is equal to C. Assume now that $4i+1 <$ $\mathrm{nbd}(G)$. Then G_i contains no non-bounding cycle. If there was one, say C, let x, y be diametrically opposite vertices on C. Since their distance from v is at most i, their distance in G_i is at most $2i$. Since C is isometric in G_i it follows that the length of C is at most $4i+1$ which is a contradiction with the choice of i.

The absence of non-bounding cycles in G_i implies that the induced embedding of G_i (obtained by replacing non-simply connected faces by discs) is spherical. Denote by V_i, E_i, and F_i the number of vertices, edges, and faces, respectively, of G_i under this embedding. Moreover, let $f_{i,j}$ $(j \geqslant 3)$ be the number of faces of G_i of size j, and let $v_{i,j}$ $(j \geqslant 1)$ be the number of vertices of G_i of degree j in G_i. Note that degrees in G_i of vertices of B_{i-1} are equal to the degrees in G. Clearly,

$$V_i = \sum_{j \geqslant 1} v_{i,j}, \qquad F_i = \sum_{j \geqslant 3} f_{i,j}, \tag{2}$$

and

$$2E_i = \sum_{j \geqslant 1} j v_{i,j} = \sum_{j \geqslant 3} j f_{i,j}. \tag{3}$$

Let us now prove the sufficiency of (a). Using the Euler's formula $V_i - E_i + F_i = 2$ for the spherical embedding of G_i and applying (2) and (3) we obtain

$$12 = (6V_i - 2E_i) - (4E_i - 6F_i) \tag{4}$$

$$= -\sum_{j \geqslant 1} (j-6) v_{i,j} - 2 \sum_{j \geqslant 4} (j-3) f_{i,j} \tag{5}$$

$$\leqslant 5 \sum_{j=1}^{5} v_{i,j} - \sum_{j \geqslant 7} (j-6) v_{i,j} - 2 \sum_{j \geqslant 4} (j-3) f_{i,j} \tag{6}$$

$$\leqslant 5 \sum_{j=1}^{5} v_{i,j} - \frac{1}{43} \sum_{j \geqslant 7} (6j+1) v_{i,j} - \frac{1}{2} \sum_{j \geqslant 4} j f_{i,j}. \tag{7}$$

In passing from (6) to (7) we used the inequalities $43(j-6) \geqslant 6j+1$ (if $j \geqslant 7$) and $4(j-3) \geqslant j$ (if $j \geqslant 4$). It follows that (7) is positive, and this implies

$$V_i - V_{i-1} \geqslant \sum_{j=1}^{5} v_{i,j} \tag{8}$$

$$> \frac{1}{215} \sum_{j \geqslant 7} v_{i,j} + \frac{1}{215} \sum_{j \geqslant 7} 6j v_{i,j} + \frac{1}{10} \sum_{j \geqslant 4} j f_{i,j} \tag{9}$$

$$\geqslant \frac{1}{215} V_{i-1} - \frac{1}{215} t + \frac{1}{215} \sum_{j \geqslant 7} 6j v_{i,j} + \frac{1}{10} \sum_{j \geqslant 4} j f_{i,j}, \tag{10}$$

where t is the number of vertices in B_{i-1} of degree six. If $u \in B_{i-1}$ is a vertex of degree six, let u' be an arbitrary neighbor of u in B_{i-2} (we assume that $i \geqslant 3$). Then we have the following three cases:

(a) $\deg(u') \geqslant 7$.

(b) $\deg(u') = 6$, all faces containing u' are triangles, but u' has a neighbor of degree seven or more.

(c) $\deg(u') = 6$ and u' lies on a face of size four or more. (Note that since $u' \in B_{i-2}$, it also lies on a non-triangular face in G_i.)

We say that u is of type a, b, or c, if u' has the above property (a), (b), or (c), respectively. Let t_a, t_b, t_c denote the number of vertices in B_{i-1} of degree six which are of type a, b, c, respectively. Every vertex in B_{i-1} of degree $j \geqslant 7$ corresponds (as the vertex u') to at most j vertices u of type a and at most $5j$ vertices u of type b. Therefore,

$$\sum_{j \geqslant 7} 6jv_{i,j} \geqslant t_a + t_b. \tag{11}$$

Similarly, every vertex $u' \in B_{i-2}$ of degree six and lying on the boundary of a non-triangular face of size $j \geqslant 4$ corresponds to at most six vertices of type c, and hence

$$\sum_{j \geqslant 4} jf_{i,j} \geqslant \frac{1}{6} t_c. \tag{12}$$

It follows by (11) and (12) that the last three terms in (10) sum to a non-negative number, therefore implying that

$$V_i \geqslant \frac{216}{215} V_{i-1}. \tag{13}$$

(The above proof applies only for $3 \leqslant i < \frac{1}{4}(\text{nbd}(G) - 1)$, but note that the case $i \leqslant 2$ is trivial.) We see that the growth of G is exponential as far as i is small enough to guarantee the spherical embedding of G_i. We have

$$V_i \geqslant \left(\frac{216}{215}\right)^i \tag{14}$$

as far as $i < \frac{1}{4}(\text{nbd}(G) - 1)$.

Similar (and even easier) calculation for G as done above for G_i, this time using the Euler's formula for the surface S, shows that

$$|V(G)| = O(g + 1). \tag{15}$$

Now (14) and (15) imply that $i = O(\log(g + 1))$ which gives the required result.

The sufficiency of (b) and (c) is shown in the same way. The details are left to the reader. ∎

Note. We intentionally left out the exact determination of the constant c since with some additional work one can significantly improve the growth estimate (14) and the bound (15). However, this would increase the length of the presentation without improving the logarithmic order of our bounds.

3. The Main Results

THEOREM 3.1. *There is a constant c such that every graph G embedded in a closed surface S such that $nbd(G) > c \log(genus(S) + 1)$ is 6-colorable.*

Proof. Suppose that the result does not hold for a graph G and the constant c of Lemma 2.1. If necessary, we take c large enough so that $c \log 2 > 3$. By Lemma 2.1 it suffices to show that a subgraph G' of G satisfies the condition (a) of the lemma. A short non-bounding cycle of G' is also a non-bounding cycle of G, giving a contradiction.

By our assumption G is not six-colorable. Therefore it contains a seven-critical subgraph G'; i.e., the chromatic number of G' is seven but each vertex deleted subgraph of G' is six-colorable. If v is a vertex of G' of degree five or less, then the graph $G' - v$ is not six-chromatic. Therefore the minimal vertex degree of G' is at least six. Consider a vertex v of degree six in G' and suppose that all the faces of G' at v are triangles. Denote by $v_1, ..., v_6$ the consecutive neighbors of v as they appear around v on the surface. Note that v_i and v_{i+1} ($1 \leqslant i \leqslant 6$, indices modulo 6) are adjacent. Suppose that v has no neighbor of large degree; i.e., vertices $v_1, ..., v_6$ have degree six. We claim that v_i and v_j are not adjacent if $i \not\equiv j \pm 1 \pmod 6$. If they were, the triangle $v_i v v_j$ either bounds (which is easily seen to be a contradiction to the fact that G' is critical) or is non-bounding (which contradicts our assumptions on the length of non-bounding cycles.

Consider now a six-coloring of $G' - v$. In every such coloring the vertices $v_1, ..., v_6$ use all six colors (otherwise we could extend the coloring to G'). It follows that among the neighbors of v_i ($1 \leqslant i \leqslant 6$) all five colors different from the color of v_i are used, each exactly once (otherwise we could re-color v_i). Therefore exchanging the colors of v_1 and v_2 gives rise to another six-coloring of $G' - v$. Since v_3 is not adjacent to v_1, the neighbors of v_3 no longer have all colors different from the color of v_3 (the previous color of v_2 is missing). We obtain a contradiction. It follows that v has a neighbor which has degree at least seven, and the proof is complete. ∎

Albertson and Stromquist [AS] asked for a similar bound as in our theorems in case of five-colorings. We partially answer their question by the following result:

THEOREM 3.2. *There is a constant c such that if G is any graph embedded in a closed surface S such that* $\mathrm{nbd}(G) > c \log(\mathrm{genus}(S)+1)$ *and the girth of G is at least four, then G has a four-coloring.*

Proof. As in the above proof we may consider a five-critical subgraph G' of G, and in view of Lemma 2.1 it suffices to show that G' satisfies property (b) of the lemma.

If v is a vertex of G' of degree three or less, then the graph $G'-v$ is not four-chromatic. Therefore the minimal vertex degree of G' is at least four. Consider a vertex v of degree four in G' and suppose that all faces of G' at v are quadrangles (note that there are no triangles by the assumption on the girth).

Denote by $v_1, ..., v_8$ the consecutive vertices on the link of v, where v_1, v_3, v_5, v_7 are the neighbors of v. Suppose that v has no neighbor and no second neighbor of degree more than four. So vertices $v_1, ..., v_8$ have degree four. Consider now a four-coloring of $G'-v$. In every such coloring the vertices v_1, v_3, v_5, v_7 use all four colors (otherwise we could extend the coloring to G'). It follows that among the neighbors of v_i $(i=1, 3, 5, 7)$ all three colors different from the color of v_i are used, each exactly once (otherwise we would re-color v_i). Consider now v_2. Besides v_1 and v_3 it has two other neighbors. Since v_2 cannot be re-colored without changing the colors at its neighbors (this would give rise to a re-coloring of v_1), either the color of v_1, or the color of v_3 appears only once among the neighbors of v_2. Therefore exchanging the colors of v_1 and v_2, or v_3 and v_2, gives rise to another four-coloring of $G'-v$. Since v_3 is not adjacent to v_1 (the girth is at least four), the neighbors of v_3 (or v_1) no longer have all colors different from its color (the previous color of v_2 is missing). We obtain a contradiction. ∎

THEOREM 3.3. *There is a constant c such that if G is any graph embedded in a closed surface S such that* $\mathrm{nbd}(G) > c \log(\mathrm{genus}(S)+1)$ *and the girth of G is at least six, then G has a three-coloring.*

Proof. Assuming G is not three-colorable, we apply Lemma 2.1(c) in its four-critical subgraph G'. All we have to show is that G' satisfies the assumptions of the lemma.

Suppose that G' does not satisfy property (c) of Lemma 2.1. Let v be a vertex of degree three in G' which belongs to hexagonal faces only and such that every vertex at distance at most three from v has degree three as well. Let H be a hexagonal face containing v and consider a three-coloring of $G'-v$ around H. Denote the vertices on H by v, v_1, v_2, v_3, v_4, v_5, respectively, and let u_i $(1 \leqslant i \leqslant 5)$ be the neighbor of v_i which is not on H (Fig. 1a). Note that since the girth of G is six, the vertices u_i are well defined, pairwise distinct, and not adjacent to v.

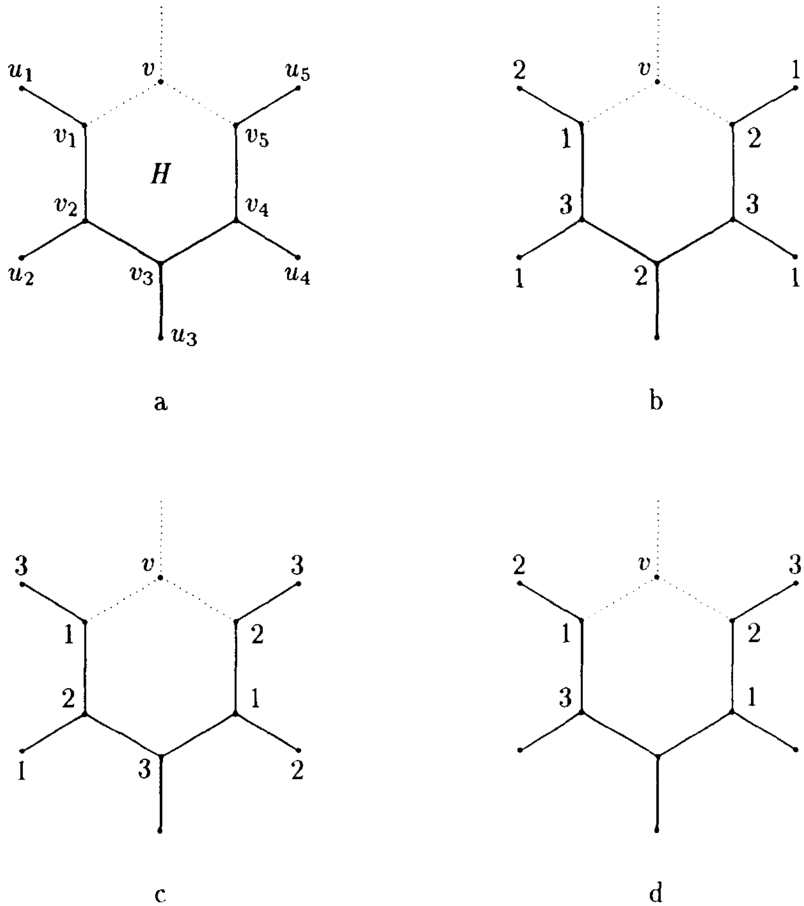

FIG. 1. Three-coloring around H.

Every three-coloring of $G' - v$ uses all three colors on the neighbors of v since it cannot be extended to G'. Assume that v_1 and v_5 are colored one and two, respectively. Then u_1 and v_2 must use both available colors (two and three) since otherwise we could re-color v_1. Similarly, v_4 and u_5 must use both one and three. Suppose first that u_1, v_2, v_4, u_5 are colored as in Fig. 1b. Then u_2 and v_3 cannot be both colored the same—both colored one gives a possibility to re-color v_2 using color two, and both colored two gives the possibility to exchange the colors of v_1 and v_2 without changing the coloring elsewhere, in each case contradicting the non-extendability of the three-coloring to G'. A similar conclusion holds for the neighbors of v_4. Therefore u_2, v_3, u_4 are colored one, two, one (or two, one, two), respectively. By the left-right symmetry we may assume one, two, one.

Consider now u_3. If it is colored three, we would re-color v_3 using color one. On the other hand, if the color of u_3 is one, we can exchange colors two, three on the vertices v_2, v_3, v_4, v_5. In each case we obtain a forbidden three-coloring.

The neighbors of v_1 and v_5 can also be colored as in Fig. 1c. Clearly, v_3 is colored three. Again in this case the neighbors u_2, v_3 of v_2 are colored differently—if both are colored three we could exchange the colors of v_1 and v_2. The same holds for the neighbors of v_4. Hence the coloring is as shown on Fig. 1c. Consider now u_3. If its color is one, then we could exchange the colors of v_2 and v_3. If u_3 is colored two we could exchange the colors of v_3 and v_4. In each case we obtain a contradiction.

Up to symmetries there is only one other possibility how to color the neighbors of v_1 and v_5 (Fig. 1d). But it is easy to see that in this case one of the hexagons containing v must belong to one of the previous cases (b), or (c). This completes the proof. ▮

References

[AS] M. O. Albertson and W. R. Stromquist, Locally planar toroidal graphs are 5-colorable, *Proc. Amer. Math. Soc.* **84** (1982), 449–457.

[AH] K. Appel and W. Haken, Every planar map is four-colorable, *Bull. Amer. Math. Soc.* **82** (1976), 711–712.

[C] R. J. Cook, Chromatic number and girth, *Period. Math. Hungar.* **6** (1975), 103–107.

[E] P. Erdős, Graph theory and probability, *Canad. J. Math.* **11** (1959), 34–38.

[Gro] H. Grötzsch, Zur Theorie der diskreten Gebielde. VII. Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel, *Wiss. Z. Martin-Luther Univ. Halle-Wittenberg Math.-Nat. Reihe* **8** (1958/59), 109–120.

[Gru] B. Grünbaum, Grötzsch's theorem on 3-colorings, *Michigan Math. J.* **10** (1963), 303–310.

[H] J. P. Hutchinson, A five-color theorem for graphs on surfaces, *Proc. Amer. Math. Soc.* **90** (1984), 497–504.

[He] P. J. Heawood, Map colour theorem, *Quart. J. Math.* **24** (1890), 332–338.

[K] H. V. Kronk, The chromatic number of triangle-free graphs, *in* "Graph Theory and Applications," Lecture Notes in Math., Vol. 309, pp. 179–181, Springer-Verlag, Berlin, 1972.

[KW] H. V. Kronk and A. T. White, A four-color theorem for toroidal graphs, *Proc. Amer. Math. Soc.* **34** (1972), 83–86.

[RY] G. Ringel and J. W. T. Youngs, Solution of the Heawood map coloring problem, *Proc. Nat. Acad. Sci. U.S.A* **60** (1968), 438–445.

[SY] R. Steinberg and D. H. Younger, Grötzsch's theorem for the projective plane, *Ars. Combin.* **28** (1989), 15–31.

[T] C. Thomassen, Five-coloring maps on surfaces, *J. Combin. Theory Ser. B* **59** (1993), 89–105.

[W] R. L. Woodburn, A 4-color theorem for the Klein bottle, *Discrete Math.* **76** (1989), 271–276.