Embedding graphs in an arbitrary surface in linear time

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Abstract

For an arbitrary fixed surface S, a linear time algorithm is presented that for a given graph G either finds an embedding of G in S or identifies a subgraph of G that is homeomorphic to a minimal forbidden subgraph for embeddability in S. A side result of the proof of the algorithm is that minimal forbidden subgraphs for embeddability in S cannot be arbitrarily large. This yields a constructive proof of the result of Robertson and Seymour that for each closed surface there are only finitely many minimal forbidden subgraphs. The results and methods of this paper can be used to solve more general embedding extension problems.

1 Introduction

There are well-known linear time algorithms to determine whether the given graph can be embedded in the plane (Hopcroft and Tarjan [11], Booth and Lueker [4]). Extensions of these algorithms return an embedding if the graph is planar [5], or exhibit a Kuratowski subgraph homeomorphic to K_5 or $K_{3,3}$ [16, 30, 31]. Recently, linear time algorithms have been devised for embedding graphs in the projective plane (Mohar [17]) and in the torus (Juvan, Marinček, and Mohar [14]).

It is known that the problem of determining the genus or the non-orientable genus of graphs is NP-hard [28, 29]. However, for every fixed surface S there is a polynomial time algorithm for checking embeddability of graphs in S. If S is orientable of genus g, Filotti et al. [8] give an algorithm with time complexity $O(n^{\alpha g+\beta})$ (α, β are constants). For every fixed surface S, an $O(n^3)$ algorithm for testing embeddability in S can be devised using graph minors [21, 25] (with a recent improvement to $O(n^2 \log n)$ [22, 23, 24]). An extension which also constructs an embedding is described by Archdeacon in [1] with running time estimated to $O(n^{10})$.

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These algorithms use the lists of forbidden minors which are known only for the plane and the projective plane while for other surfaces only finiteness of their number has been proved. Djidjev and Reif [7] presented another polynomial time algorithm for embedding graphs in general surfaces which seems to avoid the use of forbidden minors.

In the present contribution we describe a linear time algorithm (that is not based on graph minors) which finds an embedding of a given graph G into an arbitrary fixed surface S if such an embedding exists. Otherwise, the algorithm returns a minimal forbidden subgraph $H \subseteq G$ for embeddability in S. A side result of the algorithm is that H is homeomorphic to a graph with a bounded number of edges (where the bound depends only on S). This yields a constructive proof of the result of Robertson and Seymour [21] that for each closed surface there are only finitely many minimal forbidden subgraphs. A constructive proof for nonorientable surfaces has been published by Archdeacon and Huneke [2], while orientable surfaces resisted all previous attempts. (Recently also Seymour [26] found a constructive proof of that result.)

The results and methods of this paper can be used towards solving a generalization of problems of embedding graphs in surfaces — the so called *embedding extension problems* where one has a fixed embedding of a subgraph K of G in some surface and asks for embedding extensions to G or (minimal) obstructions for existence of such extensions.

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for operations on integers, whose value is $O(n)$, n being the size of the input (cf. Cook and Reckhow [6]). The same model of computation is used in known linear time planarity testing algorithms [11].

2 Basic definitions

We will consider 2-cell embeddings of graphs in closed surfaces. They can be described in a purely combinatorial way by specifying a *rotation system* $\pi = (\pi_v \,;\, v \in V(G))$ (where π_v is a cyclic permutation of edges incident with v) and a *signature* $\lambda : E(G) \to \{-1, 1\}$. The reader is referred to [9] or [20] for more details. Having an embedding Π of G, we say that G is Π-*embedded*. Each embedding Π of G determines a set of closed walks in G, called Π-*facial walks* or simply Π-*faces*, that correspond to traversals of face boundaries of the corresponding topological embedding. Suppose that a subgraph K of G is Π-embedded. An embedding $\tilde{\Pi}$

of G is an *extension* of Π if it is an embedding in the same surface as Π and the restriction of $\tilde{\Pi}$ to K is equal to Π.

Let K be a subgraph of G. A K-*bridge* in G is a subgraph of G which is either an edge $e \in E(G)\backslash E(K)$ with both endpoints in K, or it is a connected component of $G - V(K)$ together with all edges between this component and K. The vertices of $B \cap K$ are the *vertices of attachment* of B, and B is *attached to* each of these vertices. A vertex of K of degree different from 2 is a *branch vertex* of K. A *branch* of K is any path in K whose endpoints are branch vertices but no internal vertex on this path is a branch vertex. B is *local* (on e) if it is attached to a single branch e of K. The number of branches of K, denoted by $\mathrm{bsize}(K)$, is the *branch size* of K. If B is a K-bridge in G, then the *size* $\mathrm{bsize}_K(B)$ of B is the number of branches of $K \cup B$ that are contained in B. A *basic piece* of K is either a branch vertex or an *open branch* of K (i.e., a branch with its endpoints removed). If B is attached to three or more basic pieces, then B is *strongly attached*.

Suppose that K is Π-embedded. Let B be a K-bridge in G and $\tilde{\Pi}$ an extension of Π to $K \cup B$. Then there is a unique Π-face F that is not a $\tilde{\Pi}$-face, and we say that B is *embedded in* F. Each basic piece on F has one or more *appearances* (or *occurrences*) on F. The embedding of B in F is *simple* if B is not attached to distinct appearances of the same basic piece. More generally, an embedding extension is *simple* if all bridges have simple embeddings.

The difficult part of this paper is to discover obstructions for simple embedding extensions. The next result slightly simplifies this problem. It implies that we may replace every K-bridge by a small subgraph and then consider only obstructions that can be expressed as the union of entire bridges.

Theorem 2.1 (Mohar [19]) *Let $\mathcal{B}$ be the set of K-bridges in G. There is a number* c *depending only on* $\mathrm{bsize}(K)$ *such that each $B \in \mathcal{B}$ contains a subgraph $\tilde{B}$, called an* E-graph *of B, such that* $\mathrm{bsize}_K(\tilde{B}) \le$ c *and that the following holds. If $\{B_1, \dots, B_k\} \subseteq \mathcal{B}$ ($k \ge 1$) are arbitrary nonlocal K-bridges, $\tilde{B}_1, \dots, \tilde{B}_k$ their E-graphs, and if Π is an embedding of K, then any simple extension of Π to $K \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_k$ can be further extended to a simple extension of Π to $K \cup B_1 \cup \cdots \cup B_k$. Moreover, there is a linear time algorithm that replaces all K-bridges B in G with their E-graphs $\tilde{B}$.*

3 Restricted embedding extensions

If B is a K-bridge and T is the set of basic pieces of K that B is attached to, then B is of *type* T. Suppose that K is Π-embedded. In general, a bridge of type T can be embedded in two or more Π-faces, and in some faces in several different ways. To formalize the essentially different ways of embedding bridges in particular faces, we introduce the notion of embedding schemes. Let F be a Π-face. For a type T, let $\pi_1, \dots, \pi_k$ be the appearances of basic pieces from T on F. An *embedding scheme* for the type T in the face F is a subset δ of $\pi_1, \dots, \pi_k$ containing at least one appearance of each basic piece from T. δ is *simple* if each $x \in T$ has exactly one appearance in δ. There is a natural partial ordering $\preceq$ among embedding schemes induced by the set inclusion. An embedding of B in F is *δ-compatible* (shortly a *δ-embedding*) if B is attached only to appearances of basic pieces from δ.

An *embedding distribution* $\Delta(T)$ *for a type* T is a selection of embedding schemes for T, possibly in different faces. Suppose that $T_1, T_2, \dots, T_s$ are all types of K-bridges in G. An *embedding distribution* is a family $\Delta = \{\Delta(T_1), \dots, \Delta(T_s)\}$ where $\Delta(T_i)$ is an embedding distribution for the type T_i, $1 \le i \le s$. Δ is *simple* if all embedding schemes in $\Delta(T_1), \dots, \Delta(T_s)$ are simple. Let $\mathcal{B}$ be a set of K-bridges with an embedding extending the given embedding of K. We say that the embedding of $\mathcal{B}$ is *Δ-compatible* (or a *Δ-embedding*) if the embedding of each bridge $B \in \mathcal{B}$ is δ-compatible for some $\delta \in \Delta(T)$, where T is the type of B. The relation $\preceq$ naturally extends from embedding schemes to embedding distributions.

Now we introduce a formal definition of an *embedding extension problem*, abbreviated EEP. This is a quadruple $\Xi = (G, K, \Pi, \Delta)$ where G is a graph, K is a subgraph of G, Π is an embedding of K, and Δ is an embedding distribution for the K-bridges in G. The EEP is *simple* if Δ is simple. An *embedding extension* (abbreviated EE) for Ξ is an embedding extension of Π to G such that every K-bridge is Δ-embedded. An *obstruction* for Ξ is a set $\mathcal{B}$ of K-bridges or their subgraphs such that $(K \cup \mathcal{B}, K, \Pi, \Delta)$ admits no EE. The *size* $\mathrm{bsize}_K(\mathcal{B})$ of an obstruction $\mathcal{B}$ is equal to the sum of sizes of bridges in $\mathcal{B}$.

Embedding distributions are used in the following way. For every possible embedding distribution Δ we try to extend the given embedding of K to a Δ-embedding of G. Embedding distributions are selected one after another respecting the order $\preceq$. We start with the empty embedding distribution. Any bridge is an obstruction for this subproblem. In a general step, we already have obstructions for all embedding distributions $\Delta' \prec \Delta$. Let $\mathcal{B}$ denote their union. Then we try to extend each Δ-embedding of $\mathcal{B}$ to a Δ-embedding of G. Obtaining an embedding, we stop and return the embedding. Otherwise, an obstruction is obtained. Finally, obstructions for different embeddings of $\mathcal{B}$ are combined together with $\mathcal{B}$ into a single obstruction for Δ-compatible embedding extensions. We will refer to this process as the *procedure of embedding distribution of types*.

The main difficulty in the above procedure is in bounding the number of Δ-embeddings of $\mathcal{B}$. By using an operation called *compression* (cf. Section 4), we are able to achieve that all obstructions have bounded size and hence also bounded number of embeddings.

The procedure of embedding distribution of types can be generalized by introducing the union of EEPs. Suppose that we want to consider embedding extensions where we fix embeddings of some of the bridges. To formalize, we call an EEP $\Xi' = (G, K', \Pi', \Delta')$ a *subproblem* of $\Xi = (G, K, \Pi, \Delta)$ if $K' = K \cup \mathcal{B}$ for a set $\mathcal{B}$ of K-bridges, Π' is a Δ-compatible EE of Π, and Δ'-compatibility in K' corresponds to Δ-compatibility in K.

For $i = 1, \dots, N$, let $\Xi_i = (G, K_i, \Pi_i, \Delta_i)$ be subproblems of $\Xi = (G, K, \Pi, \Delta)$. Denote by $\mathcal{B}_i$ the set of K-bridges in K_i. We say that Ξ is the *union* of subproblems Ξ_i ($1 \le i \le N$) if for every set $\mathcal{B} \supseteq \cup_{i=1}^N \mathcal{B}_i$ of K-bridges in G, the restriction of Ξ to $K \cup \mathcal{B}$ admits an EE exactly when the restriction to $K \cup \mathcal{B}$ of at least one of Ξ_i does. In this case, an EE for some Ξ_i is also an EE for Ξ, while having obstructions Ω_i for Ξ_i ($1 \le i \le N$), their combination $\Omega = \cup_{i=1}^N (\Omega_i \cup \mathcal{B}_i)$ is an obstruction for Ξ.

A subproblem $\Xi' = (G, K, \Pi, \Delta')$ of $\Xi = (G, K, \Pi, \Delta)$ is *equivalent* to Ξ if for every set $\mathcal{B}$ of K-bridges in G and every Δ-compatible EE of K to $K \cup \mathcal{B}$, there is also a Δ'-

compatible EE of K to $K \cup \mathcal{B}$.

Let $\Xi = (G, K, \Pi, \Delta)$ be an EEP. Let $\mathcal{B}$ be the set of all K-bridges in G. Suppose that $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_N$. Denote by Δ_i the restriction of Δ to $\mathcal{B}_i$, $i = 1, \ldots, N$. The EEP $\Xi_i = (K \cup \mathcal{B}_i, K, \Pi, \Delta_i)$ is a *partial problem* of Ξ. We say that Ξ is the *intersection* of partial problems Ξ_i, $i = 1, \ldots, N$, if arbitrary EEs for $\Xi_1, \ldots, \Xi_N$ determine an EE Π_0 for Ξ. More precisely, if there are EEs Π_i for Ξ_i ($i = 1, \ldots, N$), then there is an EE Π_0 for Ξ such that its restriction to $K \cup \mathcal{B}_i$ coincides with Π_i, $i = 1, \ldots, N$. Having $\Pi_1, \ldots, \Pi_N$, one can determine Π_0 in linear time.

4 Simple embedding extensions

In this section we consider simple EEPs. We additionally assume that each bridge has been replaced by its small E-graph (cf. Theorem 2.1), every bridge has at least one simple embedding extending some embedding of K, there are no local bridges, multiple branches between the same pair of vertices of K have been replaced by a single one, and there are at most $4\,\mathrm{bsize}(K)$ strongly attached bridges. We refer to these assumptions as *Property (E)* of K.

Let $\Xi = (G, K, \Pi, \Delta)$ be a simple EEP where K has Property (E). Suppose that $\mathcal{B}$ is a set of K-bridges and $\Xi' = (G, K \cup \mathcal{B}, \Pi', \Delta')$ is a subproblem of Ξ. Then Ξ' is *2-restricted* if every K-bridge B in G, $B \notin \mathcal{B}$, has at most two Δ'-compatible embeddings extending the embedding Π'.

Suppose that we have a set of vertices $W_0 \subseteq V(K)$. Let W_1 be the union of W_0 and all branch vertices of K. Denote by $\mathcal{S}$ the set of connected components of $K - W_1$. Suppose that we replace the paths in $\mathcal{S}$ by new pairwise disjoint paths in $G - W_1$ joining the same ends as the original paths. Then the new subgraph K' of G is homeomorphic to K and the homeomorphism $K \to K'$ is the identity on the stars of vertices in W_1. The types of bridges with respect to K and K' are in the obvious correspondence and so are the embeddings of K and K' and the embedding schemes for their bridges. Suppose that G contains exactly the same types of K-bridges and K'-bridges. Then the replacement of K by K' is called a *compression* with respect to W_0.

Theorem 4.1 (Juvan and Mohar [15]) *There is a function $c_1 : N \times N \to N$ such that the following holds. Let $\Xi = (G, K, \Pi, \Delta)$ be a 2-restricted subproblem of an EEP, and let W_0 be a set of vertices of K. If there is no Δ-compatible EE, then there is a compression $K \mapsto K'$ with respect to W_0 such that the modified EEP $\Xi' = (G, K', \Pi, \Delta)$ admits an obstruction $\mathcal{B}$ such that*

$$\mathrm{bsize}_{K'}(\mathcal{B}) \leq c_1(|W_0|, \mathrm{bsize}(K)).$$

Moreover, there is an algorithm with time complexity $O(c_1(|W_0|, \mathrm{bsize}(K))\,|V(G)|)$ that either finds an EE for Ξ, or performs the compression $K \mapsto K'$ and returns an obstruction $\mathcal{B}$ for Ξ' as described above.

The compression combined with the procedure of embedding distribution of types is our main tool used in order to guarantee that the obstructions constructed by our algorithms are not too large.

Suppose that we have an EEP $\Xi = (G, K, \Pi, \Delta)$ and that $\mathcal{B}$ is an obstruction for all EEPs $\Xi' = (G, K, \Pi, \Delta')$ for which $\Delta' \prec \Delta$. Then we say that $\mathcal{B}$ is a *complete set of representatives* for Ξ since for every Δ-embedding of $\mathcal{B}$ and each $\delta \in \Delta(T)$, there is a δ-embedded bridge of type T (called a *representative* for δ).

The next result enables us to apply Theorem 4.1 in solving general simple EEPs.

Theorem 4.2 *Let K be a subgraph of G with Property (E). Let $\Xi = (G, K, \Pi, \Delta)$ be a simple EEP and suppose that no edge of K appears on a Π-facial walk twice in the same direction. Suppose that $\mathcal{B}_0$ is a complete set of representatives for Ξ and that $K \cup \mathcal{B}_0$ also has Property (E). Then there is a number c_3 depending only on $\mathrm{bsize}(K \cup \mathcal{B}_0)$ such that each subproblem $\Xi_0 = (G, K \cup \mathcal{B}_0, \Pi_0, \Delta_0)$ of Ξ is equivalent to the union of at most c_3 EE subproblems each of which is the intersection of at most $\mathrm{bsize}(K)/2 + 1$ partial problems each of which is the union of a bounded number of 2-restricted subproblems. The decompositions of Ξ_0 to subproblems, of these to corresponding partial problems and these to subproblems can be done in $O(c_3|V(G)|)$ time.*

Proof. (Sketch) Let $\mathcal{B}_0'$ be the set of K-bridges consisting of $\mathcal{B}_0$, all strongly attached $(K \cup \mathcal{B}_0)$-bridges, and all bridges $\mathcal{B}_{x,y}^{\circ}$, where x, y are arbitrary basic pieces of $K \cup \mathcal{B}_0$, and bridges $\mathcal{B}_{x,y}^{\circ}$ are defined as follows. Let $\mathcal{B}_{x,y}$ be the set of K-bridges in G of type $T = \{x, y\}$. If x is a branch vertex, put $x_1 = x_2 = x$. If x is an open branch, let x_1 and x_2 be vertices of attachment of bridges in $\mathcal{B}_{x,y}$ that are as close as possible to one and the other end of x, respectively. Define similarly y_1 and y_2. For $i, j \in \{1, 2\}$, we select a bridge $B_{x,y}^{i,j} \in \mathcal{B}_{x,y}$ attached to x_i that has an attachment on y as close to y_j as possible. Then $\mathcal{B}_{x,y}^{\circ}$ contains all bridges $B_{x,y}^{i,j}$ ($i, j \in \{1, 2\}$) and for each $\delta \in \Delta(T)$ such that $\mathcal{B}_{x,y}$ has no δ-embedding, $\mathcal{B}_{x,y}^{\circ}$ contains a pair of bridges from $\mathcal{B}_{x,y}$ whose δ-embeddings overlap.

Ξ_0 is the union of subproblems $\Xi' = (G, K \cup \mathcal{B}_0', \Pi', \Delta')$ taken over all Δ_0-embeddings of $\mathcal{B}_0' \backslash \mathcal{B}_0$ extending Π_0, and every 2-restricted type of $(K \cup \mathcal{B}_0)$-bridges in Ξ_0 has its representatives for embedding schemes in Δ'. It suffices to see that every such subproblem Ξ' satisfies the conclusions of the theorem.

First, we prove that Ξ' is equivalent to the union of a bounded number of subproblems of the form $\Xi'' = (G, K \cup \mathcal{B}_0'', \Pi'', \Delta'')$ where $\mathcal{B}_0''$ consists of $\mathcal{B}_0'$ and some additional bridges whose number is bounded. The proof, which also yields a linear time procedure for determining subproblems Ξ'', is rather complicated. The basic idea is to add one or two bridges for each type T of K-bridges in G such that every Δ-embedding of the extended graph K restricts embeddings of bridges of type T (possibly taking an equivalent subproblem) so that at most two embeddings are $\Delta''(T)$-compatible. The cases when T does not consist of two open branches are not difficult. However, the cases when T consists of open branches (say e, f) need very careful treatment. In most cases we can show that the extended set $\mathcal{B}_0''$ of bridges can be chosen so that it removes the double $\{e, f\}$-singularity (if it occurs at all). If the Π''-embedded bridges $\mathcal{B}_0''$ do not remove the double $\{e, f\}$-singularity, then $\{e, f\}$ is a *corner pair* for Ξ''. Since $\mathcal{B}_0$ is a complete set of representatives, there are at most $\mathrm{bsize}(K)/2$ corner pairs.

If $\{e, f\}$ is a corner pair, let $\mathcal{B}_1^{e,f}$ be the set of K-bridges in G of type $\{e, f\}$ that are not in $\mathcal{B}_0''$. Let $\mathcal{B}_2$ be the set of K-bridges that are not in $\mathcal{B}_0''$ and that are not in $\mathcal{B}_1^{e,f}$ for any corner pair $\{e, f\}$. Furthermore, let $\mathcal{B}_2^{e,f}$ contain all K-bridges from $\mathcal{B}_2$ that have an attachment on e or f and have at most one Δ''-embedding extending the embedding Π'' of $K \cup \mathcal{B}_0''$. Similarly, let $\mathcal{B}_1$ contain those bridges from $\mathcal{B}_1^{e,f}$, taken over all corner pairs $\{e, f\}$, which have at most one Δ''-embedding extending Π''. Consider the EEPs

$$\Xi_1^{e,f} = (K \cup \mathcal{B}_0'' \cup \mathcal{B}_1^{e,f} \cup \mathcal{B}_2^{e,f}, K \cup \mathcal{B}_0'', \Pi'', \Delta_1^{e,f})$$

where $\{e, f\}$ is a corner pair and $\Delta_1^{e,f}$ is the restriction of Δ'' to $\mathcal{B}_1^{e,f} \cup \mathcal{B}_2^{e,f}$. Let

$$\Xi_2 = (K \cup \mathcal{B}_0'' \cup \mathcal{B}_1 \cup \mathcal{B}_2, K \cup \mathcal{B}_0'', \Pi'', \Delta_2)$$

be the partial problem of Ξ'' restricted to $\mathcal{B}_1 \cup \mathcal{B}_2$. The next difficult step is to show that Ξ'' is the intersection of partial problems $\Xi_1^{e,f}$ (taken over all corner pairs $\{e, f\}$) and the 2-restricted EEP Ξ_2. This part of the proof is also rather complicated, but it does not add any difficulties to the algorithm.

Finally, each problem $\Xi_1^{e,f}$ is the union of a bounded number of 2-restricted EEPs. This is proved in a separate, rather difficult work by Marinček, Juvan, and Mohar [13] where also the corresponding linear time algorithm is presented. □

Corollary 4.3 *Let $\Xi = (G, K, \Pi, \Delta)$ be a simple EEP and let W_0 be a subset of vertices of K. Let d be the total number of embedding schemes in the embedding distributions $\Delta(T)$ in Δ. Suppose that K has Property (E) and that no edge of K appears on a Π-facial walk twice in the same direction. There is a function $c : N \times N \to N$ and an algorithm with time complexity $O(c(|W_0|, d)|V(G)|)$ that either finds a Δ-compatible EE or returns a subgraph K' of G obtained by a compression with respect to W_0 and a set of at most $c(|W_0|, d)$ E-graphs of K'-bridges in G that form an obstruction for the corresponding EEP $\Xi' = (G, K', \Pi, \Delta)$.*

Proof. The proof is by induction on d. Case $d = 0$ is trivial. Otherwise, let $\Delta_1, \ldots, \Delta_d$ be the embedding distributions that are strictly simpler than Δ and are maximal with this property. Inductively, we first solve the subproblem $\Xi_1 = (G, K, \Pi, \Delta_1)$ taking care of the set W_0. An EE makes us happy and we stop. Otherwise, we compress K with respect to W_0. Let K_1 be the new subgraph of G and $\mathcal{B}_1$ an obstruction of bounded size as guaranteed by the induction hypothesis. Let W_1 be the union of W_0 and the set of vertices of attachment of bridges from $\mathcal{B}_1$. Now we replace W_0 by W_1 and solve the subproblem $\Xi_2 = (G, K_1, \Pi, \Delta_2)$, taking care of the set W_1. We either stop, or we get a new graph K_2 (after a compression with respect to W_1) and an obstruction $\mathcal{B}_2$ of bounded size. In the latter case we extend W_1 into W_2 by adding all attachments of bridges from $\mathcal{B}_2$. Continuing, we either find an EE, which is a Δ-embedding as well, or we stop after d steps with a subgraph K' of K that is a compression of K with respect to W_0. We also get an obstruction $\mathcal{B}_0 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots$. Now, $\mathcal{B}_0$ is a complete set of representatives for Ξ. Since Ξ is the union of subproblems, taken over all Δ-embeddings of $\mathcal{B}_0$, and since $\mathcal{B}_0$ has bounded size, we can consecutively apply Theorem 4.2 combined with Theorem 4.1, and for each of these subproblems perform a compression with respect to attachments of E-graphs in all previously obtained obstructions. An upper bound on $c(|W_0|, d)$ is easily established. □

5 Embedding graphs in an arbitrary surface

In this section we prove the final result of this paper that embeddability of graphs in any fixed surface S can be decided in linear time. Our algorithm also constructs an embedding (if it exists), or identifies a minimal subgraph of G that cannot be embedded in S. Such a subgraph is called a *minimal forbidden subgraph* for embeddability in S. We define the *Euler genus* of S as $2 - \chi(S)$ where $\chi(S)$ is the Euler characteristic of S.

Theorem 5.1 *Let S be a fixed closed surface. There is a constant c and a linear time algorithm that for an arbitrary given graph G either:*

(a) finds an embedding of G in S, or

(b) identifies a minimal forbidden subgraph $K \subseteq G$ for embeddability in S whose branch size is bounded by c.

A corollary of Theorem 5.1 is the result of Robertson and Seymour [21] that the set of minimal forbidden minors (or subgraphs) is finite for each surface. It is worth mentioning that our proof is constructive while the proof in [21] is only existential.

Corollary 5.2 (Robertson and Seymour [21]) *For every surface S there is a finite list of graphs such that an arbitrary graph G can be embedded in S if and only if G does not contain a subgraph homeomorphic to one of the graphs in the list.*

The rest of the paper is the sketch of the proof of Theorem 5.1. Let us point out that in case (b) it suffices to find a subgraph K of bounded branch size (in terms of the Euler genus g of S) since such a subgraph is easily changed to a minimal one in constant time.

The orientable genus of G is equal to the sum of the genera of its blocks [3]. A similar result holds for the nonorientable genus [27]. Since the blocks can be determined in linear time, this enables us to reduce the problem to 2-connected graphs.

A reduction to 3-connected graphs is not possible. However, we can achieve that G is "almost 3-connected" (called 3-connected modulo K in [12]). This step needs more work. It also uses the 3-connectivity algorithm of Hopcroft and Tarjan [10]. After this step we end up with a 2-connected graph K which can be chosen so that there are no local K-bridges [12], or we stop by obtaining (b).

The algorithm continues by induction on the (Euler) genus g of S. Recursively, we have either found an embedding in a surface of (Euler) genus smaller than g (in which case we stop), or we got a 2-connected subgraph K of G that cannot be embedded in any surface with (Euler) genus smaller than g. By the induction hypothesis bsize(K) is bounded. Therefore, K has only a bounded number of embeddings in S and each of them is 2-cell. Existence of an embedding of G in S is thus equivalent to the existence of an EE with respect to a bounded number of EEPs corresponding to particular embeddings of K in S. By solving all these problems (and successively performing compressions, if necessary, and taking care that vertices of attachment of bridges in previously obtained obstructions are not changed during later compressions), we either get an embedding of G in S, or the union of obstructions for the EEPs gives a subgraph $\tilde{K}$ of bounded branch size that cannot be embedded in S. If we need $\tilde{K}$ in further processing, we just make sure that there are no local $\tilde{K}$-bridges [12].

It remains to see how we solve an EEP $\Xi = (G, K, \Pi, \Delta)$ where Δ contains all embedding schemes that are possible under the given embedding Π of K in the surface S. We will construct a sequence of graphs $K_0, K_1, \ldots$ such that $K_0 = K$ and K_{i+1} is obtained (after a compression) from K_i by adding an obstruction for simple embedding extensions. Let us describe the construction of K_{i+1} $(i = 0, 1, 2, \ldots)$ in more details. First of all, we replace each K_i-bridge in G by its E-graph (Theorem 2.1). By using Corollary 4.3, we get in linear time the set $\mathcal{B}_i$ of K_i-bridges in a compressed obstruction for simple embedding extensions of K_i to G, taken over all EEs of Π to K_i. Of course, having found an EE, we stop and by Theorem 2.1 we also get an EE of K_0 to G. Assuming that no EE has been found, and assuming inductively that the branch size of K_i is bounded, also $\mathrm{bsize}_{K_i}(\mathcal{B}_i)$ is bounded (Corollary 4.3). We now define $K_{i+1} = K_i \cup \mathcal{B}_i$ and observe that there are no K_{i+1}-bridges that are local on a branch of K_{i+1} contained in K_i. On the other hand, bridges that are local on branches from $\mathcal{B}_i$ can be eliminated by the algorithm from [12]. After doing that, we stop if $K_{i+1} = G$ or if K_{i+1} has no embeddings in S.

Note that for each i, $\mathcal{B}_i \neq \emptyset$ (or we stop with an embedding). Therefore, the above process terminates after a finite number of steps. It remains to see that the number of steps cannot be too large. This is done by proving that K_0-bridges in K_i become more and more complicated as i grows, and that their embeddings extending any embedding of K_0 in S become less and less "simple". The details are left to the full paper.

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