JOURNAL OF COMBINATORIAL THEORY, Series B **66**, 201–231 (1996)
ARTICLE NO. 0016

Separating and Nonseparating Disjoint Homotopic Cycles in Graph Embeddings

RICHARD BRUNET

Carleton University, Ottawa, Ontario, Canada K1S 5B6

BOJAN MOHAR

University of Ljubljana, Ljubljana, Yugoslavia

AND

R. BRUCE RICHTER

Carleton University, Ottawa, Ontario, Canada K1S 5B6

Received February 27, 1992

We show that if a graph G is embedded in a surface Σ with representativity ρ, then G contains at least $\lfloor(\rho-1)/2\rfloor$ pairwise disjoint, pairwise homotopic, nonseparating (in Σ) cycles, and G contains at least $\lfloor(\rho-1)/8\rfloor-1$ pairwise disjoint, pairwise homotopic, separating, noncontractible cycles.

1. INTRODUCTION

Several recent papers deal with the representativity of an embedded graph, e.g., [RoV, RV, S2, FHRRo]. This is a nonnegative integer ρ that measures how densely a graph is embedded in a surface (and will be defined precisely later). Of particular interest are results that show that large representativity forces particular structures in the embedded graph. For example,

(1) there are $\lfloor\rho/2\rfloor$ disjoint contractible cycles in the graph, all bounding discs containing a particular face [FHRRo];

(2) there are $\lfloor(3\rho)/4\rfloor$ disjoint noncontractible cycles in any embedding in the torus [S2];

(3) if $\rho > c2^{3g}$ and G is a triangulation of the sphere with g handles, then G has a spanning tree with maximum degree at most 4 [T2];

0095-8956/96 $18.00

(4) if $\rho > c2^{2g}$ and G is a 3-connected (4-connected) graph embedded in the sphere with g handles, then G has a closed spanning walk that visits each vertex at most 3 (2) times [Y]; and

(5) if $\rho > c2^{2g}$ and G is a 5-connected triangulation of the sphere with g handles, then G has a Hamilton cycle [Y].

One of the main goals of this article is to show that every embedded graph has at least $\lfloor(\rho-1)/2\rfloor$ pairwise disjoint, pairwise homotopic cycles that are not separating in the surface.

A second goal is to prove that every graph embedded in a surface with genus at least 2 has at least $\lfloor(\rho-1)/8\rfloor-1$ pairwise disjoint, pairwise homotopic cycles that are not contractible but separate the surface. Zha and Zhao have shown that if the embedding is 7-representative, then there is a noncontractible separating cycle [ZZ]; we improve this here to 6-representative.

Our results are more general than this, in that the homotopy type of the polygons can be restricted to some specific class of curves that has additional algebraic structure. The details of this additional structure will be made clear in the presentation. Although the homotopy type cannot be completely prescribed, this does make progress on questions raised by Mohar and Robertson [MR].

A referee of an earlier version of this article has pointed out that the results about noncontractible separating cycles follow from [S1], which gives a very general characterization of when an embedded graph has disjoint cycles $P_1, ..., P_k$ homotopic to specified curves $\gamma_1, ..., \gamma_k$ in the surface. We do not see a direct way to obtain our result about nonseparating cycles from [S1]. In any case, our point of view brings out some other points of independent interest.

For example, we focus on sets of curves in the surface that satisfy an analogue of Thomassen's three path property [T1] and show that such sets are in 1–1 correspondence with normal subgroups of the fundamental group of the surface. Many of the important characteristics associated with contractible curves apply in this more general setting.

The generality we employ to prove the results about the existence of many pairwise disjoint, pairwise homotopic nonseparating cycles is a natural evolution. Initially, we showed the existence of many pairwise disjoint, pairwise homotopic noncontractible cycles, using arguments very similar to those given in this paper. In order to obtain the same result but for non-separating cycles, we were led to considering special sets of curves in the surface (the "complete sets of loops" to be introduced in the next section). With that generalization in hand, we realized that the arguments (with only very minor changes) generalized even further to Theorem 6.1.

One reason for going with the full generality of Theorem 6.1 is to emphasize the fundamental nature of the three path property in conducting homotopy arguments. It is only the three path property that is needed to do many of the basic arguments, and a main theme of this article is to demonstrate that.

This work is based in large measure on part of the Ph.D. thesis of the first author [B].

2. Complete Sets of Loops

In this section, we introduce the topological concepts that we require in this work. The main point is to introduce complete sets of loops, which form the core of our later discussions.

We require some standard terminology from topology. A standard reference for this material is [Mu]. A *path* in a topological space Σ is a continuous function γ: $[0, 1] \rightarrow \Sigma$. Set $\mathbf{I} = [0, 1]$. The *image* of the path γ is $\gamma(\mathbf{I})$. Its *basepoint* is $\gamma(0)$. It is *simple* if it is an injection. A *loop* is a path γ for which $\gamma(0) = \gamma(1)$ and the loop γ is *simple* if it is injective on $[0, 1)$. If γ: $[0, 1] \rightarrow \Sigma$ is a path, then γ^{-1}: $[0, 1] \rightarrow \Sigma$ is the *inverse path* defined by $\gamma^{-1}(t) = \gamma(1-t)$.

We require two forms of homotopy of loops—those which require a fixed common base point x (which is fixed for all calculations) and those which do not. A *homotopy* between loops γ and γ' with common basepoint x is a continuous function h: $[0, 1] \times [0, 1] \rightarrow \Sigma$ such that: (1) for each $t \in [0, 1]$, $h(0, t) = \gamma(t)$, $h(1, t) = \gamma'(t)$; and (2) for each $s \in [0, 1]$, $h(s, 0) = h(s, 1) = x$. A *free homotopy* between loops γ and γ' (with possibly different basepoints) is such a continuous function h satisfying (1) above and (2') for each $s \in [0, 1]$, $h(s, 0) = h(s, 1)$.

We use the notation $\gamma_0 \sim \gamma_1$ to say that γ_0 and γ_1 are homotopic and $\gamma_0 \sim_f \gamma_1$ to say that γ_0 and γ_1 are freely homotopic. Obviously, $\gamma_0 \sim \gamma_1$ implies $\gamma_0 \sim_f \gamma_1$. The fundamental group $\pi(\Sigma, x)$ has as its elements the equivalence classes from $\sim$.

A loop γ is *contractible* if $\gamma \sim_f \gamma'$ for some constant loop γ'. A representative of the identity element of the fundamental group is contractible. A loop is *noncontractible* if it is not contractible.

If γ, γ' are two paths such that $\gamma(1) = \gamma'(0)$, then the *composition* $\gamma \circ \gamma'$ is the function defined by

$$(\gamma \circ \gamma')(t) = \begin{cases} \gamma(2t), & 0 \leqslant t \leqslant \frac{1}{2} \\ \gamma'(2t-1), & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

In particular, if γ, γ' are loops, then their composition is defined if and only if they have a common basepoint.

The following results are elementary.

LEMMA 2.1. (1) *For any loops γ, γ', and γ'' with common basepoint x,*

$$(\gamma \circ \gamma') \circ \gamma'' \sim \gamma \circ (\gamma' \circ \gamma'').$$

(2) *If γ is a loop with basepoint x and α is any path such that $\alpha(0) = x$, then $\alpha^{-1} \circ \gamma \circ \alpha$ is a loop with basepoint $\alpha(1)$ that is freely homotopic to γ.*

(3) *It follows from* (2) *that, for any loops $\gamma_1, \gamma_2, ..., \gamma_k$ with a common basepoint, $\gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_k$ is freely homotopic to $\gamma_k \circ \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_{k-1}$.*

(4) *If γ and γ' are loops with basepoint x, then γ is freely homotopic to γ' if and only if there is a loop α with basepoint x such that γ is homotopic to $\alpha^{-1} \circ \gamma' \circ \alpha$.*

A nonempty set $\mathscr{C}$ of loops is a *complete set of loops* if:

(1) $\mathscr{C}$ is closed under free homotopy;

(2) if $\gamma \in \mathscr{C}$, then $\gamma^{-1} \in \mathscr{C}$; i.e., $\mathscr{C}$ is closed under inverses; and

(3) the composition of any two loops of $\mathscr{C}$ having a common basepoint is another loop of $\mathscr{C}$.

The concept of complete sets of loops is at the heart of our main results. We note that the set $\mathscr{C}_0$ of contractible loops is a complete set of loops. Thus, complete sets of loops generalize contractibility.

The following two propositions are easy consequences of the definition. Here Γ_Σ is the set of all loops in Σ.

PROPOSITION 2.2. *Let $\mathscr{C}$ be a set of loops of Σ and let $\mathscr{E} = \Gamma_\Sigma \backslash \mathscr{C}$. Then $\mathscr{C}$ is complete if and only if the following conditions are satisfied*:

(a) *$\mathscr{E}$ is closed under free homotopy*;

(b) *let $\gamma_0, \gamma_1 \in \Gamma_\Sigma$ have common basepoints and suppose $\gamma_0 \circ \gamma_1 \in \mathscr{E}$. Then at least one of γ_0 and γ_1 is in $\mathscr{E}$.*

PROPOSITION 2.3. *Conditions* (a) *and* (b) *in Proposition* 2.2 *are equivalent to Condition* (a) *and the following condition*:

(c) *Let $\gamma \in \mathscr{E}$ and let σ be a path in Σ such that $\sigma(0) = \gamma(0)$ and $\sigma(1) = \gamma(t)$ for some $t \in [0, 1)$. Let γ', γ'' be the loops which are the compositions $\gamma|_{[0,t]} \circ \sigma^{-1}$ and $\gamma|_{[t,1]} \circ \sigma$, respectively. Then at least one of γ' and γ'' is in $\mathscr{E}$.*

Property (c) is the *three paths property* or *TPP*. This is motivated by Thomassen's work on the three path property in graphs (see [T1]).

A *complete partition of loops* of Σ (or a *complete partition of* Γ_Σ) is a partition $(\mathscr{C}, \mathscr{E})$ of the set Γ_Σ such that $\mathscr{C}$ is a complete set of loops in Σ and $\mathscr{E} = \Gamma_\Sigma \backslash \mathscr{C}$.

The next result is an important, easy fact about the composition of loops. It generalizes the well known fact about composition of a contractible loop with a noncontractible loop.

PROPOSITION 2.4. *Let $(\mathscr{C}, \mathscr{E})$ be a complete partition of Γ_Σ. Let $\gamma_0 \in \mathscr{E}$ and $\gamma_1 \in \mathscr{C}$ have a common basepoint x_0. Then $\gamma_0 \circ \gamma_1$ is an element of $\mathscr{E}$.*

Recall that $\mathscr{C}_0$ is the set of contractible loops.

PROPOSITION 2.5. *Let $\mathscr{C}$ be any complete set of loops. Then $\mathscr{C}_0 \subseteq \mathscr{C}$.*

Proof. Let $\gamma \in \mathscr{C}$. Then $\gamma^{-1} \in \mathscr{C}$ and $\gamma \circ \gamma^{-1} \in \mathscr{C}$. But $\gamma \circ \gamma^{-1}$ is freely homotopic to the constant loop; i.e., it is contractible. ∎

3. COMPLETE PARTITIONS AND THE FUNDAMENTAL GROUP

In this section we prove that complete partitions of Γ_Σ are in a natural 1–1 correspondence with normal subgroups of the fundamental group. Although there seems to be some understanding of this fact by topologists, the precise relationship given here is apparently new. So fix a basepoint x in Σ and let $(\mathscr{C}, \mathscr{E})$ be a complete partition. Let $\mathscr{C}_x = \{\gamma \in \mathscr{C} \mid \gamma(0) = x\}$.

The following observations are easy.

Observation 1. $\mathscr{C}_x \neq \emptyset$.

Observation 2. If $\gamma \in \mathscr{C}_x$ and $\gamma' \sim \gamma$, then $\gamma' \in \mathscr{C}_x$.

It follows that $\mathscr{C}_x$ partitions into a set $\pi(\mathscr{C}, x)$ of homotopy classes, so that $\pi(\mathscr{C}, x)$ is a nonempty subset of the fundamental group $\pi(\Sigma, x)$.

PROPOSITION 3.1. *$\pi(\mathscr{C}, x)$ is a normal subgroup of $\pi(\Sigma, x)$.*

Proof. Let $[\gamma], [\gamma'] \in \pi(\mathscr{C}, x)$. We first prove that $\pi(\mathscr{C}, x)$ is a subgroup of $\pi(\Sigma, x)$ by showing that $[\gamma]^{-1} \circ [\gamma'] \in \pi(\mathscr{C}, x)$. Since $[\gamma^{-1}] \circ [\gamma'] = [\gamma^{-1} \circ \gamma']$, we will be done if we prove that $\gamma^{-1} \circ \gamma' \in \mathscr{C}_x$. Since $\gamma \in \mathscr{C}$, we have that $\gamma^{-1} \in \mathscr{C}$. Since $\mathscr{C}$ is closed under composition, $\gamma^{-1} \circ \gamma' \in \mathscr{C}_x$.

To prove that $\pi(\mathscr{C}, x)$ is normal in $\pi(\Sigma, x)$, let $[\gamma] \in \pi(\Sigma, x)$, and $[\gamma'] \in \pi(\mathscr{C}, x)$. We have to prove that $[\gamma] \circ [\gamma'] \circ [\gamma]^{-1} \in \pi(\mathscr{C}, x)$. As in the last paragraph, it suffices to prove that $\gamma \circ \gamma' \circ \gamma^{-1} \in \mathscr{C}_x$. By Lemma 2.1(3),

$(\gamma \circ \gamma') \circ \gamma^{-1} \sim_f \gamma^{-1} \circ (\gamma \circ \gamma')$. But $\gamma^{-1} \circ (\gamma \circ \gamma') \sim (\gamma^{-1} \circ \gamma) \circ \gamma' \sim \gamma'$, and so $\gamma \circ \gamma' \circ \gamma^{-1} \in \mathscr{C}_x$. ▮

For the opposite direction, let H be a normal subgroup of $\pi(\Sigma, x)$. Let $\mathscr{C}_H$ denote the set of all loops γ for which there is a $[\gamma'] \in H$ such that $\gamma \sim_f \gamma'$.

PROPOSITION 3.2. *If H is a normal subgroup of $\pi(\Sigma, x)$, then $\mathscr{C}_H$ is a complete set of loops.*

Proof. Clearly, $\mathscr{C}_H$ is not empty. Since $\sim_f$ is transitive, $\mathscr{C}_H$ is closed under free homotopy. Since $\gamma \sim_f \gamma'$ implies $\gamma^{-1} \sim_f \gamma'^{-1}$, $\mathscr{C}_H$ is also closed under inverses.

Finally we show that $\mathscr{C}_H$ is closed under composition. Let $\gamma_1, \gamma_2 \in \mathscr{C}_H$ have common basepoint y. We have to prove that $\gamma_1 \circ \gamma_2 \in \mathscr{C}_H$.

For $i = 1, 2$, let $[\gamma_i'] \in H$ be such that $\gamma_i \sim_f \gamma_i'$. Let $h_i: I \times I \to \Sigma$ be a free homotopy such that $h_i(t \times \{0\}) = \gamma_i(t)$ and $h_i(t \times \{1\} = \gamma_i'(t)$, for $t \in I$. Define $\alpha_i: I \to \Sigma$ by $\alpha_i(s) = h_i((0, s))$. It follows that $\alpha_i(0) = \gamma_i(0) = y$ and $\alpha_i(1) = x$.

The loop $\delta_i = \alpha_i^{-1} \circ \gamma_i \circ \alpha_i$ is freely homotopic to γ_i and has basepoint x. Furthermore, $\delta_i \in [\gamma_i']$ (see Lemma 5.1.1).

Since H is normal and $\alpha_1^{-1} \circ \alpha_2$ is a loop with basepoint x, we have that $(\alpha_1^{-1} \circ \alpha_2)^{-1} \circ \delta_1 \circ (\alpha_1^{-1} \circ \alpha_2)$ is in $\mathscr{C}_H$. Simplifying yields $\alpha_2^{-1} \circ \gamma_1 \circ \alpha_2$ is in $\mathscr{C}_H$. Therefore, $[\alpha_2^{-1} \circ \gamma_1 \circ \alpha_2] \circ [\alpha_2^{-1} \circ \gamma_2 \circ \alpha_2] \in H$, so that $\alpha_2^{-1} \circ \gamma_1 \circ \gamma_2 \circ \alpha_2 \in \mathscr{C}_H$. By Lemma 2.1(2), this last path is freely homotopic to $\gamma_1 \circ \gamma_2$, so $\gamma_1 \circ \gamma_2 \in \mathscr{C}_H$. ▮

We now show that the operations of Propositions 3.1 and 3.2 are actually inverses.

THEOREM 3.3. *There is a bijection between the set of complete sets of loops of Σ and the set of normal subgroups of $\pi(\Sigma, x)$ given by the relations*

$$H = \pi(\mathscr{C}_H, x), \qquad \mathscr{C}_{\pi(\mathscr{C}, x)} = \mathscr{C}.$$

Proof. Let H be a normal subgroup of $\pi(\Sigma, x)$:

$$\begin{aligned}
[\gamma] \in \pi(\mathscr{C}_H, x) &\Leftrightarrow \gamma \in \mathscr{C}_H \text{ and } \gamma(0) = x \\
&\Leftrightarrow \exists [\gamma'] \in H \text{ such that } \gamma \sim_f \gamma' \text{ and } \gamma(0) = x \text{ (Lemma 2.1(4))} \\
&\Leftrightarrow \exists [\alpha] \in \pi(\Sigma, x), [\gamma'] \in H \text{ such that } [\gamma] = [\alpha^{-1}] \circ [\gamma'] \circ [\alpha] \\
&\Leftrightarrow [\gamma] \in H.
\end{aligned}$$

The proof of the second relation is similar. ▮

The complete set of loops $\mathscr{C}_0$ consisting of the contractible loops corresponds to the trivial subgroup of $\pi(\Sigma, x)$ consisting just of the identity. The complete partition $(\mathscr{C}_0, \mathscr{E}_0)$ is the *fundamental partition.*

The commutator subgroup H_s of $\pi(\Sigma, x)$ is a normal subgroup and it can be shown [GH] that the simple loops in the corresponding complete set of loops $\mathscr{C}_s$ separate Σ into two components, at least one of which is orientable. The complete partition $(\mathscr{C}_s, \mathscr{E}_s)$ of Γ_Σ is the *separating partition.*

Although it does not concern us in this work, there is another "separating partition" in the nonorientable case. Let H_n be the smallest subgroup of $\pi(\Sigma, x)$ containing H_s and $\{\alpha^2 \mid \alpha \text{ is orientation-reversing}\}$. Every simple loop γ in Σ for which $\gamma(\mathbf{I})$ separates Σ into two pieces is freely homotopic to a loop in H_n.

There is a particular characteristic shared by $\mathscr{C}_0$ and $\mathscr{C}_s$ that turns out to play an important role for us. A complete partition $(\mathscr{C}, \mathscr{E})$ is a *crossing partition* if, for any simple loop $\gamma \in \mathscr{C}$ and any loop γ', if γ' crosses γ transversely at some point, then γ' crosses γ at least twice. It is an easy exercise to show that both $(\mathscr{C}_0, \mathscr{E}_0)$ and $(\mathscr{C}_s, \mathscr{E}_s)$ are crossing partitions.

The following proposition will be very useful when dealing with nonorientable loops.

PROPOSITION 3.4. *Let Σ be a nonorientable surface, and let $(\mathscr{C}, \mathscr{E})$ be a crossing partition of Γ_Σ. Then all orientation-reversing simple loops are in $\mathscr{E}$.*

For the proof, we make use of the following notion. A loop γ_1 is *n-freely homotopic* to the loop γ_2 if $\gamma_1 \sim_f \gamma_2^n$, for some integer n. Thus, 1-freely homotopic is the same as freely homotopic and any loop is -1-freely homotopic to its inverse.

Proof. Let γ be an orientation-reversing simple loop. Then there is a Möbius band M whose boundary is a simple loop γ' that is 2-freely homotopic to γ. Let σ be a path from one point of γ' to another point of γ' and that intersects γ only once and this intersection is a transverse crossing. There is a simple loop γ'' whose image is contained in $\gamma'(\mathbf{I}) \cup \sigma(\mathbf{I})$ that intersects γ in a single point, which is a transverse crossing. ∎

We have one last complete partition to mention here. If Σ is nonorientable, then the set $\mathscr{C}_p$ of all orientation-preserving loops is a complete set of loops. This corresponds to an index-2 subgroup of the fundamental group. The complete partition $(\mathscr{C}_p, \mathscr{E}_p)$ is the *orientation-preserving partition.*

4. $\mathscr{C}$-REPRESENTATIVITY OF EMBEDDINGS

We shall now consider a graph G embedded in a surface Σ. A helpful reference for some of the basic concepts is [RoV]. The main point of this section is to generalize the notion of representativity to $\mathscr{C}$-representativity, for any complete set of loops $\mathscr{C}$. We shall prove results about $\mathscr{C}$-representativity paralleling standard results about representativity. In particular, it is finite and it attained by a simple loop. This discussion requires a detailed understanding of loops in surfaces, which is where we begin. There are some basic facts about neighbourhoods of points in a graph embedded in a surface described in [HR]. We will use these as needed without explicit reference.

A *surface* is a compact connected Hausdorff space for which every point has a neighbourhood homeomorphic to $\mathbb{R}^2$. Portions of this work generalize to more general 2-manifolds, but our interest is with surfaces.

Theorems 4.1 and 4.2 are the main nontrivial technical points central to the entire discussion. Let $\alpha: [0, 1] \to \Sigma$ be a path and let I be a subinterval of $[0, 1]$. Then the *multiplicity* of α on I is the number of ordered pairs (t, t') such that $t, t' \in I$, $t < t'$ and $\alpha(t) = \alpha(t')$. The multiplicity of a loop is the multiplicity of the loop on $[0, 1)$.

THEOREM 4.1. *Let Σ be a surface and let Φ be a closed subset of Σ. Suppose γ is a loop in Σ disjoint from Φ. Then there is a homotope γ' of γ that is disjoint from Φ and has finite multiplicity.*

There are several proofs of this that we could give here. We have chosen this one because it is completely elementary. In particular, it does not rely on knowledge of the fundamental group of a surface—it depends only on the existence of disc neighbourhoods.

Proof. For each $t \in [0, 1]$, let $\bar{D}_t$ be a closed disc in Σ with interior D_t such that $\gamma(t) \in D_t$ and $\bar{D}_t$ is disjoint from Φ.

CLAIM 1. *There is a $\delta > 0$ such that if $0 < t' - t < \delta$, then there is a disc D_s such that $\gamma([t, t']) \subset D_s$.*

If not, then for each positive integer N, there exists t_N and t'_N such that $0 < t'_N - t_N < 1/N$ and no D_s contains $\gamma([t_N, t'_N])$. There exists an infinite sequence N_j for which the sequences t_{N_j} and t'_{N_j} both converge to t^*.

There is a disc D_s containing $\gamma(t^*)$. By continuity, there is a $\delta > 0$ such that if $|t - t^*| < \delta$, then $\gamma(t) \in D_s$. Thus, for sufficiently large j, $\gamma([t_{N_j}, t'_{N_j}]) \subset D_s$, a contradiction that proves Claim 1.

Fix the positive integer N large enough so that if $0 < t' - t \leqslant 1/N$, then there is a disc D_s containing $\gamma([t, t'])$. For $j = 1, 2, ..., N$, let I_j denote the interval $[(j-1)/N, j/N]$. There is some disc D_{s_j} containing $\gamma(I_j)$. Relabel the discs $D_{s_1}, D_{s_2}, ..., D_{s_N}$ as $D_1, D_2, ..., D_N$.

In N steps, we shall find the homotope of γ that has finite multiplicity. Let $\alpha: [0, 1/N] \to \Sigma$ be a simple path in D_1 joining $\gamma(0)$ with $\gamma(1/N)$. (If these points happen to be the same, then we can choose α to be a simple loop.) Let γ_1 be the loop obtained by traversing α on $[0, 1/N]$ and γ on $[1/N, 1]$. Because α and $\gamma|_{[0, 1/N]}$ are both paths in the disc D_1 having the same end points, they are homotopic. Therefore, γ_1 and γ are homotopic.

Now suppose $i \geqslant 1$ and we have γ_i homotopic to γ, $\gamma_i(\mathbf{I})$ disjoint from Φ and γ_i has finite multiplicity on $[0, i/N]$. We now show how to obtain γ_{i+1}.

Identify the closed disc $\bar{D}_{i+1}$ with the unit disc in the plane, which we use for metric reference. Since $\gamma(I_{i+1}) \subset D_{i+1}$, and $\gamma(I_{i+1})$ is compact, there is some real number $r < 1$ such that $\gamma(I_{i+1})$ is contained in the open disc of radius r. Notice that $\gamma_i(i/N)$ is in this disc of radius r.

Claim 2. *There are at most finitely many components of* $\bar{D}_{i+1} \backslash \gamma_i([0, i/N])$ *that have a point in the disc of radius* r.

For otherwise, there is a point r^* of $\gamma_i([0, i/N])$ for which every neighbourhood contains a point in each of infinitely many such components. (Pick one point r_n from each of infinitely many such components. Let r^* be a limit point of the infinite set of r_n.) We shall show that the finite multiplicity of γ_i on $[0, i/N]$ does not allow this.

Let $0 \leqslant t_1 < t_2 < \cdots < t_p \leqslant i/N$ be those $t \in [0, i/N]$ such that $\gamma_i(t) = r^*$. For each $j = 1, 2, ..., p$, let $J(j, \varepsilon) = [t_j - \varepsilon, t_j + \varepsilon] \cap [0, i/N]$ and let $\mathbf{J}(\varepsilon) = \bigcup_{j=1}^{p} J(j, \varepsilon)$. We claim that for some $\varepsilon > 0$, γ_i is injective on $\mathbf{J}(\varepsilon) \backslash \{t_1, t_2, ..., t_p\}$.

For if it were not, then for each m, there would exist $t'_m, t''_m \in \mathbf{J}(1/m)$ such that $t'_m \neq t''_m$ and $\gamma_i(t'_m) = \gamma_i(t''_m) \neq r^*$. But then the multiplicity of γ_i on $[0, i/N]$ is infinite.

Let $s = 0$ if both $t_1 > 0$ and $t_p < i/N$, let $s = 1$ if either $t_1 = 0$ or $t_p = i/N$, but not both, and let $s = 2$ if $t_1 = 0$ and $t_p = i/N$. We shall define an embedding of the graph $K_{1, 2p-s}$ in $\bar{D}_{i+1}$. We map the central vertex to r^*, the leaves to $\gamma_i(t_i \pm \varepsilon)$ and the edges to $\gamma_i((t_i - \varepsilon, t_i))$ and $\gamma_i((t_i, t_i + \varepsilon))$. (Suitable care needs to be employed if $s > 0$.)

There is a disc $\Delta \subset D_{i+1}$ and a homeomorphism $h\colon \Delta \to \mathbb{R}^2$ such that $h(r^*)$ is the origin and $h(K_{1, 2p-s} \cap \Delta)$ is $2p - s$ straight rays from the origin to infinity. For a sufficiently small circular disc Δ' in $\mathbb{R}^2$, Δ' intersects $h(\gamma_i([0, i/N]))$ only in $h(r^*)$ and bits of the $2p - s$ straight rays. This corresponds to a neighbourhood of r^* in D_{i+1} that intersects only finitely many components of $D_{i+1} \backslash \gamma_i([0, i/N])$, completing the proof of Claim 2.

By Claim 2, $\gamma_i([0, i/N])$ meets the closures of at most finitely many components of $D_{i+1} \backslash \gamma_i([0, i/N])$. Let these components be $R_1, R_2, ..., R_k$. For $j = 1, 2, ..., k$, let t_j^* be the largest $t \in I_{i+1}$ such that $\gamma_i(t)$ is in the closure of R_j.

There is a j_1 such that $\gamma_i(i/N) = \gamma(i/N)$ is in the closure of R_{j_1} and $t_{j_1}^* > i/N$. Suppose we have already defined the positive integers $j_1, j_2, ..., j_m$. If $t_{j_m}^* = (i+1)/N$, then stop. Otherwise, there is a j_{m+1} such that $\gamma(t_{j_m}^*)$ is in the closure of $R_{j_{m+1}}$ and $t^*_{j_{m+1}} > t^*_{j_m}$. For some $m^* \leqslant k$, we shall have $t^*_{j_{m^*}} = (i+1)/N$. Set $t^*_{j_0} = i/N$.

For $m = 1, 2, ..., m^*$, there is a simple path $\alpha_m: [t^*_{j_{m-1}}, t^*_{j_m}] \to \bar{R}_{j_m}$, where $\bar{R}_{j_m}$ is the closure of R_{j_m}, such that $\alpha_m(t^*_{j_{m-1}}) = \gamma(t^*_{j_{m-1}})$, $\alpha_m(t^*_{j_m}) = \gamma(t^*_{j_m})$ and α_m is otherwise disjoint from $\gamma_i([0, i/N])$. Define γ_{i+1} to be γ_i on $[0, i/N]$, α_m on $[t^*_{j_{m-1}}, t^*_{j_m}]$, $m = 1, 2, ..., m^*$, and γ on $[(i+1)/N, 1]$. Obviously, γ_{i+1} has finite multiplicity on $[0, (i+1)/N]$, is disjoint from Φ and is homotopic to γ_i (and therefore to γ), as required.

Finally, γ_N is the loop with finite multiplicity that is disjoint from Φ and is homotopic to γ. ∎

The following is a straightforward induction on the multiplicity, based on the TPP Proposition 2.3.

Corollary 4.1.1. *Let Σ be a surface, let $(\mathscr{C}, \mathscr{E})$ be a complete partition of Γ_Σ such that $\mathscr{E} \neq \emptyset$, and let Φ be a closed subset of Σ for which there is a loop $\gamma \in \mathscr{E}$ with $\gamma(\mathbf{I})$ disjoint from Φ. Then $\mathscr{E}$ contains a simple loop γ' with $\gamma'(\mathbf{I})$ disjoint from Φ.*

The other technical result we need is the following, whose proof is due to Bob Brown. We are grateful to Helga Schirmer for her efforts in relaying the messages to get this proof.

Let γ be a loop and let $0 \leqslant t' < t'' < 1$. The two paths obtained by restricting γ to $[t', t'']$ and $[t'', 1] \cup [0, t']$ are the *subpaths of γ induced by t' and t''*. Thus, if γ_1 and γ_2 are the two subpaths of γ induced by t' and t'', then $\gamma_1(\mathbf{I}) \cup \gamma_2(\mathbf{I}) = \gamma(\mathbf{I})$ and $\gamma \sim_f \gamma_1 \circ \gamma_2$.

For a loop γ and a path σ such that $\sigma(0) = \gamma(t')$ and $\sigma(1) = \gamma(t'')$, let γ_1 and γ_2 be the two subpaths of γ induced by t' and t''. For some $\varepsilon \in \{1, -1\}$, both $\gamma_1 \circ \sigma^\varepsilon$ and $\gamma_2 \circ \sigma^{-\varepsilon}$ are loops. These two loops are the *θ-decomposition of γ with respect to σ.*

Theorem 4.2. *Let $(\mathscr{C}, \mathscr{E})$ be a complete partition of Γ_Σ. Let γ_1 be a simple loop in $\mathscr{C}$ and let γ_2 be a loop in $\mathscr{E}$. Suppose there exist distinct t', t'' such that $\gamma_2(t')$ and $\gamma_2(t'')$ are both in $\gamma_1(\mathbf{I})$. Then there exist distinct a, b such that $\gamma_2(a), \gamma_2(b) \in \gamma_1(\mathbf{I})$ and, for at least one of the two subpaths, say γ_2', of γ_2 induced by a and b, $\gamma_2'((a, b))$ is disjoint from γ_1 and the two loops in the θ-decomposition of γ_1 with respect to γ_2' are both in $\mathscr{E}$.*

Proof. There are at most countably many subintervals $I_n = [a_n, b_n]$ of $[0, 1]$ such that $\gamma_2(I_n)$ meets $\gamma_1(\mathbf{I})$ in just the endpoints. Let σ_n be the path $\gamma_2: I_n \to \Sigma$. Let $N \subseteq \Sigma$ be either an open cylinder or an open Möbius strip with equator γ_1.

We claim that at most finitely many of the sets $\sigma_n([a_n, b_n])$ are not contained in N. For suppose not. Then let A be the (infinite) set of integers

n for which $\sigma_n(I_n)$ is not contained in N. For each $n \in A$, let $t_n \in I_n$ be such that $\sigma_n(t_n) \notin N$. Let t^* be a limit point of $T = \{t_n \mid n \in A\}$. If $t^* \in (a_k, b_k)$ for some $k \in \{1, 2, ...\}$, then (a_k, b_k) is an open set containing t^* and at most one of the points in T, a contradiction. Therefore, $\gamma_2(t^*) \in \gamma_1(\mathbf{I})$. Thus, N is an open set containing $\gamma_2(t^*)$, but N contains none of the points in $\gamma_2(T)$, which is impossible.

Let $\sigma_1, ..., \sigma_k$ be the paths that have an image point not in N. For $I = 1, 2, ..., k$, let $\delta_{i,1}$ and $\delta_{i,2}$ be the θ-decomposition of γ_1 with respect to σ_i. We note that γ_1 is freely homotopic to the composition of $\delta_{i,1}$ and $\delta_{i,2}$. Therefore, either both $\delta_{i,1}$ and $\delta_{i,2}$ are in $\mathscr{E}$ or neither is.

In order to obtain a contradiction, we suppose that, for each $i = 1, 2, ..., k$, both $\delta_{i,1}$ and $\delta_{i,2}$ are in $\mathscr{C}$.

We suppose $\delta_{i,1}$ traverses σ_i^{-1} first and then follows the subpath α_i of γ_1. Let ϕ_1 be the loop obtained from γ_2 by replacing the portion along I_1 with α_1^{-1}.

Since γ_2 is freely homotopic to the composition $\phi_1 \circ \delta_{1,1}$ and $\delta_{1,1}$ is in $\mathscr{C}$, it follows from Proposition 2.2 that ϕ_1 is in $\mathscr{E}$.

We now repeat the process. Having got $\phi_j \in \mathscr{E}$, we obtain ϕ_{j+1} by replacing the portion of ϕ_j corresponding to I_{j+1} with α_{j+1}^{-1}. Thus, ϕ_j is freely homotopic to the composition $\delta_{1,j+1} \circ \phi_{j+1}$. It follows that $\phi_{j+1} \in \mathscr{E}$. Do this until we have $\phi_k \in \mathscr{E}$.

Note that $\phi_k(\mathbf{I}) \subset N$. The fundamental group of N is cyclic and generated by $[\gamma_1]$. (We may choose $\gamma_2(t)$ to be the basepoint for the computations involving the fundamental group.) Therefore, ϕ_k is homotopic to γ_1^r, for some integer r. But then $\phi_k \in \mathscr{C}$, a contradiction. Therefore, some $\delta_{i,1}$ is in $\mathscr{E}$, as claimed. ∎

Let G be a graph embedded in a surface Σ and let $\gamma \in \Gamma_\Sigma$. Then $\mathrm{cr}(\gamma, G)$ is the number of $t \in [0, 1)$ such that $\gamma(t) \in G$. Given a complete partition $(\mathscr{C}, \mathscr{E})$ of Γ_Σ with $\mathscr{E} \neq \emptyset$, the *$\mathscr{C}$-representativity* of G is $\rho_\varepsilon(G) = \min\{\mathrm{cr}(\gamma, G) \mid \gamma \in \mathscr{E}\}$. If $(\mathscr{C}, \mathscr{E})$ is the fundamental partition $(\mathscr{C}_0, \mathscr{E}_0)$, then this is just the usual representativity.

We next show that $\mathscr{C}$-representativity is finite. Then we show that the $\mathscr{C}$-representativity is attained by a simple loop that goes through only vertices of G.

COROLLARY 4.2.1. *For any complete partition $(\mathscr{C}, \mathscr{E})$ of Γ_Σ with $\mathscr{E} \neq \emptyset$, and any embedding of G in Σ, the $\mathscr{C}$-representativity of G is finite.*

Proof. We show the existence a simple loop γ in $\mathscr{E}$ for which $\mathrm{cr}(\gamma, G)$ is finite. By Corollary 4.1.1 there is a simple loop γ in $\mathscr{E}$. If $\mathrm{cr}(\gamma, G)$ is finite, then we are done. Otherwise, there is some closed edge e of G such that γ goes through e infinitely often.

There is a closed disc Δ in Σ that meets G only in e, with only the ends of e in the boundary of Δ. The boundary of Δ is the image of a simple loop γ_0 that is contractible and, therefore, is in $\mathscr{C}$. Clearly γ must intersect $\gamma_0(\mathbf{I})$ at least twice, since $\gamma(\mathbf{I})$ is not contained in Δ.

By Theorem 4.2, there is a simple loop γ' in $\mathscr{E}$ made up of a simple subpath of γ and a simple subpath of γ_0. Obviously, γ' meets e in at most two points, namely the ends of e. We repeat this for each edge of G (of which there are only finitely many) to obtain a simple loop in $\mathscr{E}$ that meets G only finitely often. ∎

We are now prepared for the final touch.

THEOREM 4.3. *Let G be a graph embedded in a surface Σ and let $(\mathscr{C}, \mathscr{E})$ be a complete partition of Γ_Σ. Then there is a simple loop $\gamma \in \mathscr{E}$ such that* $\operatorname{cr}(\gamma, G) = \rho_{\mathscr{E}}(G)$ *and* $\gamma(\mathbf{I}) \cap G \subseteq V(G)$.

Proof. We begin by showing the existence of a simple loop in $\mathscr{E}$ that attains the $\mathscr{C}$-representativity. If $\rho_{\mathscr{E}}(G) = 0$, then the result follows from Corollary 4.1.1, with $\Phi = G$. Thus, we can assume $\rho_{\mathscr{E}}(G) > 0$. Let $\gamma \in \mathscr{E}$ be such that $\operatorname{cr}(\gamma, G) = \rho_{\mathscr{E}}(G)$.

Now consider the case that $\rho_{\mathscr{E}}(G) = 1$. Then there is a single face F of G such that $\gamma(\mathbf{I})$ is contained in the closure of F. Let x be the single point in $\gamma(\mathbf{I}) \cap G$. We can assume $\gamma(0) = x$. There is a closed disc Δ in Σ containing x such that $\Delta \cap G$ is just x and an appropriate number of rays emanating from x. Let γ_1 be a simple loop whose image is the boundary of Δ.

Then γ meets γ_1 at least twice, so, by Theorem 4.2, there is a subinterval $I = [a, b]$ of $[0, 1]$ such that $\gamma(I)$ has only its ends in $\gamma_1(\mathbf{I})$ and both the loops made up of $\gamma\colon I \to \Sigma$ and the subpaths of γ_1 are in $\mathscr{E}$. Thus, $\gamma(I)$ is disjoint from Δ, except for its ends.

If $\gamma(a) = \gamma(b)$, then $\gamma\colon I \to \Sigma$ is a loop in $\mathscr{E}$ that is disjoint from G, contradicting the assumption that $\rho_{\mathscr{E}}(G) = 1$. Therefore, $\gamma(a)$ and $\gamma(b)$ are distinct points of Δ.

Let $\sigma\colon [0, 1] \to \Delta$ be a simple path that joins $\gamma(b)$ to $\gamma(a)$ and meets G only at x if at all. Let $\sigma'\colon [0, 1] \to (F \backslash \Delta) \cup \gamma(\{a, b\})$ be a simple path with ends $\gamma(a)$ and $\gamma(b)$. (The set $F \backslash \Delta$ is open, $\gamma(a)$ and $\gamma(b)$ are in the closure of the component containing $\gamma(I)$, so there is a simple path in F joining them.) By the TPP, either σ' composes with $\gamma\colon I \to \Sigma$ to make a loop in $\mathscr{E}$ or σ' composes with σ to make a simple loop in $\mathscr{E}$. The former is impossible, since the composition is disjoint from G. Therefore, the latter occurs and there is a simple loop in $\mathscr{E}$ meeting G in only one point.

We now suppose $\rho_{\mathscr{E}}(G) \geqslant 2$.

CLAIM 1. *For any face F of G, $\gamma(t)$ is in the boundary of F for at most two $t \in [0, 1)$.*

Otherwise, let $0 \leqslant t_1 < t_2 < t_3 < 1$ be three times at which γ meets the boundary ∂F of F. For $i, j \in \{1, 2, 3\}$, if $\gamma(t_i) = \gamma(t_j)$, then let $\alpha_{i,j}$ be a constant path, with image $\gamma(t_i)$. Otherwise, let α_{ij} be a simple path in F joining $\gamma(t_i)$ and $\gamma(t_j)$.

Let $\gamma_{1,2}$ and $\gamma_{2,3}$ be γ restricted to $[t_1, t_2]$ and $[t_2, t_3]$, respectively, and let $\gamma_{1,3}$ be γ restricted to $[t_3, 1] \cup [0, t_1]$. If $\alpha_{1,2} \circ \alpha_{2,3} \circ \alpha_{1,3} \in \mathscr{E}$, then $\rho_{\mathscr{E}}(G) = 0$, since this curve is freely homotopic to one that does not meet G at all. Thus, by the TPP, at least one of the three loops $\gamma_{i,j} \circ \alpha_{i,j}^{-1}$, $1 \leqslant i < j \leqslant 3$, must be in $\mathscr{E}$. But each has fewer crossings with G than γ, a contradiction to the choice of γ. Therefore, each face F satisfies $\mathrm{cr}(\gamma, \partial F) \leqslant 2$ and Claim 1 is proved.

If some (open) face F meets $\gamma(\mathbf{I})$, then the boundary of F must satisfy $\mathrm{cr}(\gamma, \partial F) \geqslant 2$. Therefore, any face F that meets $\gamma(\mathbf{I})$ has its boundary meeting $\gamma(\mathbf{I})$ at exactly two different times. We now show that these two different times must in fact be at two different places as well.

For suppose $0 \leqslant t_1 < t_2 < 1$ are such that $\gamma(t_1) = \gamma(t_2) \in \partial F$. Then, by the TPP, at least one of the two loops γ restricted to $[t_1, t_2]$ and its complement is in $\mathscr{E}$ and has fewer crossings with G, a contradiction.

So let F be a face of G such that $F \cap \gamma(\mathbf{I})$ is not empty and let x and y be the distinct points in ∂F such that $\gamma(t_1) = x$ and $\gamma(t_2) = y$. Let α be a simple path in F joining x and y.

By the TPP, one of the two paths in γ with ends x and y, together with α, is in $\mathscr{E}$. Suppose it is the loop using the path across F. This loop is freely homotopic to one that does not meet G at all, showing $\rho_{\mathscr{E}}(G) = 0$, a contradiction. Therefore, it is the other one. Repeat this for every face that meets $\gamma([0, 1])$ and the result is a simple loop that attains the representativity.

If the simple loop found above that attains the representativity does not meet G only in vertices, then it meets G in some edge e. We will eliminate this intersection without destroying any of the other properties. Repeating this step yields a simple loop in $\mathscr{E}$ that attains the representativity and meets G only in vertices.

Let Δ be a closed disc in Σ that meets G only in e and its endpoints, with e being contained in the interior of Δ. By Theorem 4.2, there is a simple loop in $\mathscr{E}$ whose image is contained in $\gamma(\mathbf{I}) \cup \Delta$ and does not go into the interior of Δ. This loop can be chosen so as to meet G in no more points than γ does. Therefore, it meets G in exactly the same number of points—the intersection with e being traded for an intersection with an end of e. This completes the proof. ∎

5. The Calculus of Face Chains

A simple loop γ that meets an embedded graph G only at vertices describes an alternating sequence $v_0, F_1, v_1, ..., F_n, v_n$ of vertices and faces of G such that $v_0 = v_n$, $\gamma(\mathbf{I}) \cap G = \{v_1, v_2, ..., v_n\}$ and

$$\gamma(\mathbf{I}) \subset \left(\bigcup_{i=1}^{n} F_i \right) \cup \{v_1, v_2, ..., v_n\}.$$

Such sequences are the combinatorial structures with which we shall work in the remainder of this paper.

A *face chain* is an alternating sequence $v_0, F_1, v_1, ..., F_n, v_n$ of vertices and faces of an embedded graph G such that, for each $i = 1, 2, ..., n$, v_{i-1} and v_i are in the boundary ∂F_i of F_i. The *length* of the face chain is n, and the face chain is *closed* if $v_0 = v_n$.

Given a face chain $\Lambda = v_0, F_1, v_1, ..., F_n, v_n$, a path α_Λ is obtained by taking the composition of simple paths in each F_i joining v_{i-1} and v_i, for $i = 1, 2, ..., n$. The face chain is a *face-representation* of α_Λ and the *length of the face-representation* is n. (There is some ambiguity here. If either F_i is not homeomorphic to an open disc or one or both of v_{i-1} and v_i is repeated in the boundary walk of F_i, then the choice of the simple path in F_i is not determined up to homotopy. Thus, to be represented by the face chain means there are choices of these simple paths which yield a path freely homotopic to γ. Mostly, these distinctions will not concern us.)

Let $(\mathscr{C}, \mathscr{E})$ be a complete partition of Γ_Σ for some surface Σ. The loop γ is *freely $\mathscr{C}$-homotopic* to the loop γ' if there is a loop $\gamma'' \in \mathscr{C}$ with the same basepoint as γ' such that $\gamma \sim_f \gamma' \circ \gamma''$. We write $\gamma \sim_{\mathscr{C}} \gamma'$ to denote that γ is freely $\mathscr{C}$-homotopic to γ'.

Lemma 5.1. *The relation $\sim_{\mathscr{C}}$ is an equivalence relation.*

Proof. We begin with some additional facts about free homotopy.

Lemma 5.1.1. *Let h: $[0, 1] \times [0, 1] \to \Sigma$ be a free homotopy between the loops $\alpha(t) = h(0, t)$ and $\beta(t) = h(1, t)$. Let σ: $[0, 1] \to \Sigma$ be the path defined by $\sigma(s) = h(s, 0) = h(s, 1)$. Then α and $\sigma \circ \beta \circ \sigma^{-1}$ are homotopic (with fixed basepoint $\alpha(0)$).*

Proof. We can define the homotopy $\hat{h}$ by

$$\hat{h}(s, t) = \begin{cases} h(3t, 0), & 0 \leqslant t \leqslant s/3 \\ h(s, (3t - s)/(3 - 2s)), & s/3 \leqslant t \leqslant 1 - s/3 \\ h(-3t + 3, 1), & 1 - s/3 \leqslant t \leqslant 1. \end{cases}$$

This is a homotopy between α and (a specific parametrization of) the loop $\sigma \circ \beta \circ \sigma^{-1}$. ∎

The other result we need is easy.

LEMMA 5.1.2. *Let α and β be paths with the same endpoints. Let γ and δ be paths such that $\gamma(1) = \alpha(0)$ and $\delta(0) = \alpha(1)$. If α and β are homotopic (keeping the endpoints fixed), then $\gamma \circ \alpha$ is homotopic to $\gamma \circ \beta$ and $\alpha \circ \delta$ is homotopic to $\beta \circ \delta$ (keeping the endpoints fixed).*

Now back to the proof of Lemma 5.1. Reflexivity is trivial. For symmetry, suppose $\alpha \sim_{\mathscr{C}} \beta$. Then there is a $\gamma \in \mathscr{C}$ such that $\alpha \sim_f \beta \circ \gamma$. By Lemma 5.1.1, there is a path σ such that $\alpha \sim \sigma \circ \beta \circ \gamma \circ \sigma^{-1}$. Now Lemma 5.1.2 yields that

$$\sigma^{-1} \circ \alpha \circ \sigma \circ \gamma^{-1} \sim \sigma^{-1} \circ \sigma \circ \beta \circ \gamma \circ \sigma^{-1} \circ \sigma \circ \gamma^{-1} \sim \beta.$$

But $\sigma^{-1} \circ \alpha \circ \sigma \circ \gamma^{-1}$ is freely homotopic to $\alpha \circ \sigma \circ \gamma^{-1} \circ \sigma^{-1}$. Since γ^{-1} is in $\mathscr{C}$ and is freely homotopic to $\sigma \circ \gamma^{-1} \circ \sigma^{-1}$, this last loop is also in $\mathscr{C}$ and so $\beta \sim_{\mathscr{C}} \alpha$, as required.

We conclude with transitivity. If $\alpha \sim_{\mathscr{C}} \beta$ and $\beta \sim_{\mathscr{C}} \gamma$, then, by symmetry, there exist $\delta, \varepsilon \in \mathscr{C}$ such that $\beta \sim_f \alpha \circ \delta$ and $\beta \sim_f \gamma \circ \varepsilon$. By transitivity of $\sim_f$, $\alpha \circ \delta \sim_f \gamma \circ \varepsilon$. Thus, $(\gamma \circ \varepsilon) \sim_{\mathscr{C}} \alpha$, so by symmetry, $\alpha \sim_{\mathscr{C}} (\gamma \circ \varepsilon)$. Thus, there is a $\delta' \in \mathscr{C}$ such that $\alpha \sim_f (\gamma \circ \varepsilon) \circ \delta'$. But $(\gamma \circ \varepsilon) \circ \delta' \sim_f \gamma \circ (\varepsilon \circ \delta')$ and $\varepsilon \circ \delta' \in \mathscr{C}$. Thus, $\alpha \sim_{\mathscr{C}} \gamma$, as required. ∎

We let $[\gamma]_{\mathscr{C}}$ denote the set of loops freely $\mathscr{C}$-homotopic to γ.

Now let G be a graph embedded in Σ. Suppose γ is a loop face-represented by a closed face chain of G. The *$\mathscr{C}$-length* $l(\gamma, \mathscr{C})$ of γ is the shortest length of any face-representation of any member of $[\gamma]_{\mathscr{C}}$.

By Proposition 2.4, if γ is in $\mathscr{E}$, then any γ' for which $\gamma \sim_{\mathscr{C}} \gamma'$ is also in $\mathscr{E}$. Therefore, for any $\gamma \in \mathscr{E}$ that is face-represented by a closed face chain, $\rho_{\mathscr{E}}(G) \leqslant l(\gamma, \mathscr{C})$.

Let $\Lambda = v_0, F_1, v_1, ..., F_n, v_n$ be a closed chain and let $\Lambda' = w_0, F'_1, w_1, ..., F'_k, w_k$ be a face chain such that w_0 is incident with some F_i and w_k is incident with some F_j. For ease of exposition, assume $1 \leqslant i < j \leqslant n$. (Up to a cyclic permutation of Λ, this is always the case.) Then there are two face chains in Λ whose first and last faces are F_i and F_j. Clearly, we can combine each of these with Λ' to get a closed face chain containing Λ'.

The following result is the arithmetic heart of the proof that there are $\rho/2$ pairwise disjoint, pairwise homotopic nonseparating cycles.

THEOREM 5.2. *Let G be embedded in a surface Σ and let $(\mathscr{C}_1, \mathscr{E}_1)$, $(\mathscr{C}_2, \mathscr{E}_2)$ be two complete partitions such that $\mathscr{E}_2 \subseteq \mathscr{E}_1$. Let γ be a loop in $\mathscr{E}_2$*

whose $\mathscr{C}_1$-length l is finite and let $\Lambda = v_0, F_1, ..., F_l, v_l$ be a closed face chain that represents some loop in $[\gamma]_{\mathscr{C}_1}$. Let $\Lambda' = w_0, F'_1, ..., F'_k, w_k$ be a face chain of length $k \geqslant 0$ such that w_0 and w_k are each incident with some face in Λ. (If $k = 0$, then we assume that w_0 is incident with distinct faces in the face chain Λ.) If either

$$k \leqslant \left\lfloor \frac{\rho_{\mathscr{E}_1}(G) + \rho_{\mathscr{E}_2}(G) - l - 1}{2} \right\rfloor - 1 \tag{1}$$

or both w_0 and w_1 are in $\{v_1, v_2, ..., v_n\}$ and

$$k \leqslant \left\lfloor \frac{\rho_{\mathscr{E}_1}(G) + \rho_{\mathscr{E}_2}(G) - l - 1}{2} \right\rfloor, \tag{2}$$

then the face chain formed by Λ' and the shorter face chain (or either if they have equal length) of Λ between w_0 and w_k has length $< \rho_{\mathscr{E}_1}(G)$ and, therefore, represents a loop that is in $\mathscr{C}_1$.

Proof. Let $\gamma^* \in \mathscr{E}_2$ be a loop face-represented by Λ.

(1) Let i and j be indices such that w_0 and w_k are incident with F_i and F_j, respectively. Choose the labelling so that $1 \leqslant i < j \leqslant n$ and $\Lambda_1 = w_0, F_i, v_i, F_{i+1}, ..., F_j, w_k$ is the shorter of the two face chains from Λ joining w_0 and w_k. Let Λ_2 be the other face chain (including F_i and F_j) in Λ joining w_0 and w_1.

Clearly, Λ_1 has length $j - i + 1$ and Λ_2 has length $l - (j - i) + 1$. Since Λ_1 is not longer than Λ_2, $j - i \leqslant l/2$. Therefore, the face chain $\Lambda' \cup \Lambda_1$ obtained by concatenating Λ_1 and Λ' has length at most $l/2 + 1 + k$. Using the estimate for k given in the hypothesis yields that this chain has length at most

$$\frac{\rho_{\mathscr{E}_1}(G) + \rho_{\mathscr{E}_2}(G) - 1}{2},$$

which is less than $\rho_{\mathscr{E}_2}(G)$, since $\mathscr{E}_2 \subseteq \mathscr{E}_1$ implies $\rho_{\mathscr{E}_2}(G) \geqslant \rho_{\mathscr{E}_1}(G)$. Therefore, if γ_1 is a loop face-represented by $\Lambda' \cup \Lambda_1$, then $\mathrm{cr}(\gamma_1, G) < \rho_{\mathscr{E}_2}(G)$, and so γ_1 is in $\mathscr{C}_2$.

On the other hand, $\Lambda_2 \cup \Lambda'$ and $\Lambda_1 \cup \Lambda'$ obviously use every face in Λ' twice, F_i and F_j at most twice, and every other face of Λ once. Thus, these two face chains have total length at most $2k + l + 2$, which, by hypothesis, is at most $\rho_{\mathscr{E}_1}(G) + \rho_{\mathscr{E}_2}(G) - 1$.

Let γ_2 be a loop face-represented by $\Lambda' \cup \Lambda_2$. Since γ^* is freely homotopic to $\gamma_1 \circ \gamma_2$ (care being taken with orientations—γ_1 and γ_2 should traverse Λ' in opposite directions), the TPP implies that at least one of γ_1 and γ_2 is in $\mathscr{E}_2$. Since it is not γ_1, it must be γ_2. Thus, $\Lambda' \cup \Lambda_2$ must have length at least $\rho_{\mathscr{E}_2}(G)$, so the inequality at the end of the preceding paragraph shows $\Lambda' \cup \Lambda_1$ has length at most $\rho_{\mathscr{E}_1}(G)-1$. Therefore, γ_1 is in $\mathscr{C}_1$, as required.

The proof of (2) is the same, except that now $\Lambda' \cup \Lambda_1$ and $\Lambda' \cup \Lambda_2$ traverse F_i and F_j a total of once each. ∎

6. The Main Result

In this section we prove the main result, from which the existence of $\rho/2$ pairwise disjoint, pairwise homotopic nonseparating cycles follows easily. To ease the cumbersome notation, let $(\mathscr{C}, \mathscr{E})$ be a complete partition of Γ_Σ and let G be a graph embedded in Σ. A set of *$\mathscr{C}$-parallels* is a set of pairwise disjoint cycles of G, for which there are simple loops having the cycles as images and which are pairwise freely $\mathscr{C}$-homotopic.

A loop γ' is *n-freely $\mathscr{C}$-homotopic* to a loop γ if $\gamma' \sim_{\mathscr{C}} \gamma^n$. We are really only interested in the cases 1- and 2-freely $\mathscr{C}$-homotopic. Our main result is the following.

Theorem 6.1. *Let G be a* 3*-connected graph embedded in a surface Σ with representativity at least* 3. *Let $(\mathscr{C}_1, \mathscr{E}_1)$ be a crossing partition and let $(\mathscr{C}_2, \mathscr{E}_2)$ be a complete partition such that $\mathscr{E}_2 \subseteq \mathscr{E}_1$. Let $\gamma \in \mathscr{E}_2$ and let $l = l(\gamma, \mathscr{C}_1)$.*

(1) *If γ is orientation-preserving, then G has a set of at least $\lfloor (\rho_{\mathscr{E}_1}(G) + \rho_{\mathscr{E}_2}(G) - l - 1)/2 \rfloor$ $\mathscr{C}_1$-parallels, all in $[\gamma]_{\mathscr{C}_1}$.*

(2) *If γ is orientation-reversing, then G has a set of at least $\lfloor (\rho_{\mathscr{E}_1}(G) + \rho_{\mathscr{E}_2}(G) - l - 1)/4 \rfloor$ $\mathscr{C}_1$-parallels, all in $[\gamma^2]_{\mathscr{C}_1}$, i.e., all* 2*-freely $\mathscr{C}_1$-homotopic to γ.*

Proof. Let $\Lambda = v_0, F_1, v_1, ..., F_l, v_l$ be the simple closed face chain representing a loop in $[\gamma]_{\mathscr{C}}$, which we may take without loss of generality to be γ. The point of the assumption that G is 3-connected and the representativity is at least 3 is to ensure that every vertex v of G has a *wheel neighbourhood*; i.e., the union of the closed faces incident with v meets the graph in a wheel, with a possibly subdivided rim. (See [RoV].)

We shall only prove (1). The proof of (2) is similar. Let $M = \lfloor (\rho_{\mathscr{E}_1}(G) + \rho_{\mathscr{E}_2}(G) - l - 1)/2 \rfloor$ and suppose $M \geqslant 1$. We shall construct a set $\{C_1, C_2, ..., C_M\}$ of $\mathscr{C}_1$-parallels in $[\gamma]_{\mathscr{C}_1}$. The C_i with odd indices will be constructed on the right-hand side of γ, while the C_i with even indices will be on the left-hand side. The outline for the proof is:

I. Construction of C_1.

A. Construction of a loop σ_1 freely homotopic to γ such that $\sigma_1(\mathbf{I}) \subseteq G$.

B. Construction of a simple loop γ_1 freely $\mathscr{C}_1$-homotopic to γ such that $\gamma_1(\mathbf{I}) \subseteq \sigma_1(\mathbf{I})$. Then $C_1 = \gamma_1(\mathbf{I})$.

II. Construction of C_2.

A. Construction of a loop σ_2 freely homotopic to γ such that $\sigma_2(\mathbf{I}) \subseteq G$.

B. Construction of a simple loop γ_2 freely $\mathscr{C}_1$-homotopic to γ such that $\gamma_2(\mathbf{I}) \subseteq \sigma_2(\mathbf{I})$. Then $C_2 = \gamma_2(\mathbf{I})$.

C. C_1 and C_2 are disjoint.

III. Construction of C_n, given $C_1, ..., C_{n-1}$.

A. Construction of a loop σ_n freely homotopic to γ_{n-2} such that $\sigma_n(\mathbf{I}) \subseteq G$.

B. Construction of a simple loop γ_n freely $\mathscr{C}_1$-homotopic to γ such that $\gamma_n(\mathbf{I}) \subseteq \sigma_n(\mathbf{I})$. Then $C_n = \gamma_n(\mathbf{I})$.

C. C_n and C_{n-2} are disjoint.

D. C_n and C_j are disjoint, $j = 1, 2, ..., n-3, n-1$.

So now we begin with I, the construction of C_1.

A. The construction of σ_1. Consider the portions of the boundaries of $F_1, F_2, ..., F_l$ on the right-hand side of γ. Specifically, γ splits each of the closed discs F_i into two closed discs. As we traverse γ, one side is naturally the left-hand side and the other is the right-hand side. Each of these closed discs is bounded by the portion of γ in F_i and part of the boundary of F_i. Since G is a wheel-neighbourhood embedding, distinct faces intersect either not at all or in a vertex or in an edge and its ends. For each $i = 1, 2, ..., l$, let $\alpha_i: [0, 1] \to \partial F_i$ be a simple path from v_{i-1} to v_i on the right-hand side of γ. Then let σ_1 be the composition $\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_l$.

B. Constructing γ_1. It is easy to see that σ_1 is a loop that is homotopic to γ. If it is a simple loop, then we are done: $\gamma_1 = \sigma_1$. If it is not simple, then there exist t' and t'' such that $0 \leqslant t' < t'' < 1$ such that $\sigma(t') = \sigma(t'')$. With no loss of generality, we can assume $\sigma(t')$ is a vertex of G. We can suppose v is incident with F_i and F_j, with $i \neq j$. Apply Theorem 5.2 to find that one of the closed face chains in Λ starting and ending with F_i and F_j has length $< \rho_{\mathscr{E}_1}(G)$ and, therefore, represents a loop in $\mathscr{C}_1$. But this loop is freely homotopic to one of the subloops of σ that starts and ends at v. Thus, σ contains a subloop $\beta_1 \in \mathscr{E}_2$ that is freely $\mathscr{C}_1$-homotopic to σ.

There are at most finitely many ordered pairs (t', t'') such that $0 \leqslant t' < t'' < 1$ and $\sigma(t') = \sigma(t'')$ is a vertex of G. The number of such pairs is the *vertex multiplicity* of σ. Obviously, β_1 has smaller vertex multiplicity. Therefore, we can repeat this argument finitely often until we arrive at a simple loop γ_1 that is freely $\mathscr{C}_1$-homotopic to γ.

II. The construction of C_2.

A. The construction of σ_2. Of course we assume $M \geqslant 2$. Now consider the wheel-neighbourhood N_i of v_i, $i = 1, 2, ..., l$. The subpath of the loop γ from v_{i-1} through v_i to v_{i+1} splits N_i into a closed left-hand side $N_{i,L}$ and a closed right-hand side, which meet exactly on the subpath.

The subgraph of $G - \{v_{i-1}, v_{i+1}\}$ contained in $N_{i,L}$ contains a vertex adjacent to v_i, as otherwise there is a single face incident with all of v_{i-1}, v_i, and v_{i+1}, which shows there is a closed face chain of length $l - 1$ that represents a loop freely homotopic to γ. This contradicts the hypothesis that the $\mathscr{C}_1$-length of γ is l.

Let η_i: $[0, 1] \rightarrow N_{i,L}$ be a simple path whose image is the boundary of $N_{i,L}$ in G from v_{i-1} to v_i. Let t_i be the least $t > 0$ for which $\eta_i(t)$ is a vertex x_i of G adjacent to v_i and let t^i be the largest $t < 1$ such that $\eta_i(t)$ is a vertex y_i of G.

Clearly, $y_{i-1} = x_i$, where the indices are read modulo l. Let α_i: $[0,1] \rightarrow N_{i,L}$ be the subpath of η_i restricted to $[t_i, t^i]$, so α_i is a simple path from x_i to y_i. The composition σ_2 of $\alpha_1, ..., \alpha_l$ is a loop freely homotopic to γ.

B. Construction of γ_2. Suppose σ_2 is not a simple loop. Then there exist $0 \leqslant t_1 < t_2 < 1$ such that $\sigma_2(t_1) = \sigma_2(t_1)$. Since each α_i is simple and every vertex has a wheel neighbourhood, it must be that there exist i and j and numbers t' and t'', not both either 0 or 1, such that $\alpha_i(t') = \alpha_j(t'')$. Clearly, we can choose t' and t'' so that $\alpha_i(t') = \alpha_j(t'') = v \in V(G)$.

There is a face F' of G incident with both v_i and v and there is a face F'' of G incident with both v_j and v. Let Λ' be the face chain v_i, F', v, F'', v_j and recall $\Lambda = v_0, F_1, v_1, ..., F_l, v_l$ is the face chain representing γ. Since $M \geqslant 2$, we can apply Theorem 5.2(2) to Λ and Λ', so there is a closed face chain in $\Lambda \cup \Lambda'$ through Λ which is so short that it must represent a loop in $\mathscr{C}_1$. The other closed face chain represents a loop in $\mathscr{E}_2$. But each of these is freely homotopic to one of the subloops of σ_2 from v to v. Let β_1 be the subloop of σ_2 that is in $\mathscr{E}_2$.

Clearly, the vertex multiplicity of β_1 is less than that of σ_2. Therefore, in finitely many steps we obtain γ_2 that has 0 vertex multiplicity and is, by Lemma 5.1, freely $\mathscr{C}_1$-homotopic to γ. It traverses the cycle C_2.

C. C_2 is disjoint from C_1. Suppose to the contrary that they have a vertex v in common. Since $\gamma_2(\mathbf{I}) \subseteq \bigcup_{i=1}^{l} N_{i,L}$ and $\gamma_1(\mathbf{I})$ is contained in the corresponding right-hand sides, there exist i and j such that v is a vertex of both N_i and N_j. There exist faces F' and F'' such that F' is in $N_{i,L}$ and is incident with both v_i and v and F'' is in $N_{i,R}$ and is incident with both v_j and v.

Apply Theorem 5.2(2) to the face chain $\Lambda'' = v_i, F', v, F'', v_j$ and the original face chain Λ. Of the two closed face chains through Λ'', one is so short that it represents a loop in $\mathscr{C}_1$. This face chain contains another closed face chain through Λ'' that represents a simple loop. This simple loop is crossed transversely only once by γ, contradicting the assumption that $(\mathscr{C}_1, \mathscr{E}_1)$ is a crossing partition. Thus, C_2 is disjoint from C_1.

III. Construction of C_n, given $C_1, ..., C_{n-1}$.

A. Construction of σ_n. We assume as inductive hypotheses that $M \geqslant n \geqslant 3$ and that, for $j = 3, 4, ..., n-1$, for each vertex v of C_j, there is a face F_v incident with both v and a vertex $\hat{v}$ in C_{j-2}—if j is odd, then F_v is on the left-hand side of γ_j and the right-hand side of γ_{j-2}, while if j is even, then F_v is on the right-hand side of γ_j and the left-hand side of γ_{j-2}. For each pair of vertices v, v' of C_j, let $\alpha_j(v, v')$ be the simple subpath of γ_j joining v and v' (in that order, so $\gamma_j = \alpha_j(v, v') \circ \alpha_j(v', v)$). In F_v choose a simple path β_v from v to $\hat{v}$. Then we also assume that $\alpha_j(v, v')$ is $\mathscr{C}_1$-homotopic to $\beta_v \circ \alpha_{j-2}(\hat{v}, \hat{v}') \circ \beta_{v'}^{-1}$. (This means that there are loops in $\mathscr{C}_1$ that can be attached to the former path such that the resulting path is homotopic (with fixed endpoints) to the latter path.) The base of the inductive construction is $n = 1$ and $n = 2$, for which these additional considerations are vacuous.

We now show how to construct σ_n freely homotopic to γ_{n-2}. For sake of definiteness, we shall assume n is odd. The argument is slightly different in the two cases—we shall indicate where the differences occur. Let $0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ be those t such that $\gamma_{n-2}(t) \in V(G)$. For each $i = 1, 2, ..., m$, let $w_i = \gamma_{n-2}(t_i)$. We are constructing on the right-hand side of γ_{n-2}—by Proposition 3.4, γ_{n-2} is orientation preserving. (In case n is even, we would construct σ_n on the left-hand side of γ_{n-2}.)

For each $i = 1, 2, ..., m$, let N_i denote the wheel neighbourhood of w_i and let $N_{i,R}$ denote the closed disc in N_i on the right-hand side of γ_{n-2}. Thus, $N_{i,R}$ is bounded by a subpath of γ_{n-2} and a path Q_i in G. We must deal with the possibility that some of the $N_{i,R}$ consist of a single face.

The ends a_i and b_i of Q_i are obviously among $w_1, w_2, ..., w_m$, labelled so that in γ_{n-2}, the order of traversal is a_i, w_i, b_i. Since G is 3-connected and 3-representative, no single face is incident with all of the w_i, and, therefore, some (at least two) of these vertices, say $w_{i_1}, w_{i_2}, ..., w_{i_r}$, are such that w_{i_s} is incident with an edge in $N_{i_s,R}$ that is not in $\gamma_{n-2}(\mathbf{I})$. We choose the labelling so that $0 < i_1 < i_2 < \cdots i_r \leqslant m$.

Now we are ready to construct σ_n. For $s = 1, 2, ..., m$, let μ_s be a simple path traversing Q_{i_s} from a_{i_s} to b_{i_s}. Let x_s be the first vertex adjacent to w_{i_s} after a_{i_s} that μ_s encounters and let y_s denote the last vertex encountered by μ_s before b_{i_s}. Now let η_s denote the subpath of μ_s from x_s to y_s.

A little thought shows that $y_{s-1} = x_s$, for $s = 1, 2, ..., r$, with the indices read modulo r. We set σ_n to be the composition of $\eta_1, \eta_2, ..., \eta_r$. By the construction, it is obvious that σ_n is freely homotopic to γ_{n-2}.

B. The construction of γ_n. Before we just go ahead and turn σ_n into a simple loop, we must be sure to deal with one special situation first, which will guarantee that the simple loop is disjoint from C_{n-2}. Suppose that some vertex w_j of C_{n-2} is traversed by σ_n. Then there is a vertex w_{i_s} and a face F that is incident with both w_j and w_{i_s}. Let β be a simple path in F joining w_j and w_{i_s}.

We first rule out the possibility that F is on the left-hand side of γ_{n-2} at w_j. For we can, as before, find a face chain from each of w_j and w_{i_s} back to C_1, and obtain a loop that is in $\mathscr{C}_1$. This loop "lifts" back to a loop κ including β and a portion of γ_{n-2} that is in $\mathscr{C}_1$. But then κ is simple and crosses γ_{n-2} transversely only once, contradicting the assumption that $(\mathscr{C}_1, \mathscr{E}_1)$ is a crossing partition.

Therefore, F is on the right-hand side of γ at w_j and it follows that $j = i_{s'}$ for some s'. We can assume that $1 \leqslant s < s' \leqslant r$ and we can deduce that s and s' are not consecutive in the cyclic ordering $(1, 2, ..., r)$. Choose such an s' so that $s' - s$ is minimized.

In particular, $w_{i_{s'-1}}$ does not occur in the boundary of the face F. This means that, in the wheel neighbourhood of $w_{i_{s'}}$, the face F is not that face incident with $w_{i_{s'-1}}$, so that w_{i_s} is traversed by $\eta_{s'}$.

There are face chains Λ_s and $\Lambda_{s'}$ from each of w_{i_s} and $w_{i_{s'}}$ to Λ, using faces on the left-hand side of γ_{n-2}. Let Λ' be the face chain made up from F, Λ_s, and $\Lambda_{s'}$. By Theorem 5.2(1), this is so short that, together with part of Λ, it face represents a loop in $\mathscr{C}_1$. Moreover, it lifts to show that either the subpath of σ_n from $w_{i_{s'}}$ to w_{i_s} or the subpath of σ_n from w_{i_s} to $w_{i_{s'}}$, together with a path across F, makes a loop in $\mathscr{C}_1$.

If a and b are the neighbours of $w_{i_{s'}}$ in ∂F, then these are repeated vertices in σ_n. For one or the other of these, there is a subloop of σ_n with the neighbour as basepoint that is both in $\mathscr{C}_1$ and contains the traversals of both w_{i_s} and $w_{i_{s'}}$. This is a "special" subloop.

Now suppose there is some vertex v of G that is visited more than once by σ_n. We note it is not visited more than r times. Let us suppose $\sigma_n(t_1) = \sigma_n(t_2) = v$ for some $0 \leqslant t_1 < t_2 < 1$. Then σ_n restricted to each of $[t_1, t_2]$ and $[0, t_1] \cup [t_2, 1]$ is a loop whose composition is in $\mathscr{E}_2$.

CLAIM 1. *One of these two loops is in* $\mathscr{C}_1$.

This claim is the heart of the whole matter. Suppose v occurs in $\eta_p(\mathbf{I})$ and $\eta_q(\mathbf{I})$, with $1 \leqslant p < q \leqslant r$. Then v is incident with a face F_1 that is also incident with w_{i_p} and with a face F'_1 that is also incident with w_{i_q}. Now we can apply the inductive assumption to find faces $F_2, F_3, ..., F_{(n-1)/2}$ and $F'_2, F'_3, ..., F'_{(n-1)/2}$ so that the face chain $v, F_1, w_{i_p}, F_2, \hat{w}_{i_p}, F_3, ..., F_{(n-1)/2}, v^*$ joins v to a vertex v^* in C_1 and the face chain $v, F'_1, w_{i_q}, F'_2, \hat{w}_{i_q}, F'_3, ..., F'_{(n-1)/2}, v^+$ joins v to a vertex v^+ in C_1.

Thus, the face chain Λ_v from v^* to v^+ obtained by concatenating these face chains has length $n-1$. Since $n \leqslant M$, Theorem 5.2(1) applies. (If n is even, a similar argument, which we omit, is required to show that Theorem 5.2(1) applies.) Therefore, there is a face chain Λ_v^* containing Λ_v and a portion of the original face chain Λ that represents a loop κ_1 in $\mathscr{C}_1$. Now we can use the inductive assumption to find a loop κ_2 that is freely $\mathscr{C}_1$-homotopic to κ_1 (and so it is in $\mathscr{C}_1$, but it only uses the portions of Λ_v down to C_3 and then cuts across γ_3).

Repeating this $(n-1)/2$ times, we get a loop $\kappa_{(n-1)/2}$ that consists of a simple path in F_1, a simple path in F'_1, and a portion of γ_{n-2}; $\kappa_{(n-1)/2}$ is in $\mathscr{C}_1$. But by construction of σ_n, the loop $\kappa_{(n-1)/2}$ is homotopic to one of the subloops of σ_n. Therefore, one of the two loops in the statement of the claim is in $\mathscr{C}_1$ and Claim 1 is proved.

Because each vertex of G is visited at most finitely often by σ_n, in finitely many steps we can remove loops in $\mathscr{C}_1$ from σ_n and obtain a simple loop γ_n that is in $\mathscr{E}_2$, being sure to start by removing the "special" subloops.

The proof will be complete when we show that the inductive properties still hold. First, it is obvious that every vertex in $C_n = \gamma_n(\mathbf{I})$ is incident with a face that is on the left-hand side of γ_n that is also incident with some one of the w_{i_s} on the right-hand side of γ_{n-2}. Therefore, we need only prove that if v and v' are any two vertices of C_n, then each of the portions of γ_n between them is $\mathscr{C}_1$-homotopic to the corresponding portion of γ_{n-2}.

But this is also obvious from the construction; there are only finitely many $\mathscr{C}_1$ loops in σ_n that have been avoided in obtaining γ_n. In the portion of the traversal of γ_n under consideration, replace these $\mathscr{C}_1$ loops to obtain a subpath of σ_n from v to v' that is homotopic to the path in γ_{n-2}. Therefore, the path in γ_n is $\mathscr{C}_1$-homotopic to this path, as required.

C. C_n is disjoint from C_{n-2}. This was guaranteed by the removal of the "special" subloops of σ_n.

D. C_n is disjoint from C_j, $j = 1, 2, ..., n-3, n-1$. Let j be largest such that C_n and C_j have a common vertex v. If j and n have different parity (i.e., if j is even), then there is a face chain from v through $C_{n-2}, C_{n-4}, ..., C_1$ and through $C_{j-2}, C_{j-4}, ...$ to a vertex incident with a face of Λ. The last faces of these chains are on different sides of γ. These face chains combine to a single face chain Λ' to which Theorem 5.2(1) applies. Thus, there is a closed face chain containing part of Λ that is so short it represent a loop in $\mathscr{C}_1$. In fact, we can find within this first face chain a face chain that contains a part of Λ and the last faces mentioned previously, but which represents a simple loop. This face chain is so short that the simple loop in $\mathscr{C}_1$.

But this simple loop crosses γ transversely exactly one, a contradiction to the assumption that $(\mathscr{C}_1, \mathscr{E}_1)$ is a crossing partition.

If, on the other hand, j is odd, then there are face chains from v through $C_{n-2}, ..., C_1$ and through $C_{j-2}, ..., C_1$. These combine with a face chain within Λ to produce a face chain so short that it represents a loop in $\mathscr{C}_1$. Again, we can find within this a face chain that contains the first two faces of the ones down from v and it represents a simple loop in $\mathscr{C}_1$. This loop crosses γ_{n-2} transversely exactly once, again contradicting the fact that $(\mathscr{C}_1, \mathscr{E}_1)$ is a crossing partition. ∎

7. Consequences of the Main Theorem

There are obviously many possible consequence of Theorem 6.1, as we have many possible choices for $\mathscr{C}_1$, $\mathscr{C}_2$, and γ. One somewhat surprising result is the following.

Corollary 7.1. *Let G be a graph embedded in an orientable surface Σ and let $(\mathscr{C}, \mathscr{E})$ be any complete partition. Then G contains a set of $\lfloor(\rho(G)-1)/2\rfloor$ pairwise disjoint, pairwise freely homotopic cycles, each traversed by a simple loop in $\mathscr{E}$.*

We just remark that to prove Corollary 7.1, we must have $\rho(G) \geqslant 3$ for the conclusion to have any content, in which case we can assume G is 3-connected (see [RoV]). Thus, Theorem 6.1 applies and we can choose γ to be in $\mathscr{E}$ so that it attains the $\mathscr{C}$-representativity. The following (which is the main concrete result of this work) is obtained by choosing $(\mathscr{C}, \mathscr{E})$ to be the separating partition $(\mathscr{C}_s, \mathscr{E}_s)$.

Corollary 7.2. *Let G be a graph embedded in an orientable surface Σ. Then G contains a set of $\lfloor(\rho(G)-1)/2\rfloor$ pairwise disjoint, pairwise freely homotopic nonseparating cycles.*

In a similar vein, we have the following.

Corollary 7.3. *Let G be a graph embedded in a nonorientable surface Σ. Let $(\mathscr{C}, \mathscr{E})$ be a complete partition such that $\mathscr{E}$ contains only orientation-reversing loops. Then G has a set of $\lfloor(\rho(G)-1)/4\rfloor$ pairwise disjoint pairwise homotopic cycles, each traversed by a loop which is 2-freely homotopic to a loop in $\mathscr{E}$.*

If the nonorientable genus of Σ is at least 2, then each of the cycles in Corollary 7.3 separates the surface into two pieces, neither of which is homeomorphic to a disc. There is more about separating cycles in the next section.

We conclude this section with some remarks about algorithms. One of the most important issues that needs resolution is to know how a complete partition might be "given."

Thomassen's three path algorithm [T1] shows that if $(\mathscr{C}, \mathscr{E})$ is a complete partition and there is a polynomial algorithm for determining if a (simple loop that traverses a) cycle P is in $\mathscr{E}$, then there is a polynomial algorithm for finding a shortest cycle in $\mathscr{E}$. Thomassen describes a polynomial algorithm to determine if a cycle is separating and a polynomial algorithm to determine if a cycle is essential. Therefore, our proofs shows there is a polynomial time algorithm to find the $\rho/2$ pairwise disjoint, pairwise homotopic nonseparating cycles guaranteed in Corollary 7.2.

But for other, more exotic, complete partitions, the situation is less clear. It would be quite interesting to know if membership in $\mathscr{E}$ can always be determined in polynomial time for any complete partition $(\mathscr{C}, \mathscr{E})$.

8. Noncontractible Separating Cycles

In this section, we prove an analogue of Corollary 7.2 for noncontractible separating cycles in the embedded graph. Recall that $(\mathscr{C}_0, \mathscr{E}_0)$ is the fundamental partition, so that $\mathscr{C}_0$ consists of the contractible loops.

Theorem 8.1. *Let G be a 3-connected embedding with representativity ρ in an orientable surface of genus at least 2. Let $(\mathscr{C}, \mathscr{E})$ be any complete partition, let $\gamma \in \mathscr{E}$ and let $l = l(\gamma, \mathscr{C}_0)$. Then G contains*

$$\left\lfloor \frac{\rho + \rho_{\mathscr{E}}(G) - l - 1}{8} \right\rfloor - 1$$

pairwise disjoint, pairwise freely homotopic cycles, each separating a homotope of γ from some other loop in $\mathscr{E}_0$.

Proof. Let t be the largest positive integer such that $2t \leqslant (\rho + \rho_{\mathscr{E}} - l - 1)/2$. In order for the theorem to have any content, we must have $2t \geqslant 8$ and, therefore, $\rho \geqslant 17$. By theorem 6.1, there are $2t$ $\mathscr{C}_0$-parallels in G, each homotopic to γ. Let them be $C_1, C_2, ..., C_{2t}$, labelled so that, for each $j = 2, ..., 2t$, C_{j-1} and C_j bound a cylinder $\mathbf{C}_j$ containing all the cycles $C_1, ..., C_{j-2}$.

If γ is separating, then the cycles $C_1, ..., C_{2t}$ more than satisfy the conclusion of the theorem. Thus, we may assume γ is not separating.

Let $v_0, f_1, v_1, f_2, ..., f_k, v_k$ be a shortest face chain disjoint from the interior of $\mathbf{C}_1$ with $v_0 \in V(C_1)$ and $v_k \in V(C_2)$. Let γ' denote the simple path through these faces and their connecting vertices, so that γ' joins a vertex of $f_1 \cap C_1$ to a vertex of $f_k \cap C_2$.

Now we look for disjoint paths in G homotopic to γ', disjoint from the interior of $\mathbf{C}_1$, and only their endpoints are in $C_1 \cup C_2$. (In this context, a homotopy allows the endpoints to vary, although they must remain on C_1 and C_2.) Let W_1 be the walk from v_0 to v_k through the boundaries of $f_1, f_2, ..., f_k$ on one side of γ'.

Suppose this walk has a repeated vertex v, so v is incident with f_i and f_j, with $1 \leqslant i < j \leqslant k$. If $j \neq i+1$, then the face chain $v_0, f_1, ..., v_{i-1}, f_i, v, f_j, v_j, ..., f_k, v_k$ joins C_1 to C_2 and is shorter than $f_1, ..., f_k$. This is impossible, so $j = i+1$. As G is 3-connected and $\rho \geqslant 3$, the intersection of two faces is either empty, a vertex, or an edge with its two ends. Therefore, f_i and f_{i+1} must intersect in an edge and the walk W_1 traverses this edge twice, as $(..., u, e, v, e, u, ...)$. Simply deleting these two traversals of e from the walk (for all occurrences of such traversals) produces a path P_1 homotopic to γ'. Note that P_1 is disjoint from the interior of $\mathbf{C}_1$ and P_1 has one end on C_1 and the other end on C_2. It is not clear that P_1 necessarily is internally disjoint from C_1 and C_2—in fact this need not be the case. But any such intersections must take place within the cylinder containing γ and bounded by C_3 and C_4. Therefore, the fact that the face chain $v_0, f_1, ..., f_k, v_k$ is shortest implies that simply taking P_1 to start at its last intersection with C_1 and finish at its first intersection with C_2 yields a path still homotopic to γ'.

By an argument very similar to that used in the proof of Theorem 6.1 to get the loop σ_2 (on the way to obtaining C_2), we get a walk W_2 homotopic to γ' and W_2 is on the side of γ' opposite to P_1, with the ends being chosen so that W_2 meets C_1 and C_2 only at its ends. We claim that W_2 is disjoint from P_1 and contains a path P_2 homotopic to γ'.

As to disjointness, suppose there is a vertex v of P_1 also in W_2. Then v is in the boundary of the face f_i and is in the wheel neighbourhood of the vertex w in common between f_{j-1} and f_j. Suppose, without loss of generality, that $i \geqslant j$. If $i \neq j, j+1$, then there is a face chain shorter than $v_0, f_1, ..., f_k, v_k$ joining C_1 and C_2, which is impossible. If $i = j$ or $j+1$, then there is a simple loop from v to w, through f_j and f_i back to v. This loop goes through at most three vertices and faces, crosses γ' transversely exactly once, and is disjoint from the interior of $\mathbf{C}_1$. Therefore, it is noncontractible, so $\rho \leqslant 3$, another contradiction. Therefore, W_2 and P_1 are disjoint.

Observe that both W_2 and P_1 are homotopic to γ' (keeping the endpoints in the homotopy on C_1 and C_2). Suppose W_2 has a repeated vertex v, with v in the wheel neighbourhoods of w and x. In order not to have a shorter sequence of faces joining C_1 and C_2, w and x can be separated by no more than two faces in the chain $v_0, f_1, ..., f_k, v_k$. This produces a face chain through v, w, x of length at most 4. As $\rho \geqslant 17$, this must be contained in a disc. Deleting the portion of W_2 that is in this disc produces a shorter walk that is homotopic to γ'. Continuing in this way, we end up with a path P_2, contained in W_2, that is homotopic to γ'. This path, P_2, has one end in C_1 and the other in C_2, but is otherwise disjoint from the cyclinder $\mathbf{C}_1$.

For $i=1, 2$, let u_i be the vertex of P_i in C_1 and let v_i be the end of P_i in C_2. There are paths Q_1 and Q_2 in C_1 and C_2, respectively, with Q_1 joining u_1 and u_2 and Q_2 joining v_1 and v_2, such that $P_1 \cup P_2 \cup Q_1 \cup Q_2$ is a contractible cycle.

The first noncontractible separating cycle we get is the cycle $C_1^* = P_1 \cup P_2 \cup R_1 \cup R_2$, where R_1 is the path complementary to Q_1 in C_1 joining u_1 and u_2, and R_2 is the analogous path in C_2. It is clear that this is separating, as one side is the region F consisting of $\mathbf{C}_1$ and the disc bounded by $P_1 \cup P_2 \cup Q_1 \cup Q_2$. Cut out F and cap the cycle C_1^* with a disc on both pieces, giving surfaces Σ_1 and Σ_2, with the former containing F. Then Σ is the connected sum of Σ_1 and Σ_2, so $\mathfrak{g}(\Sigma) = \mathfrak{g}(\Sigma_1) + \mathfrak{g}(\Sigma_2)$.

The only question is, what is $\mathfrak{g}(\Sigma_1)$? It is at least 1, since it contains the noncontractible loop γ that we started with. It is not more than 1, since there is no noncontractible loop that is not freely homotopic to γ and disjoint from γ. Hence, it must be exactly 1. Since $\mathfrak{g}(\Sigma) > 1$, we see that $\mathfrak{g}(\Sigma_2) \geqslant 1$ and, therefore, C_1^* is an noncontractible separating cycle.

In order to construct the remaining separating parallels, recall we have the $2t$ parallels $C_1, ..., C_{2t}$. We suppose we have $2j$ disjoint homotopic paths $P_1, ..., P_{2j}$, each joining a vertex of C_1 to a vertex of C_2, but otherwise disjoint from $\mathbf{C}_1$. The labelling is such that $P_1, P_3, ...$ occur in this order going away from γ' and $P_2, P_4, ...$ occur in this order going away from γ' and on the other side of γ' from $P_1, P_3, ...$ If $j < t-1$, then we describe how to obtain P_{2j+1} and P_{2j+2}. We also assume that, for every $i \leqslant j$, each vertex of P_{2i-1} is incident with a face in a face chain, disjoint from the interior of the cylinder bounded by C_1 and C_2, of length at most $i-1$ ending at a face incident with a vertex of P_1; for the vertices of P_{2i}, they are in such a chain of length at most $i-1$ ending at a face incident with a vertex of P_2.

The paths P_{2j+1} and P_{2j+2} are to be found in the wheel neighbourhoods of the vertices of P_{2j-1} and P_{2j}, respectively, making sure we go on the "outer" side. The concerns which need to be addressed are the following:

(1) we actually construct paths;

(2) the new paths are disjoint from the previous paths and from each other;

(3) the new paths are homotopic to the previous paths; and

(4) no internal vertex of any path is in $C_1 \cup C_2$.

The construction is essentially the same as that for the $\mathscr{C}_1$-parallels presented in the proof of Theorem 6.1. We must determine the walk that will contain the path. So, for example, we consider the possibility that two vertices v, w of P_{2j-1} are incident with a common face f on the outer side of P_{2j-1}. There are face chains of length at most $j-1$ from each of v and w to P_1, ending at vertices incident with faces f_r and f_s, respectively, with $1 \leqslant r \leqslant s \leqslant k$. Then there is a face chain through $f_1, ..., f_r, v, f, w$, and $f_s, f_{s+1}, ..., f_k$ of length at most $2j-2+1+r+k-s+1$ joining C_1 to C_2. This must be at least k, so $2j \geqslant s-r$. On the other hand, there is a closed face chain through v, f, w, and $f_r, f_{r+1}, ..., f_s$ which has length at most $2j-2+1+s-r+1 \leqslant 4j \leqslant 4(t-2) < \rho$. Therefore, the loop face-represented by this face chain is contractible. In this way, we get the same nesting effect as in the proof of Theorem 6.1 and can describe the required walk W which is homotopic to P_{2j-1}.

The remaining items are handled by similar arguments. It is useful to keep in mind that the best general inequality we can get for $4t$ is $4t \leqslant \rho - 1$. In one case, we actually are right at this inequality, so that we cannot guarantee any more disjoint paths in this homotopy class.

Each of the remaining noncontractible separating cycles C_j^*, $j \geqslant 2$, is obtained from the cycles C_{2j-1} and C_{2j} and the paths P_{2j-1} and P_{2j}, in a manner similar to that for C_1^*. We must be careful to make sure C_j^* is a cycle and that it is homotopic to C_1^*. If P_{2j-1} and P_{2j} both meet each of C_{2j-1} and C_{2j} in a single point, then there is no difficulty.

Suppose, then, for example, that P_{2j} has at least two vertices in common with C_{2j}. (There are really four cases here, but they are all handled in the same manner.) Let x be the last vertex (as we traverse P_{2j} from C_2 to C_1) in P_{2j} that is in C_{2j}. There is a face chain of length at most j joining x to a vertex w_i incident with both f_i and f_{i+1} (faces of the original chain joining C_1 and C_2). There is a second face chain of length at most j joining x to a vertex of the face chain of length l that is contained in $\mathbf{C}_1$. Combining these two face chains with the part of the face chain from w_i to C_1 (which has length i), we get a face chain of length at most $2j+i$ from C_1 to C_k. This must be at least k, so that $2j \geqslant k-i$.

On the other hand, using the other part of the face chain from C_1 to C_2, we get a face chain between two vertices of the face chain of length l of length at most $2j+(k+1-i)$. If this has length at most $(\rho+\rho_{\mathscr{E}}-l-1)/2-1$, then Theorem 5.2 implies that there is a face chain using this one and part of the one representing γ that is represents a contractible loop. This happens certainly if $4j+1 \leqslant (\rho+\rho_{\mathscr{E}}-l-1)/2-1$, which is the limiting factor in the number of separating parallels that we get. ∎

Corollary 8.1.1. *Let G be a graph embedded with representativity ρ in an orientable surface Σ of genus at least two. Then there is a set of $\lfloor(\rho-1)/8\rfloor-1$ pairwise disjoint, pairwise homotopic noncontractible separating cycles.* ∎

9. Miscellaneous Improvements and Results

In this section, we discuss the existence of a single noncontractible separating cycle. In the nonorientable case, Corollary 7.3 assures us that such a cycle exists whenever $\rho \geqslant 5$ and $\tilde{g}(\Sigma) \geqslant 2$—we do not know how to improve this. However, in the orientable case, Theorem 8.1 requires $\rho \geqslant 17$ to get the first noncontractible separating cycle. This is not as good as the bound of 7 in [ZZ]. However, if one only wishes just one noncontractible separating cycle, the argument of [ZZ] can be improved, as shown below, to apply to 6-representative embeddings in orientable surfaces.

Theorem 9.1. *Let G be embedded in an orientable surface Σ of genus at least 2. If G is 6-representative, then G has a noncontractible separating cycle.*

Proof. As discussed earlier, we may assume G is 3-connected. Let ρ be the representativity of G and let $v_0, f_1, ..., f_\rho, v_\rho$ be a face chain face-representing a noncontractible loop γ. If $\rho \geqslant 7$, then we let C_1 be a cycle through the boundaries of the f_i on one side of γ (as in the proof of Theorem 6.1). (So if γ is separating, then we are done, so we can assume γ is not separating.)

We then get C_2 as in the proof of Theorem 6.1, so C_2 goes through the boundaries of the wheel neighbourhoods of the vertices through which γ passes, on the other side of γ.

If $\rho = 6$, then we let $v_1, v_2, ..., v_6$ be the vertices through which γ passes, in this order. Choose C_1 to go through v_1, v_3, v_5 and the boundaries of the wheel neighbourhoods of v_2, v_4, v_6—all on one side of γ. Choose C_2 to go through v_2, v_4, v_6 and the boundaries of the wheel neighbourhoods of v_1, v_3, v_5, all on the other side of γ. It is easy to see that these are disjoint homotopic noncontractible cycles.

Let $x_0, g_1, ..., g_k, x_k$ be a shortest face chain joining a vertex x_0 of C_1 and a vertex x_k of C_2 that is disjoint from the interior of the cylinder bounded by C_1 and C_2. Let γ' be the path face-represented by the g_j, so γ' meets the graph exactly at $x_0, ..., x_k$. We pick one side of γ' on which to construct P_1; P_2 is constructed on the other side of γ'.

Consider the wheel neighbourhood of x_{k-1}. Suppose some other face g incident with x_{k-1} is incident with some vertex u of C_2, with g on the specified side of γ'. As we rotate around x_{k-1}, starting at the portion of γ' joining x_{k-1} to x_k and staying on the specified side of γ', let g be selected so that as we continue from g in this rotation, there is no face between g and g_{k-1} that has a vertex in C_2. We redefine our face chain so that g_k is this face g. Now, with γ' going through the new g_k and keeping the same side, there are no faces incident with both x_{k-1} and a vertex of C_2 on that side of γ'. We perform the same operation at x_1, so that no face incident with x_1 on the specified side of γ' is incident with a vertex of C_1.

Now, let P_1 be the path through the face boundaries on the side of γ' opposite to the specified side. (Of course, P_1 might not be a path, but is only trivially not a path—see the proof of Theorem 8.1.) We proceed exactly as in the proof of Theorem 8.1 to construct the path P_2. However, the arguments presented in the proof of Theorem 8.1 do not adequately deal with the possibility that W_2 returns to C_1 or hits C_2 twice. In the context of Theorem 8.1, this did not matter, since we had this all within the cylinder bounded by C_3 and C_4.

Suppose W_2 has two vertices u and v incident with C_1. Then we can assume $u = x_0$ and v is incident with a face f incident with x_0 (recall no face incident with x_1 that is used in creating W_2 is incident with a vertex of C_1). Let x_0 be incident with the original face f_i and v with f_j. Then $|i-j| \leqslant 2$, by representativity. This gives a face chain of length at most 4, representing a contractible loop, so we can simply delete the first part of W_2 and get a new walk in the same homotopy class. Continuing in this way, we may assume W_2 has only one vertex in C_1.

At the other end, there is the same problem to consider, but it requires a little more delicacy. Suppose W_2 meets C_2 at the vertex v before the end of W_2, which is at u. By the construction, the face incident with v must be incident also with x_k. Let v and x_k be incident with faces in the wheel neighbourhoods of v_i and v_j, respectively. By representativity, $|i-j| \leqslant 3$. This gives a face chain of length at most 6, so if $\rho \geqslant 7$, the face-represented loop is contractible and we proceed as in the preceding paragraph. (This is the argument of [ZZ].)

If $\rho = 6$ and $|i-j| \leqslant 2$, then we are done anyway. If $|i-j| = 3$, then our careful construction of C_1 and C_2 shows that we can save one face in getting to one of v_i and v_j, because in the construction of C_2, we did not use both the wheel neighbourhood at v_i and the wheel neighbourhood at v_j. Therefore, the face chain has length at most 5, and we get the required contractible face-represented loop.

The rest of the proof proceeds exactly as in the proof of Theorem 8.1 for the first noncontractible separating cycle. ▮

Finally, we have a simple, but related fact.

THEOREM 9.2. *Let C_1 and C_2 be disjoint homotopic noncontractible cycles in a graph G embedded in a surface Σ, and let $l = l(C_1, \mathscr{C}_0)$. Then G contains l totally disjoint paths, each contained within the cylinder bounded by C_1 and C_2 and each having one end in C_1 and the other end in C_2.*

Proof. Let $P_1, ..., P_k$ be a maximum collection of such disjoint paths in the cylinder Q bounded by C_1 and C_2. By Menger's theorem, there are k vertices $v_1, ..., v_k$ such that every path in Q from C_1 to C_2 (from now on called a (C_1, C_2)-*path*) goes through at least one of the v_i. Without loss of generality, we assume v_i is on P_i.

For $i = 1, ..., k$ and $j = 1, 2$, let the path P_i have end u_{ij} on C_j. The cyclic order of the u_{i1} in C_1 is the same as that of the u_{i2} on C_2. We assume these orders are $(u_{1j}, u_{2j}, ..., u_{kj})$, for $j = 1, 2$.

We claim there is a face f_i within Q incident with both v_i and v_{i+1}. To see this, let Q_{ij} denote the part of C_j between u_{ij} and $u_{(i+1)j}$. Let P_{i1} and P_{i2} be the segments of P_i from C_1 to v_i and from v_i to C_2, respectively. There is no path P in the graph from an internal vertex of $P_{i1} \cup Q_{i1} \cup P_{(i+1)1}$ to an internal vertex of $P_{i2} \cup Q_{i2} \cup P_{(i+1)2}$ so that P (except for its endpoints) is contained in the open disc bounded $P_i \cup P_{i+1} \cup Q_{i1} \cup Q_{i2}$. For if such a P exists, then there is a (C_1, C_2)-path disjoint from the $\{v_1, ..., v_k\}$. It follows that the required face f_i exists.

The face chain $v_1, f_1, v_2, ..., v_k, f_k, v_1$ face-represents a loop that is homotopic to C_1 and meets the graph only at $v_1, ..., v_k$. By definition of l, we have $k \geqslant l$, so we are done. ∎

References

[AVR] M. Anderson, R. B. Richter, and R. P. Vitray, Representativity and discs in embedded graphs, preprint.

[B] R. Brunet, "On the Homotopy of Polygons in Graph Embeddings," Ph.D. thesis, Carleton University, 1993.

[FHRRo] J. R. Fiedler, J. P. Huneke, R. B. Richter, and N. Robertson, Computing the orientable genus of projective graphs, *J. Graph Theory* **20** (1995), 297–308.

[GH] M. J. Greenberg and J. R. Harper, "Algebraic Topology: A First Course," Mathematics Lecture Notes Series, Addison–Wesley, New York, 1981.

[HR] P. Hoffman and B. Richter, Embedding graphs in surfaces, *J. Combin. Theory Ser. B* **36** (1984), 65–84.

[MR] B. Mohar and N. Robertson, Disjoint essential circuits in toroidal maps, *in* "Planar Graphs" (W. T. Trotter, Ed.), Dimacs Series in Discrete Math. and Theor. Comp. Sci. Vol. 9, Amer. Math. Soc., Providence, RI, 1993, pp. 109–130.

[Mu] J. R. Munkres, "Topology: A First Course," Prentice–Hall, Englewood Cliffs, NJ, 1975.

[RV] R. B. Richter and R. Vitray, Contractible and noncontractible curves in surfaces, preprint.

[RoV] N. Robertson and R. Vitray, Representativity of surface embeddings, *in* "Paths, Flows, and VLSI-Layout" (B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver, Eds.), pp. 293–328, Springer-Verlag, Berlin, 1990.

[S1] A. Schrijver, Disjoint circuits of prescribed homotopies in a graph on a compact surface, *J. Combin. Theory Ser. B* **51** (1991), 127–159.

[S2] A. Schrijver, Graphs on the torus and geometry of numbers, *J. Combin. Theory Ser. B* **58** (1993), 147–158.

[T1] C. Thomassen, Embeddings of graphs with no short noncontractible cycles, *J. Combin. Theory Ser. B* **48** (1990), 155–177.

[T2] C. Thomassen, Trees in triangulations, *J. Combin. Theory Ser. B* **60** (1994), 56–62.

[Y] X. Yu, Disjoint paths, planarizing cycles, and spanning walks, preprint.

[ZZ] X. Zha and Y. Zhao, On non-null separating circuits in embedded graphs, *Contemp. Math.* **147** (1993), 349–362; *in* "Graph Structure Theory" (N. Robertson and P. D. Seymour, Eds.), Amer. Math. Soc., Providence, RI, 1993.