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Systems of Curves on Surfaces

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It is proved that for each compact (bordered) surface Σ and each integer k there is a constant N with the following property: If Γ is a family of pairwise nonhomotopic closed curves on Σ such that any two curves from Γ intersect in at most k points, then Γ contains at most N curves.

1. INTRODUCTION

Let Σ be a (bordered) compact surface and k an integer. Suppose that we have a set Γ of pairwise nonhomotopic simple (closed) curves in Σ with the property that any two curves from Γ intersect in at most k points. It is proved that Γ cannot contain too many curves; i.e., there is a number N depending only on Σ and k such that $|\Gamma| \leqslant N$ (Theorems 3.3 and 3.4). The same holds for nonsimple curves as well (Theorem 3.5). This simple result does not seem to have a straightforward proof. It can be applied in the study of properties of graphs on surfaces. An example of such an application is presented in the last section. A special case when Σ is the torus is considered in Section 4 where we find linear upper bounds on the number of curves in Γ. On the other hand, our bounds for surfaces of genus greater than 1 are probably far from being optimal. However, examples from Section 5 show that the bounds will not be very small, in general. In the last part of the paper we add an application of Theorem 3.3. We present a short proof of the fact that for each compact surface Σ and an integer $k \geqslant 0$, there are only finitely many minor minimal embeddings in Σ of face-width k. This result has been verified previously (only for closed surfaces) with similar techniques but with longer proofs [MN, GRS]. Another application of Theorem 3.3 was obtained by Mohar and Robertson [MR] who considered the structure of nongenus embeddings of graphs in surfaces and proved that, up to certain generalized Whitney-type switchings, there are only a bounded number of types of nongenus embeddings in any fixed surface.

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Systems of simple closed curves on surfaces have been considered very early from the homotopy or homology point of view. However, we were not able to trace many results in the flavor of our work. The special case when curves are disjoint was solved by Malnič and Mohar [MM], while Fisk and Mohar [FM] proved Theorem 3.3 under a stronger hypothesis that each curve intersects other curves at most k times altogether.

Epstein [E] (extending the work of Baer [B1, B2]) proved that two freely homotopic simple closed curves on a surface are indeed isotopic. In particular, if they are disjoint, they bound a cylinder. It is also folklore that the algebraic intersection number of two curves (that gives only a lower bound on the number of points in the "geometric" intersection) depends only on the homology classes of curves. It is interesting to mention that our results (e.g., Theorem 3.3) are no longer true if we consider systems of simple closed curves with bounded algebraic intersection.

2. THE COMPLEXITY OF GRAPH EMBEDDINGS

Let Σ be a compact surface. We denote its interior by $\operatorname{int}\Sigma$ and its boundary by $\partial\Sigma$. The Euler characteristic of Σ is denoted by $\chi(\Sigma)$, and g_Σ is its genus (orientable or nonorientable). Graphs in this paper are finite and undirected. Loops and multiple edges are permitted. By V_G we denote the vertex set and by E_G the edge set of a graph G.

Let G be a graph (possibly disconnected) embedded in Σ such that $G \cap \partial\Sigma = \emptyset$. A well-known consequence of the Schoenflies' theorem is that G has a *regular neigborhood* N_G in Σ. This is a compact surface in Σ, formed by "small" disjoint discs around each vertex plus pairwise disjoint connecting "strips" along the edges. By $F_1, F_2, ..., F_r$ we denote the *faces* of G, i.e., the connected components of $\Sigma \backslash G$, and by $\hat{F}_1, \hat{F}_2, ..., \hat{F}_r$ the corresponding bordered compact surfaces obtained by *cutting* Σ *along* G, that is, the closures of connected components of $\Sigma \backslash N_G$. For each *dissecting component* $\hat{F}_i$ and its corresponding face F_i we have $\hat{F}_i \subset F_i$. Let

$$\psi_\Sigma = \max\{0, 1 - 2\chi(\Sigma)\}.$$

We define the *complexity* $\phi_\Sigma(G)$ of the dissection of Σ along G as

$$\phi_\Sigma(G) = \sum_{i=1}^{r} \psi_{\hat{F}_i}.$$

Clearly, $\phi_\Sigma(G) \geqslant 0$, and $\phi_\Sigma(G) = 0$ if and only if G is 2-cell embedded in Σ.

We will need to know how the complexity is changed under various operations on graphs, in particular under the addition or deletion of vertices and edges. Let v be a vertex and e an edge of a graph G. Then

$G-v$ is the vertex-deleted subgraph of G and $G-e$ is the subgraph with the deleted edge e. Suppose that we cut Σ along $G-e$. Let F be the face of $G-e$ containing e. We may assume that the intersection $\hat{e}=e\cap\hat{F}$ with the dissecting component $\hat{F}$ of F is a single arc. Note that $\hat{e}$ connects two distinct points on the boundary of $\hat{F}$ even if e is a loop in the graph. The edge e is *separating w.r.t.* $G-e$ if the cutting of $\hat{F}$ along $\hat{e}$ disconnects $\hat{F}$. If e is separating, then it is *strongly separating w.r.t.* $G-e$ if none of the dissecting components $\hat{F}'$ and $\hat{F}''$ of $\hat{F}$ is a disc, and it is *weakly separating w.r.t.* $G-e$ otherwise. If e is nonseparating, then it is *strongly nonseparating w.r.t.* $G-e$ if the cutting of $\hat{F}$ along $\hat{e}$ does not yield a disc, and *weakly nonseparating w.r.t.* $G-e$ otherwise (the latter case can occur only if $\hat{F}$ is a cylinder or a Möbius band). Let v be an isolated vertex of G. We say that v is *planarly isolated w.r.t.* $G-v$ if v belongs to a disc in the dissection along $G-v$.

LEMMA 2.1. *Let G be a graph embedded in a compact surface Σ such that $G\cap\partial\Sigma=\varnothing$. If v is an isolated vertex of G, then*

$$\phi_\Sigma(G)=\begin{cases}0, & G=\{v\} \text{ and } \Sigma \text{ is the 2-sphere}\\ 1, & G=\{v\} \text{ and } \Sigma \text{ is the projective plane}\\ \phi_\Sigma(G-v)+2, & v \text{ is not planarly isolated w.r.t. } G-v\\ \phi_\Sigma(G-v)+1, & v \text{ is planarly isolated w.r.t. } G-v.\end{cases}$$

If e is an edge of G, then

$$\phi_\Sigma(G)=\begin{cases}\phi_\Sigma(G-e)-2, & e \text{ is strongly nonseparating w.r.t. } G-e\\ \phi_\Sigma(G-e)-1, & e \text{ is weakly nonseparating w.r.t. } G-e\\ \phi_\Sigma(G-e)-1, & e \text{ is strongly separating w.r.t. } G-e\\ \phi_\Sigma(G-e), & e \text{ is weakly separating w.r.t. } G-e.\end{cases}$$

Proof. The dissecting components of $G-v$ and of G are the same, except for one component $\hat{F}$ for which the corresponding component $\hat{F}'$ w.r.t. G is obtained by removing an open disc around v. Hence $\chi(\hat{F}')=\chi(\hat{F})-1$, and this yields the first part of the lemma.

To prove the second part, let $\hat{e}\subseteq\hat{F}$ in the dissection w.r.t. $G-e$. Suppose first that $\hat{e}$ is nonseparating. Denote by $\hat{F}'$ the surface obtained after dissecting $\hat{F}$ along $\hat{e}$. Since $\hat{e}$ connects two distinct boundary points of $\hat{F}$ we have $\chi(\hat{F}')=\chi(\hat{F})+1$. Also, $\hat{F}$ is not a disc. It follows that

$$\phi_\Sigma(G)-\phi_\Sigma(G-e)=\psi_{\hat{F}'}-\psi_{\hat{F}}=\psi_{\hat{F}'}-(1-2\chi(\hat{F}'))-2.$$

Hence the difference in embedding complexities is -1 if $\hat{F}'$ is a disc, and -2 otherwise.

Next, suppose $\hat{e}$ separates $\hat{F}$ into $\hat{F}'$ and $\hat{F}''$. Then $\chi(\hat{F}')+\chi(\hat{F}'')=\chi(\hat{F})+1$ and

$$\phi_\Sigma(G)-\phi_\Sigma(G-e)=\psi_{\hat{F}'}+\psi_{\hat{F}''}-\psi_{\hat{F}}. \tag{1}$$

If none of $\hat{F}'$, $\hat{F}''$ is a disc (and hence neither $\hat{F}$ is a disc), (1) gives $\psi_{\hat{F}'}+\psi_{\hat{F}''}=\psi_{\hat{F}}-1$. Consequently, the difference in embedding complexities is -1. Finally, let one of $\hat{F}'$, $\hat{F}''$, say $\hat{F}'$, be a disc. Then $\psi_{\hat{F}'}=0$ and $\hat{F}''$ is homeomorphic to $\hat{F}$. Hence the complexity does not change. ∎

Let G and H be graphs embedded in Σ so that G and H intersect in finitely many points. By $G \sqcup H$ we denote the graph obtained by subdividing each edge of G at points where it intersects H, and the same in H, and finally taking the union of these graphs.

COROLLARY 2.2. *Let G and H be arbitrary graphs in Σ disjoint from $\partial\Sigma$, such that $V_H \subseteq V_G$ and such that G and H intersect in finitely many points. Then $\phi_\Sigma(G \sqcup H) \leqslant \phi_\Sigma(G)$, and the equality holds if and only if each edge e of $G \sqcup H$ that is not contained in G is weakly separating w.r.t. G.*

Proof. We can construct $G \sqcup H$ from G by first subdividing some edges of G and then successively adding new edges that are not contained in G. Edge subdivisions do not change the complexity of the dissection. Successive application of Lemma 2.1 then shows that $\phi_\Sigma(G \sqcup H) \leqslant \phi_\Sigma(G)$.

To show when the equality holds, let $G_0, G_1, ..., G_k = G \sqcup H$ be the sequence of intermediate graphs, where G_0 is a subdivision of G and other graphs are obtained by successively adding edges $e_1, e_2, ..., e_k$ (in that order). If some e_i is weakly separating w.r.t. G, then it is obviously weakly separating w.r.t. G_{i-1}. Hence, if all edges of H are weakly separating w.r.t. G, then $\phi_\Sigma(G \sqcup H)=\phi_\Sigma(G)$. For the converse observe that for each index i, $G+e_i \subseteq G \sqcup H$. Hence $\phi_\Sigma(G \sqcup H) \leqslant \phi_\Sigma(G+e_i) \leqslant \phi_\Sigma(G)$. Thus $\phi_\Sigma(G \sqcup H) = \phi_\Sigma(G)$ implies that all edges of H that are not contained in G are weakly separating w.r.t. G by Lemma 2.1. ∎

3. SYSTEMS OF SIMPLE CLOSED CURVES

Let γ and γ' be simple closed curves in Σ with finitely many points of intersection. The cardinality of the intersection is the (*geometric*) *intersection number* of γ and γ'. As a consequence of the Schoenflies' theorem we may classify each intersection as either a *touching* or a *crossing*. A finite collection Γ of simple curves in Σ such that each pair of curves in Γ intersects in finitely many points is called a *system of simple curves in Σ*. We define its *complexity* $\phi_\Sigma(\Gamma)$ as $\phi_\Sigma(\Gamma)=\phi_\Sigma(G_\Gamma)$, where G_Γ is the naturally defined graph of Γ. (Note that if $\gamma \in \Gamma$ is disjoint from other curves in Γ, then γ gives rise to a vertex with one loop in G_Γ.) The following result is an immediate consequence of Lemma 2.1 and Corollary 2.2. Let us recall that a simple closed curve $\gamma \in \Sigma$ is *separating* if $\Sigma \backslash \gamma$ is disconnected.

LEMMA 3.1. *Let Γ be a (possibly empty) system of simple closed curves in the interior of a compact surface Σ and let $\gamma \notin \Gamma$ be a noncontractible simple closed curve in* int Σ *which intersects G_Γ in a finite number of points.*

(a) *If $\gamma \cap G_\Gamma = \emptyset$, then*

$$\phi_\Sigma(\Gamma \cup \{\gamma\}) = \begin{cases} \phi_\Sigma(\Gamma) + 1, & \gamma \text{ is separating} \\ \phi_\Sigma(\Gamma), & \gamma \text{ is nonseparating.} \end{cases}$$

(b) *If $\gamma \cap G_\Gamma \neq \emptyset$, then $\phi_\Sigma(\Gamma \cup \{\gamma\}) \leqslant \phi_\Sigma(\Gamma)$. The equality holds if and only if each edge of $G_{\Gamma \cup \{\gamma\}}$ contained in γ is weakly separating w.r.t. G_Γ.*

Let Γ be a system of simple closed curves. The curves from Γ are in *general position* if no curve from Γ is freely homotopic to another curve from Γ or to its inverse. A system of curves Γ is a *k-system* ($k \geqslant 0$) if any two curves from Γ intersect at most k times. We first extend a result from [MM, Proposition 3.7] to bordered compact surfaces.

LEMMA 3.2. *Let Σ be a compact surface with $b \geqslant 0$ boundary components, and let Γ be a* 0-*system of noncontractible simple closed curves in general position on Σ. Then*

$$|\Gamma| \leqslant \max\{1, 3(g_\Sigma - 1) + 2b\}.$$

Proof. Without loss of generality we may assume that Γ is disjoint from $\partial\Sigma$. The lemma is obvious if $b = 0$ and $g_\Sigma \leqslant 1$, when Γ consists of at most one curve. Assume that either $g_\Sigma \geqslant 2$ or $b > 0$, and let Γ_1, Γ_2 be the subfamilies of one-sided and two-sided curves from Γ, respectively. Dissecting Σ along Γ gives rise to a collection of bordered compact surfaces Σ_1, Σ_2, ..., Σ_r with the total number of $b + 2|\Gamma| - |\Gamma_1|$ boundary components. Moreover, $\chi(\Sigma) = \sum_{i=1}^r \chi(\Sigma_i)$. By pasting discs to boundary components, we get r closed surfaces. Hence

$$\chi(\Sigma) + b + 2|\Gamma| \leqslant 2r + |\Gamma_1|. \tag{2}$$

Suppose that Σ is orientable. Then $\Gamma_1 = \emptyset$. Moreover, $r \leqslant b - \chi(\Sigma)$ since the number of cylinders among the dissecting components with nonnegative Euler characteristic is bounded by b. Now (2) gives the claimed bound. Suppose that Σ is nonorientable. Then $|\Gamma_1| \leqslant g_\Sigma$. Moreover, $r \leqslant b + g_\Sigma - \chi(\Sigma)$ since among the dissecting components with nonnegative

Euler characteristic there are at most $b+g_\Sigma$ cylinders and Möbius bands. Again, (2) implies the claim. ∎

Note that the bound of the above lemma is sharp if Σ is not simply connected.

Let Γ_0 be a 0-system of noncontractible simple closed curves in general position on Σ. By Lemma 3.2 we have $|\Gamma_0| \leqslant \max\{1, 3(g_\Sigma - 1) + 2b\}$. Let s_0 be the number of separating curves in Γ_0. Lemma 3.1 shows that the complexity does not change when we add $|\Gamma_0| - s_0$ pairwise disjoint non-separating curves on the surface; by further adding s_0 pairwise disjoint separating ones we increase the complexity up to $\phi_\Sigma(\Gamma_0) = \psi_\Sigma + s_0 \leqslant \psi_\Sigma + |\Gamma_0|$.

Our next theorem says that for a fixed $k \geqslant 0$, a k-system of simple closed curves in general position on a fixed surface Σ cannot be too large.

THEOREM 3.3. *Let Σ be a (bordered) compact surface and $k \geqslant 0$ an integer. There is a constant $N_{k,\Sigma}$ such that if Γ is a k-system of simple closed curves in general position on Σ, then*

$$|\Gamma| \leqslant N_{k,\Sigma}.$$

Proof. Without loss of generality we may assume that all curves in Γ are disjoint from $\partial\Sigma$. This can be achieved by a small homotopy change of curves in Γ without changing the number of intersections. Furthermore, we may assume that all intersections are crossings and that no three curves intersect in a common point. Also, since there is at most one contractible curve in Γ we henceforth assume that the curves in Γ are noncontractible (and that Σ is not the 2-sphere).

Consider a maximal subfamily $\Gamma_0 \subseteq \Gamma$ of pairwise disjoint curves. Let $n_0 = |\Gamma_0|$ and $c_0 = \psi_\Sigma + \max\{1, 3(g_\Sigma - 1) + 2b\}$. The remark after Lemma 3.2 shows that $\phi_\Sigma(\Gamma_0) \leqslant c_0$.

Every remaining curve from $\Gamma \backslash \Gamma_0$ intersects at least one curve from Γ_0. Now let us successively add to Γ_0 curves from $\Gamma \backslash \Gamma_0$ such that, at each step, the added curve contains an "edge" that is not weakly separating relative to the graph of the previous curves. Since by Lemma 3.1 the complexity strictly decreases, at most c_0 such curves can be added. Denote by Γ' the obtained k-system of $n' \leqslant n_0 + c_0$ curves. We can estimate the number of edges of $G_{\Gamma'}$ recursively as follows: by adding a new curve to the system of x_i curves at ith step, we introduce at most kx_i new vertices (because all intersections are crossings by assumption), and hence at most $2kx_i$ new edges. Therefore,

$$|E_{G_{\Gamma'}}| \leqslant n_0 + 2k \sum_{i=1}^{c_0} (n_0 + i - 1) = n_0 + 2kn_0c_0 + kc_0(c_0 - 1).$$

Finally, consider the curves from $\Gamma\backslash\Gamma'$. Each such curve γ has only weakly separating edges w.r.t. $G_{\Gamma'}$ and intersects $G_{\Gamma'}$ at most $kn' \leqslant k(n_0+c_0)$ times. Therefore, γ splits up into at most $k(n_0+c_0)$ segments which traverse the dissecting surfaces of $G_{\Gamma'}$, and each of them separates a disc from its corresponding dissecting surface. Let us think of the edges of $G_{\Gamma'}$ as being "doubled" in the boundary components of the dissecting surfaces of Γ', that is, $G_{\Gamma'}$ gives rise to a disjoint union of cycles with the total number of $2\,|E_{G_{\Gamma'}}|$ edges. Suppose γ is given an orientation describing its traversal. Then each segment of γ inherits an orientation which can be identified by an ordered pair of boundary edges in the corresponding dissecting surface. Consequently, γ defines a sequence of at most $k(n_0+c_0)$ such ordered pairs which is unique up to cyclic permutation or inverse ordering. We declare two curves $\gamma, \gamma' \in \Gamma\backslash\Gamma'$ to be *similar* if, up to cyclic permutation or inverse ordering, the corresponding sequences of ordered pairs are the same. Suppose that there are N pairwise similar curves from $\Gamma\backslash\Gamma'$. If we compare "the same segment in the sequence" of three different curves we see that at least two of these segments can be moved homotopically onto each other, keeping the endpoints on the boundary; this holds because each segment separates a disc from the corresponding dissecting surface. It follows that out of N similar curves at least $N/2$ are "locally homotopic" along the first segment, at least $N/4$ along the first two segments, etc. Consequently, at least $N/2^{k(n_0+c_0)}$ are "locally homotopic" along all segments. But this means that they are homotopic. Since our curves are pairwise nonhomotopic by assumption, we have $N \leqslant 2^{k(n_0+c_0)}$. This gives a bound on the number of curves from $\Gamma\backslash\Gamma'$ in the same similarity class. But the number of equivalence classes of pairwise similar curves in $\Gamma\backslash\Gamma'$ is bounded because the sequences are bounded in length by $k(n_0+c_0)$, and the elements of each sequence are taken from the edge-set of $G_{\Gamma'}$ which is also bounded by a constant. Hence $|\Gamma|$ is smaller than some constant $N_{k,\Sigma}$, and the proof is complete. ∎

Instead of free homotopy classes we can also consider homotopy of curves with fixed endpoints x_0, x_1 (possibly $x_0 = x_1$). The proof of Theorem 3.3 goes through also in this case (except that we start with Γ_0 consisting of a single curve from Γ).

THEOREM 3.4. *Let Σ be a (bordered) compact surface, let $x_0, x_1 \in \Sigma$ (possibly $x_0 = x_1$), and let $k \geqslant 0$ be an integer. There is a constant $N^*_{k,\Sigma}$ such that if Γ is a k-system of simple curves from x_0 to x_1 that are pairwise nonhomotopic relative to their endpoints, then*

$$|\Gamma| \leqslant N^*_{k,\Sigma}.$$

Theorems 3.3 and 3.4 can be extended to systems of nonsimple curves. Each closed curve γ in Σ can be parametrized as a mapping $\gamma: S^1 \to \Sigma$, where S^1 is the unit circle in $\mathbf{C}$. If γ, γ' are distinct closed curves, we define

$$\mathrm{cr}(\gamma, \gamma') = |\{(z, z') \in S^1 \times S^1 \mid \gamma(z) = \gamma'(z')\}|.$$

We also define the number of self-intersections of γ as

$$\mathrm{cr}(\gamma, \gamma) = \tfrac{1}{2}|\{(z, z') \in S^1 \times S^1 \mid z \neq z', \gamma(z) = \gamma(z')\}|.$$

We say that a finite collection Γ of closed curves is a *k-system* of closed curves if $\mathrm{cr}(\gamma, \gamma) \leqslant k$ and $\mathrm{cr}(\gamma, \gamma') \leqslant k$ for any curves $\gamma, \gamma' \in \Gamma$. The curves in Γ are in *general position* if no curve of Γ is homotopic to another curve from Γ, or its inverse.

THEOREM 3.5. *Let Σ be a (bordered) compact surface and $k \geqslant 0$ an integer. There is a constant $N'_{k,\Sigma}$ such that if Γ is a k-system of (possibly nonsimple) closed curves on Σ in general position (or pairwise nonhomotopic relative to a fixed base point), then*

$$|\Gamma| \leqslant N'_{k,\Sigma}.$$

Proof. We first consider homotopy with a fixed base point $r \in \Sigma$. Let $\gamma \in \Gamma$. Choose a spanning tree T in $G_{\{\gamma\}}$ rooted at $r \in V_{G_{\{\gamma\}}}$. Each edge $e = xy$ of $G_{\{\gamma\}}$ determines a simple closed curve $\gamma(e)$ which is obtained by a small homotopy change from the closed walk in $T \cup e$ from r to x, then along e to y and back to r along edges of T. If $e_1, ..., e_l$ are edges of $G_{\{\gamma\}}$ in the same order as they appear on γ, then γ is homotopic to the product $\gamma(e_1)\,\gamma(e_2) \cdots \gamma(e_l)$. Note that $l \leqslant 2k$. Therefore, if $\gamma_1, ..., \gamma_s$ are simple closed curves, there are at most $(s+1)^{2k} - 1$ pairwise nonhomotopic curves γ in Γ such that each of the simple "subcurves" $\gamma(e_1), ..., \gamma(e_l)$ is homotopic to one of $\gamma_1, ..., \gamma_s$. It follows that a k-system of pairwise nonhomotopic closed curves with $1 + 2^{2k} + 3^{2k} + \cdots + (s+1)^{2k}$ elements determines a k-system of pairwise nonhomotopic simple closed curves $\gamma_1, \gamma_2, ..., \gamma_s$ such that each of γ_i $(i = 1, ..., s)$ is a "subcurve" of a distinct element of Γ. Now Theorem 3.4 completes the proof.

Let us now show how we deduce the free homotopy case from the result for homotopy with a base point. We may assume that G_Γ is connected. Choose $\gamma_0 \in \Gamma$ and $r \in \gamma_0$. Let $\mathscr{H}$ be the intersection graph of Γ, i.e., $V_{\mathscr{H}} = \Gamma$, and γ, γ' are adjacent if they intersect. Let $\mathscr{B}$ be a breadth-first-search spanning tree of $\mathscr{H}$ rooted at γ_0. If $\gamma_0, \gamma_1, ..., \gamma_{2t}$ is a path in $\mathscr{B}$, then $\gamma_0, \gamma_2, ..., \gamma_{2t}$ are pairwise disjoint. Each γ_i contains a noncontractible simple "subcurve" γ'_i. If $t \geqslant (k+2) \max\{1, 3(g_\Sigma - 1) + 2b\} =: d$, Lemma 3.2 implies that $k+3$ curves among $\gamma'_0, \gamma'_2, ..., \gamma'_{2t}$ are all homotopic to a simple curve γ. This implies that $k+1$ among the corresponding curves γ_i are homotopic to powers of γ, and since $\operatorname{cr}(\gamma_i, \gamma_i) \leqslant k$, each is homotopic to one of $\gamma, \gamma^2, ..., \gamma^k$ (or its inverse). Hence two of them would be homotopic. Therefore, $t < d$.

If $\gamma \in \Gamma$, let $\gamma_0, \gamma_1, ..., \gamma_j = \gamma$ be a path in the tree $\mathscr{B}$. Then we define a curve δ from $\gamma \cap \gamma_{j-1}$ to r as follows: We follow γ_{j-1} to a point of intersection with γ_{j-2}, next we follow γ_{j-2} to γ_{j-3}, etc. Let $\bar{\gamma}$ be the curve γ concatenated with δ and δ^{-1}. This can be done in such a way that the curves δ for different curves γ intersect only close to the vertices of G_Γ. Using the above estimate on t, one can show that for $\gamma, \gamma' \in \Gamma$ we have $\operatorname{cr}(\bar{\gamma}, \bar{\gamma}') \leqslant 4d \cdot 3k = 12dk$. The resulting system of curves $\{\bar{\gamma} \mid \gamma \in \Gamma\}$ contains pairwise nonhomotopic curves relative to their base point r, and this completes the proof. ∎

4. THE TORUS

In this section we derive almost sharp bounds on the size of k-systems of simple closed curves in general position for the case when $\Sigma = \Sigma_1$ is the torus.

Let γ and γ' be simple closed curves in Σ_1 with finitely many intersecting points. Each crossing of γ with γ' can be classified either as *positive* or *negative* with respect to a fixed chosen global orientation of Σ_1, and the *algebraic intersection number* $\alpha(\gamma, \gamma')$ of γ and γ' is defined to be the number of positive crossings minus the number of negative crossings. See, e.g., [ZVC] for details. It is well known that the algebraic intersection number can be computed from the standard representation of curves relative to some fixed chosen generators of the first homology group. Since the fundamental group of the torus is commutative, it is isomorphic to the first homology group $H_1(\Sigma_1)$. Moreover, the homology classes coincide with the free homotopy classes. Let a and b be the generators of $H_1(\Sigma_1)$. We write $\gamma \in (m, n)$ to denote that γ is in the same class as $ma + nb$, where $m, n \in \mathbf{Z}$. If $\gamma' \in (m', n')$ then

$$\alpha(\gamma, \gamma') = \det \begin{bmatrix} m & n \\ m' & n' \end{bmatrix}.$$

Moreover, the class (m, n) contains a simple closed curve as a representative if and only if $\gcd(m, n) = 1$. Consequently, simple noncontractible curves γ and γ' are in general position if an only if $\alpha(\gamma, \gamma') \neq 0$.

PROPOSITION 4. *Let Γ be a system of noncontractible simple closed curves in general position on the torus. If $|\alpha(\gamma, \gamma')| \leqslant k$ for each pair $\gamma, \gamma' \in \Gamma$, then*

$$|\Gamma| \leqslant 2k+2.$$

Proof. Let $\Gamma = \{\gamma_1, ..., \gamma_N\}$. Denote by $\alpha_{ij} = \alpha(\gamma_i, \gamma_j)$, and let (m_i, n_i) be the class of γ_i, $i = 1, 2, ..., N$. We may assume that $|\alpha_{12}| = k$, and that $(m_1, n_1) = (1, 0)$. Also, by considering either a homology class or its inverse, we may assume $n_i \geqslant 0$ for all indices. In fact, we have $n_j > 0$ for all $j \geqslant 2$ because the curves are in general position and simple.

Now $|\alpha_{12}| = k$ gives $n_2 = k$, and $|\alpha_{1j}| \leqslant k$ gives $0 < n_j \leqslant k$ for all $j \geqslant 2$. Consider $|\alpha_{2j}| = |m_2 n_j - k m_j| \leqslant k$. Then

$$\frac{m_2 n_j}{k} - 1 \leqslant m_j \leqslant \frac{m_2 n_j}{k} + 1. \tag{3}$$

If $n_j = k$, there are three possibilities for m_j, case $j = 2$ being one of them. Since $\gcd(m_2, k) = 1$, (3) implies that there are at most two possibilities for m_j if $n_j \neq k$. Altogether there are at most $2k+2$ possible classes. ∎

COROLLARY 4.2. *Let Γ be a k-system of simple closed curves in general position on the torus. Then*

$$|\Gamma| \leqslant 2k+3.$$

The bounds $2k + O(1)$ of Proposition 4.1 and Corollary 4.2 are not sharp. They can easily be improved to a bound $\frac{3}{2}k + O(1)$. On the other hand, the homology representatives $(1, i)$ $(i = 0, ..., k)$ show that the exact bound is at least $k + O(1)$.

It is worth mentioning that the same approach does not work on general surfaces, even if they are orientable. Let Σ be an orientable closed surface of genus $g = g_\Sigma$, and let γ and γ' be simple closed curves in Σ. Their homology classes can be represented by a canonical system of generators (see [ZVC] for definitions) as $(a_1, b_1, a_2, b_2, ..., a_g, b_g)$ and $(a'_1, b'_1, a'_2, b'_2, ..., a'_g, b'_g)$, respectively. Then the algebraic intersection number of γ and γ' equals

$$\alpha(\gamma, \gamma') = \sum_{i=1}^{g} \det \begin{bmatrix} a_i & b_i \\ a'_i & b'_i \end{bmatrix}$$

[ZVC, Proposition 3.6.3]. If $g \geqslant 2$, then it is easy to find examples of arbitrarily many simple closed curves with pairwise bounded algebraic intersection numbers that are pairwise nonhomologic and hence pairwise freely nonhomotopic. All one has to do is to find appropriate vectors

$(a_1^{(j)}, b_1^{(j)}, ..., a_g^{(j)}, b_g^{(j)})$, $j = 1, 2, ...$. In order to make sure that the chosen vectors correspond to simple curves, one can use the result of Poincaré [P] (cf. also [M, S]) who proved that there exists a simple closed curve as a representative of the homology class $(a_1, b_1, ..., a_g, b_g)$ if and only if $\gcd(a_1, b_1, ..., a_g, b_g) = 1$.

5. EXAMPLES

The bound $N_{k,\Sigma}$ on the cardinality of a k-system of simple closed curves in general position from Section 3 is probably far from being sharp. However, the following constructions show that the bounds are not small, in general.

First we show that for a fixed k and a large enough g, there are k-systems of simple closed curves in general position in the surface of genus g containing at least $c_k g^{\lfloor k/4 \rfloor}$ curves, where c_k is an appropriate constant.

PROPOSITION 5.1. *For every $k > 0$ and g large enough there exists a k-system of simple closed curves in general position on a closed surface of genus g containing $\binom{n}{\lfloor k/2 \rfloor}$ curves, where $n = \lfloor (\sqrt{25 + 48(g-1)} - 5)/2 \rfloor$.*

Proof. A construction for $k \leqslant 5$ is easy and is left to the reader. Suppose now that $k \geqslant 6$ and $g \geqslant (k^2 + 12k + 59)/48$. Then $n \geqslant k/2$. Let $h = g - n$. Simple arithmetic shows that $h \geqslant (n-3)(n-4)/12$ and, hence, the complete graph K_n has an embedding in the closed orientable surface of genus h. Fix such an embedding. For each vertex v of K_n cut out two discs which are "close" to v and in the same face of the chosen embedding of K_n. Connect them by a handle. The resulting surface Σ has genus g. Finally, let γ_v be a simple closed curve at v that is homotopic to a noncontractible curve in the added handle.

Each cycle C of K_n determines a closed curve γ'_C on Σ that consecutively follows the edges of C and at each vertex $v \in V_C$ follows the curve γ_v. Note that there is only one way to traverse γ_v such that γ'_C does not cross itself at v. By a small perturbation at each vertex, γ'_C can be changed into a simple curve γ_C. Moreover, by small changes of curves along the edges we can achieve that any pair of curves $\gamma_C, \gamma_{C'}$ $(C \neq C')$ intersects only in small neighborhoods of vertices in $V_C \cap V_{C'}$, and at most twice for each such vertex. Hence curves corresponding to all cycles of K_n of length at most $k/2$ form a k-system.

If C and C' are cycles of K_n with $V_C \neq V_{C'}$, then γ_C and $\gamma_{C'}$ are easily seen to be nonhomologic in Σ. Let $l = \lfloor k/2 \rfloor$. For every l-subset U of V_{K_n}, choose a cycle C_U with $V_{C_U} = U$. Hence, $\Gamma = \{\gamma_{C_U} \mid U \subseteq V_{K_n}, |U| = l\}$ forms a k-system of curves in general position, and $|\Gamma| = \binom{n}{l}$. ∎

Let us remark that the above construction can be slightly improved. By inserting $\binom{n-1}{2}$ handles at each vertex of K_n, we can achieve that any two curves in Γ intersect at most once at each common vertex of the corresponding cycles. Now we can take all k-subsets of V_{K_n} and improve the bound from $c_1 g^{\lfloor k/4 \rfloor}$ to $c_2 g^{\lfloor k/3 \rfloor}$ (where c_1 and c_2 depend on k only). Further improvements are possible.

Our second construction shows that for an arbitrary $g > 0$ there exists a k-system of simple closed curves in general position on a closed orientable surface of genus g containing at least $c_g k^g$ curves, where c_g is an appropriate constant.

PROPOSITION 5.2. *For each $g > 0$ and $k \geqslant 0$ there is a k-system of noncontractible simple closed curves in general position on a closed orientable surface of genus g containing more than $(k/g)^g$ curves.*

Proof. Take a compact surface of genus 0 with $2g$ boundary components $Q_1, Q_2, ..., Q_{2g}$. Connect Q_{2i-1} with Q_{2i} by adding a handle, $i = 1, 2, ..., g$, to obtain a closed orientable surface Σ of genus g. Let $n = (n_1, n_2, ..., n_g)$, where $0 \leqslant n_i \leqslant \lfloor k/g \rfloor$, and let C_n be disjoint "concentric circles" at the "planar" part of Σ, indexed by the above g-tuples. To each of the $r = (\lfloor k/g \rfloor + 1)^g$ circles C_n we associate a simple closed curve γ_n as follows. Fix a global positive orientation of Σ. The curve γ_n traverses an arc on C_n in the positive direction; then it goes to Q_1, wraps around the handle at Q_1 n_1 times in the positive direction. Then it goes along the added handle to Q_2 and returns to C_n; then it goes on to the second handle, etc. It is easy to see that the curves can be made simple. Moreover, the curves can be chosen in such a way that the following holds. If $n = (n_1, n_2, ..., n_g)$, $m = (m_1, m_2, ..., m_g)$, then γ_n and γ_m intersect at the ith handle at most $|n_i - m_i| \leqslant \lfloor k/g \rfloor$ times; the segments of these curves between the corresponding circles and handles are "parallel" (close to each other) and have, therefore, no other intersections. Hence, each pair of these curves intersects at most k times. Since the curves are obviously pairwise nonhomologic, they are also pairwise nonhomotopic. This completes the proof. ∎

6. MINIMAL TRIANGULATIONS

Let Σ be a compact surface. A curve γ on Σ is *essential* if

(a) γ is closed and homotopically nontrivial, or

(b) γ is open with both endpoints in the same boundary component B such that after pasting a disc D_B to B and by extending γ with a path in D_B connecting its endpoints, a noncontractible closed curve is obtained, or

(c) γ is open with endpoints in distinct components of $\partial\Sigma$.

A triangulation $\mathcal{T}$ of a compact surface is *k-minimal* ($k \geqslant 3$) if each edge is contained in an essential k-cycle or in an essential path of length k, and all essential cycles and paths in $\mathcal{T}$ have length at least k. Barnette and Edelson [BE1, BE2] proved that any closed surface admits only finitely many 3-minimal triangulations. A simple proof of this result was independently given by Gao, Richmond, and Thomassen [GRT] (unpublished), and later by Nakamoto and Ota [NO], who obtained linear bounds on the size of a 3-minimal triangulation in terms of the genus of the surface. Malnič and Mohar proved the finiteness of 4-minimal triangulations on orientable closed surfaces [MM]. That the class of k-minimal triangulations is finite for any k and all closed surfaces has first been proved by Malnič and Nedela [MN]. Recently, Gao, Richter, and Seymour [GRS] gave a different and shorter proof. Results of our paper yield a very short proof for an arbitrary k, generalized to bordered compact surfaces.

THEOREM 6.1. *For each compact* (*bordered*) *surface* Σ *and an integer* $k \geqslant 3$ *there is a constant* $c_{k,\Sigma}$ *such that every* k*-minimal triangulation of* Σ *has at most* $c_{k,\Sigma}$ *edges.*

Proof. Let $\mathcal{T}$ be a k-minimal triangulation of Σ. Let E_0 be the set of edges that belong to some essential k-cycle in $\mathcal{T}$. Denote by E_Γ the set of edges which belong to some k-cycle in the free homotopy class Γ. Then $E_0 = E_{\Gamma_1} \cup E_{\Gamma_2} \cup \cdots \cup E_{\Gamma_N}$ for some N (depending on $\mathcal{T}$), where each Γ_i is nontrivial. Since each subgraph formed by E_Γ contains a k-cycle from Γ, the number N is bounded by Theorem 3.3. We shall prove that for every $\Gamma \neq 1$, $|E_\Gamma|$ is bounded by some constant depending just on k. For each $e \in E_\Gamma$ choose a k-cycle $C_e \in \Gamma$ containing e, and let $\mathcal{C}_\Gamma = \{C_e \mid e \in E_\Gamma\}$. Clearly, $|E_\Gamma| \leqslant k\,|\mathcal{C}_\Gamma|$. We shall give a bound for $|\mathcal{C}_\Gamma|$.

Consider an arbitrary pair of cycles $C, C' \in \mathcal{C}_\Gamma$ such that $C \cap C' \neq \emptyset$. There is a disc (with possibly some identifications on its boundary from the outside) which is bounded by a segment of C and a segment of C' (cf. [E]). If both segments cross at their common ends, we perform a homotopic switch of the segments across the disc to obtain k-cycles $C_1, C_1' \in \mathcal{C}_\Gamma$ such that $C_1 \cup C_1' = C \cup C'$. The number of crossings is reduced by 2. (Note that C_1 and C_1' are cycles and their length is k since the triangulation is k-minimal.) Induction on the total number of crossings in $\mathcal{C}_\Gamma$ makes possible to assume that any two cycles in $\mathcal{C}_\Gamma$ cross each other (possibly along a segment) at most once. We claim that there is a constant $f(k)$ such that any k-cycle in $\mathcal{C}_\Gamma$ intersects at most $f(k)$ other k-cycles in $\mathcal{C}_\Gamma$.

Suppose that this is not the case. Then there is a cycle $C \in \mathcal{C}_\Gamma$ and a vertex $v \in V_C$ such that more than $f(k)/k$ of the k-cycles in $\mathcal{C}_\Gamma$ meet C at v. Denote this family by $\mathcal{C}_v$. Let Γ be one-sided. Since homotopic one-sided curves cross an odd number of times, we may assume that each cycle from

$\mathscr{C}_v$ crosses C at v (possibly along a segment), and this is the only crossing. By a small homotopy change of these cycles we obtain a bouquet of simple closed curves in Γ. Each bounds an (open) disc with C (cf. [E]). Hence, one of these discs contains at least half of the curves. Let Γ be two-sided. Each pair of k-cycles in $\mathscr{C}_\Gamma$ is noncrossing. By a small homotopy change we obtain pairwise disjoint simple closed curves in Γ. Since each pair bounds a cylinder (cf. [E]), there is a cylinder which contains all of them. Moreover, there is a cylinder bounded by two of these curves, such that after identification at v, the resulting disc contains at least half of the other curves.

Consequently, there is a disc D bounded by segments $a-b$ $(a=v)$ of two distinct k-cycles in $\mathscr{C}_v$ which contains $a-b$ segments of at least $f(k)/2k$ other cycles from $\mathscr{C}_v$. Let K be the subgraph in D formed by the above $a-b$ segments of length at most k. Assuming $f(k)$ is sufficiently large, e.g., $f(k)>(4k)^{2k-1}(2k-2)!\,(2k)!$, it is easy to see (cf. [FM, Lemma 3.2] for details) that K contains vertices A and B joined by (at least) $2k$ internally disjoint paths $L_1, ..., L_{2k}$ in K. We may assume that in the local rotation at A, the paths $L_1, ..., L_{2k}$ leave A in this order and that $L_1 \cup L_{2k}$ bounds a disc D_L (with a possible identification if $A=B=v$) such that $L_1, ..., L_{2k} \subseteq D_L$. Consider an edge $e \in E_{\mathscr{T}}$ on the link(A) between L_k and L_{k+1}. The cycle C_e through e cannot leave D_L at A. Therefore, it intersects $L_1 \cup L_{2k}$ in a vertex different from A and B. Hence, C_e is of length greater than k, a contradiction. The claim is proved.

Each k-cycle in $\mathscr{C}_\Gamma$ intersects at most $f(k)$ other k-cycles in $\mathscr{C}_\Gamma$. Hence, at least $|\mathscr{C}_\Gamma|/f(k)$ are pairwise disjoint. It is easy to see (cf. [MN]) that in a k-minimal triangulation, there are at most $k+1$ pairwise disjoint k-cycles in the same homotopy class. Hence, $|\mathscr{C}_\Gamma| \leqslant (k+1)\,f(k)$.

Let E_1 be the set of edges of $\mathscr{T}$ that belong to an essential path of length k with its endpoints on the same boundary component of Σ. Two essential curves of type (b) are *homotopic* if they have endpoints on the same boundary component B and their extensions through D_B are homotopic relative to the base point in the "center" of D_B. Theorem 3.4 (with $x_0=x_1$ being the center of D_B) shows that there is only a bounded number of homotopy classes of such essential paths of length k in $\mathscr{T}$. By considering an arbitrary class Γ, we can apply the same proof as above to show that $|E_\Gamma|$ is bounded. The same proof works also for the edge set E_2 of those edges that belong to some essential path of length k joining distinct boundary components.

Since by k-minimality $|E_{\mathscr{T}}| \leqslant |E_0|+|E_1|+|E_2|$, the proof is complete. ∎

Let M be an integer. Theorem 6.1 can be extended to the case when for each homotopy class Γ of simple essential curves in Σ we prescribe a number $k(\Gamma) \leqslant M$ such that the following holds: Each essential cycle or path in $\mathscr{T}$

has length at least $k(\Gamma)$, where Γ is its homotopy class, and each edge is contained in an essential curve Q in the graph of length exactly $k(\Gamma)$, where Γ is the homotopy class of Q.

Let G be a graph embedded in a compact surface Σ. We assume that the interior of each edge is either disjoint from $\partial\Sigma$ or completely contained in $\partial\Sigma$. If γ is an essential curve in Σ, we define $\mathrm{cr}(\gamma, G)$ as $|\gamma \cap G|$ if γ is of type (a), and as $|\gamma \cap G| - \frac{1}{2}|\partial\gamma \cap G|$ in cases (b) and (c), where $\partial\gamma$ consists of the endpoints of γ. The *face-width* (or *representativity*) $\mathrm{fw}(G)$ of G in a non-simply connected surface Σ is defined as

$$\mathrm{fw}(G) = \min\{\mathrm{cr}(\gamma, G) \mid \gamma \text{ essential curve in } \Sigma\}.$$

Let us remark that it suffices to consider only simple essential curves which intersect G in vertices only and that homeomorphic embeddings have the same face-width.

An embedding of G in Σ is *minor-minimal* of given face-width $\mathrm{fw}(G) > 0$, if for every $e \in E_G$ we have $\mathrm{fw}(G/e) = \mathrm{fw}(G) - 1$ and either $\mathrm{fw}(G - e) = \mathrm{fw}(G) - 1$ (if e is not on $\partial\Sigma$) or $\mathrm{fw}(G - e) = \mathrm{fw}(G) - \frac{1}{2}$ (if e is on $\partial\Sigma$). Note that we contract only edges e that do not change the surface, i.e., e is not a loop, and not an edge in the interior of Σ with both endpoints on $\partial\Sigma$.

To each minor-minimal embedding G of face-width $k > 1$ we associate a triangulation $T(G)$ by taking the barycentric subdivision of G. Recall that the barycentric subdivision of an embedded graph is performed as follows. In the "middle" of each edge we put an additional vertex. Also, we choose an additional vertex in each open face F of G (if $F \cap \partial\Sigma \neq \emptyset$, we choose the vertex from $\partial\Sigma \cap F$). In addition, we connect each vertex associated to a face with all (new and old) vertices in its facial walk. Note that the two new edges between a vertex in $F \cap \partial\Sigma$ and its neighboring old vertices in $\partial\Sigma$ are both contained in $\partial\Sigma$.

By applying the barycentric subdivision as in [MM, MN], Theorem 6.1 gives an elementary proof that there are only finitely many minor-minimal embeddings of given face-width on any compact surface (which otherwise follows from the proof of the Graph Minor Theorem of Robertson and Seymour [RS]).

Corollary 6.2. *For any compact surface there are only finitely many minor-minimal embeddings of a given face-width.*

Proof. Let k be the given face-width. Cases $k \leqslant 1$ are left to the reader. For $k > 1$, it is easy to see that the barycentric subdivision of a minor-minimal embedding in Σ of face-width k is a $2k$-minimal triangulation of Σ. Now Theorem 6.1 applies. ∎

As pointed out in [GRS], Corollary 6.2 is equivalent to Theorem 6.1 for closed surfaces. Let us remark that this equivalence holds for bordered compact surfaces as well.

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