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Disjoint Essential Cycles

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Graphs that have two disjoint noncontractible cycles in every possible embedding in surfaces are characterized. Similar characterization is given for the class of graphs whose orientable embeddings (embeddings in surfaces different from the projective plane, respectively) always have two disjoint noncontractible cycles. For graphs which admit embeddings in closed surfaces without having two disjoint noncontractible cycles, such embeddings are structurally characterized.

1. INTRODUCTION

Graphs in this paper are finite and undirected. They may contain loops and parallel edges. A *cycle* C_n of length $n \geqslant 1$ in a graph G is a subgraph of G on n cyclically adjacent vertices. The cycle of length 1 is just a loop, and a cycle of length 2 is a subgraph of G consisting of two vertices and a pair of parallel edges between them.

Dirac [2] (cf. also [5]) proved that a 3-connected graph G contains no two disjoint cycles if and only if one of the following cases occurs: G is a wheel $K_1 * C_n$ $(n \geqslant 3)$ with 3 or more spokes, $G = K_5$, or G has at least 6 vertices and contains vertices $x, y, z \in V(G)$ which cover all the edges of G. In the last case, $G = K_{3,k}$ $(k \geqslant 3)$ or G is a graph obtained from $K_{3,k}$ by adding 1, 2, or 3 edges between the vertices in the color class of $K_{3,k}$ containing 3 vertices. Dirac's result can be generalized to arbitrary graphs. Since the removal of vertices of degree 0 or 1, and the suppression of vertices of degree 2 in a graph do not change the number of cycles, we may without loss of generality treat only the case when the minimal vertex degree is at least 3. A graph G with the minimal degree 3 or more does not contain two disjoint cycles if and only if one of the following cases occurs:

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(a) G has a vertex $x \in V(G)$ such that $G - x$ is a forest.

(b) G has a vertex $x \in V(G)$ such that $G - x$ is a simple cycle and G has no loops at x, i.e., G is a wheel with the spokes allowed to be multiple edges.

(c) $G = K_5$.

(d) There are vertices $x, y, z \in V(G)$ such that $G - \{x, y, z\}$ is edgeless, there are no loops at x, y, z, and no parallel edges between $\{x, y, z\}$ and $V(G) \setminus \{x, y, z\}$. (But parallel edges between x, y, z are allowed.)

In the study of properties of graphs embedded in (closed) surfaces, it is important to know as much as possible about the separating (homology) properties and the homotopy properties of cycles of the graph. A cycle C of a graph G embedded in some surface is *essential* if C is noncontractible on the surface. Every graph that is 2-cell embedded in a surface distinct from the 2-sphere contains an essential cycle. Moreover, the essential cycles of a 2-cell embedded graph G on Σ determine a d-dimensional subspace of the cycle space of G, where d is equal to twice the genus of Σ if Σ is orientable, or d is the genus of Σ in the nonorientable case. However, it can happen that any two essential cycles intersect. Such examples are the graphs $K_{3,k}$ which have arbitrarily large genus and nonorientable genus (for k large) but do not have disjoint essential cycles. An interesting outcome of our results is that these graphs (and simple extensions of them) are more or less the only such examples.

We characterize graphs which have two disjoint essential cycles in every embedding in surfaces. It is shown that these are precisely the graphs that cannot be embedded in the projective plane with exception of the graphs $K_{3,k}$ ($k \geqslant 5$) and simple extensions of these graphs (Theorem 3.1). Similar characterizations are given for the class of graphs whose orientable embeddings (embeddings in a surface different from the projective plane, respectively) always have two disjoint essential cycles (Theorem 4.2). It turns out that among the projective planar graphs such graphs are precisely those which can be embedded in the projective plane with representativity three or more.

In the second part of the paper we present a structural characterization of graphs embedded in surfaces (maps) without disjoint essential cycles. This settles the disjoint essential cycles problem initiated in [6].

2. BASIC DEFINITIONS

Let K be a subgraph of G. A *K-component* in G is a subgraph of G which is either an edge $e \in E(G) \setminus E(K)$ (together with its endpoints) which has both endpoints in K, or it is a connected component of $G - V(K)$ together with all edges (and their endpoints) between this component and K. (K-components are sometimes also called *K-bridges* in G.) We say that a K-component B in G is *attached* to a vertex x of K if $x \in V(B \cap K)$. For $X \subseteq V(G)$, an *X-component* is an H-component where H is the edgeless graph with vertex set equal to X.

A subgraph K of G is a *$K_{2,3}$-graph* in G if K consists of two vertices x, y and three internally disjoint (x, y)-paths P_1, P_2, P_3, and there is a K-component B that is attached to internal vertices of P_1, P_2, and P_3. (See Fig. 1a which shows such a subgraph K together with the part of B attached to the paths P_i.) Similarly, K is a *K_4-graph* in G if it is homeomorphic to K_4, and there is a K-component that is attached to all four vertices of degree 3 in K (Fig. 1b). A subgraph of G is a *K-graph* if it is either a $K_{2,3}$-graph or a K_4-graph. The following well-known facts will be used in the sequel (cf. [3]).

LEMMA 2.1. *Let K be a $K_{2,3}$-graph in G. If G is embedded in a surface Σ, then at least two of the cycles of K are essential.*

LEMMA 2.2. *Let K be a K_4-graph in G. If G is embedded in a surface Σ, then at least two "triangles" of K are essential cycles on Σ.*

COROLLARY 2.3. *If G contains two disjoint K-graphs, then it contains two disjoint essential cycles in every embedding.*

Let G be a graph and k an integer. If $G = G_1 \cup G_2$ with $E(G_1) \cap E(G_2) = \emptyset$, $|E(G_1)| \geqslant k$, $|E(G_2)| \geqslant k$, and $|V(G_1) \cap V(G_2)| = k$, then we say that the vertex set $V(G_1) \cap V(G_2)$ is a *k-separator* of G. If G has at least $k+1$ vertices and contains no l-separator for $l < k$, then G is *k-connected.* Let us remark that our definition of k-connectivity slightly differs from the usual one if $k \geqslant 4$ but in case when $k \leqslant 3$, it agrees with other definitions,

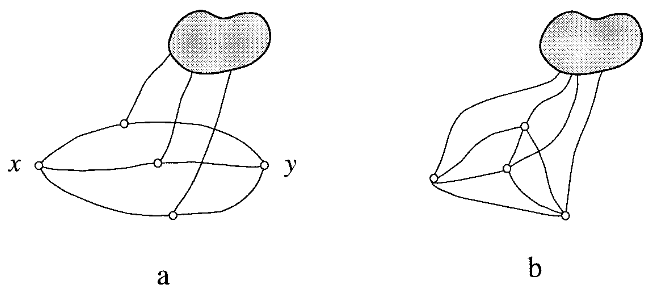

FIG. 1. K-graphs.

also taking care of loops and parallel edges. A vertex of degree 3 in G together with the incident edges is a *triad* in G.

Let G be a graph embedded in a surface Σ. The minimal number of intersections with G of any noncontractible simple closed curve γ on Σ is denoted by $\rho(G)$ and is called the *representativity* (or the *face-width*) of the embedded graph. By elementary topology, $\rho(G)$ is also the minimum number of intersections of γ with G where γ is any noncontractible simple closed curve which passes through vertices and faces only, and which uses no vertex or face more than once. The reader is referred to [9] for more details about representativity of embeddings.

A graph is *planar* if it admits an embedding in the plane. By a *plane* graph we refer to a planar graph together with an embedding in the plane. If G is a graph embedded in some surface and H is a subgraph of G such that every cycle in H is contractible on the surface, then H is said to be *plane embedded.* In such a case, there is a closed disk in the surface that contains G.

The structure of maps can be described by means of "patches" in a surface. A closed disk D in Σ is a *k-patch* if $|G \cap \partial D| = k$ and every pair of points in $G \cap \partial D$ is connected by a path in $G \cap D$ that is internally disjoint from ∂D. It is allowed that ∂D intersects G in the middle of an edge but we may subdivide such an edge and hence assume that $\partial D \cap G \subseteq V(G)$. A patch D is *well connected* if D contains a $(G \cap \partial D)$-component that is attached to all vertices of $G \cap \partial D$. A *patch structure* of an embedded graph G in Σ is given by a set of patches $D_1, ..., D_p$ in Σ such that

(i) For $1 \leqslant i < j \leqslant p$, patches D_i and D_j have disjoint interiors and $\partial D_i \cap \partial D_j \subseteq G$.

(ii) $G \subseteq D_1 \cup \cdots \cup D_p$.

Let G be a graph embedded in Σ. Suppose that $D \subset \Sigma$ is a k-patch where $k \leqslant 3$. Let H be the graph on Σ obtained as follows. If $k \leqslant 1$, delete $G \cap \operatorname{int} D$. If $k = 2$, replace $G \cap \operatorname{int} D$ by an edge in D joining the vertices of G on ∂D. If $k = 3$, replace $G \cap \operatorname{int} D$ by a triad in D that is joined to the vertices of G on ∂D (see Fig. 2). Then the graph H in Σ is said to be

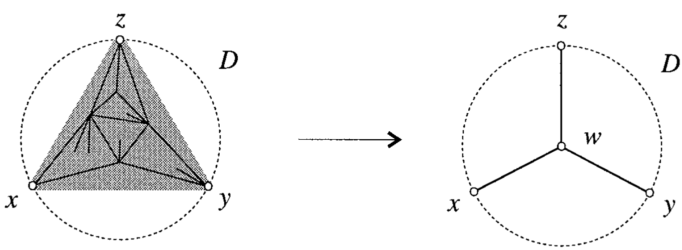

FIG. 2. An elementary reduction of order 3.

obtained from G by an *elementary reduction* of *order* k. A special case of an elementary reduction of order 3 is the well known *ΔY-exchange* that replaces a facial triangle by a triad. An elementary reduction is *nontrivial* if it changes the isomorphism class of the graph. If H' is obtained from G by a sequence of elementary reductions, we say that G is a *plane extension* of H' in Σ. If G and H are abstract graphs (not considered as being embedded in a surface), we say that G is a *planar extension* of H if G and H can be embedded in the same surface Σ so that G is a plane extension of H in Σ.

The following lemma is easy to verify.

LEMMA 2.4. *Let G be a plane extension of a graph H in Σ. Then the maximal number of disjoint essential cycles in G is equal to the maximal number of disjoint essential cycles in H.*

We will also need the following result.

LEMMA 2.5. *Let G be a graph embedded in a surface Σ, and let K be a subgraph of G and B be a K-component. If B contains an essential cycle, then there is an essential cycle in B which contains at most one vertex of K.*

Proof. Let C be an essential cycle in B. If C satisfies our goal of having at most one vertex in K, we are done. Otherwise, let x and y be two of the vertices of $V(C) \cap V(K)$. Then $C = P_1 \cup P_2$ where P_1, P_2 are paths in B from x to y. Since B is a K-component, P_1 and P_2 both contain vertices that do not belong to K, and so there is a (shortest) path P in $B - V(K)$ that joins P_1 and P_2. Since C is essential, one of the two cycles in $C \cup P$ that is distinct from C must be essential. That cycle misses either x or y, and therefore uses fewer vertices of K than C. By repeating this procedure, we eventually get the required cycle in B. ∎

3. UNAVOIDABLE PAIR OF ESSENTIAL CYCLES

Let G be a graph embedded in a surface Σ. If Σ is the 2-sphere, then G contains no essential cycles, and if Σ is the projective plane, any two essential cycles intersect. To characterize graphs which have two disjoint essential cycles in every embedding we must therefore exclude projective planar graphs. The set of graphs having an embedding without two disjoint essential cycles is minor closed, i.e., if a graph G has such an embedding, every minor of G also has such an embedding.

Let x, y, z be vertices of degree k forming one bipartite class of the graph $K_{3,k}$. For each pair of the vertices x, y, z we add an arbitrary number (possibly zero) of edges between these two vertices. The set of all graphs obtained from $K_{3,k}$ in this way will be denoted by $\mathscr{K}^*_{3,k}$.

THEOREM 3.1. *Let G be a graph that cannot be embedded in the projective plane. Then one of the following holds*:

(a) *In every embedding of G in any surface there are two disjoint essential cycles.*

(b) *G contains distinct vertices $x, y, z \in V(G)$ such that every $\{x, y, z\}$-component is a planar graph and there are at least five $\{x, y, z\}$-components that contain all three vertices x, y, z. In this case, an embedding of G in a closed surface Σ has no two disjoint essential cycles if and only if every $\{x, y, z\}$-component is plane embedded, i.e., the embedding of G is a plane extension on Σ of some graph $K \in \mathscr{K}^*_{3,k}$ $(k \geqslant 5)$.*

COROLLARY 3.2. *A graph G has two disjoint essential cycles in every embedding in any surface if and only if it cannot be embedded in the projective plane and it is not a planar extension of some graph $K \in \mathscr{K}^*_{3,k}$ $(k \geqslant 5)$.*

Proof. No graph embedded in the projective plane and no plane extension of a graph from $\mathscr{K}^*_{3,k}$ contains disjoint essential cycles. The rest is clear by Theorem 3.1. ∎

Proof of Theorem 3.1. Since G cannot be embedded in the projective plane, it contains as a minor one of the minor minimal graphs H that do not embed in the projective plane. For most of the graphs H we will show that they satisfy (a). Then also G satisfies (a). There will be only one case, $H = K_{3,5}$, when H will fulfil (b) and not (a). But in this case we will show that if G has an embedding without disjoint essential cycles, then this embedding is a plane extension of an embedded $K \in \mathscr{K}^*_{3,k}$ as claimed in (b).

It is known [4, 1] that there are exactly 35 forbidden minors for the projective plane. Twelve of them (the graphs denoted by A_1, A_5, B_3, C_1, C_2, C_{11}, D_1, D_4, E_1, E_6, E_{42}, F_6 in [4]) are not 3-connected. They are obtained as $\leqslant 2$-amalgamations of Kuratowski graphs (K_5 and $K_{3,3}$), and all contain two disjoint K-graphs. We are done by Corollary 2.3.

Another twelve of the graphs are 3-connected, but they contain a 3-separator for which none of the parts they separate is a triangle or a triad. They are shown in Fig. 3. The graphs C_7, E_{19}, D_{12}, E_{11}, E_{27}, D_9, and G_1 all contain disjoint K-graphs, and we are done. Consider now the graph B_1. The edge joining the two bottom vertices is contained in three nonseparating triangles. Therefore at least one of them is essential. But for each of them, there is a K_4-graph disjoint from it. By Lemma 2.2 we get another essential cycle.

In the graph D_3, the cycles 3 4 5 6 and 4 5 6 7 (see Fig. 4 for vertex labels) are both disjoint from a K_4-graph. Therefore both are contractible. On the other hand, they lie in the $K_{2,3}$-graph on vertices 3, 4, 5, 6, 7. By Lemma 2.1, this is not possible. The graph E_5 is obtained from D_3 by a

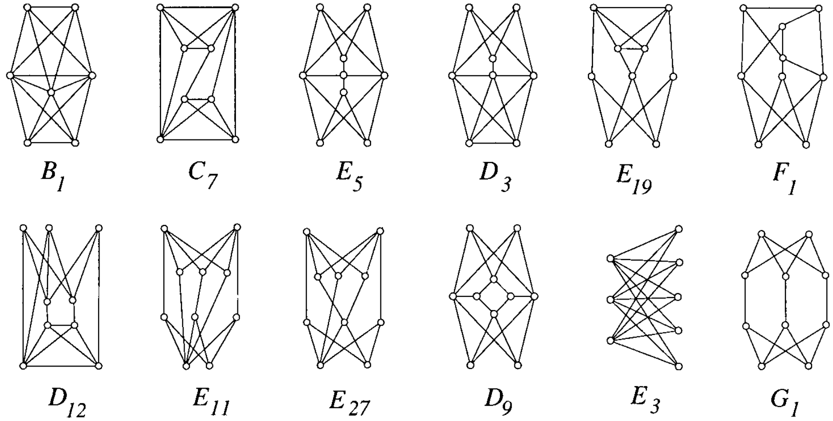

FIG. 3. Forbidden minors.

ΔY-exchange (replacing a triangle by a triad). Having an embedding of E_5 without two disjoint essential cycles we would therefore be able to get such an embedding of D_3 whose nonexistence we already verified.

In the graph F_1, the cycle 4 5 8 7 (see Fig. 4 for labels) is disjoint from a $K_{2,3}$-graph. Therefore it is contractible. The cycle 4 5 6 7 is disjoint from the complementary $K_{2,3}$-graph. So it is contractible. Now we have a contradiction as above: two cycles in the $K_{2,3}$-graph on vertices 4, 5, 6, 7, 8 are contractible.

The remaining graph to consider is the graph $E_3 = K_{3,5}$. Let G be a graph embedded in some surface containing $K_{3,5}$ as a minor. We claim that G does not contain two disjoint essential cycles if and only if G is a plane extension of some graph $K \in \mathscr{K}^*_{3,k}$, $k \geqslant 5$. One direction is easy: a plane

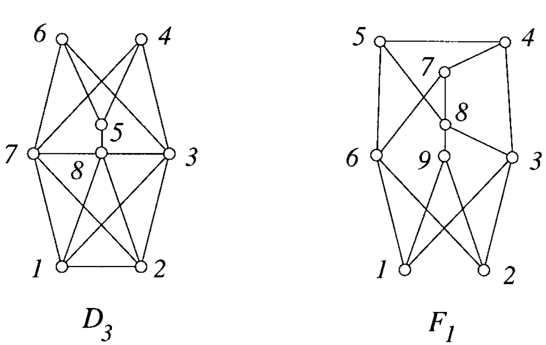

FIG. 4. D_3 and F_1.

extension of K cannot contain disjoint essential cycles. Conversely, suppose that $K_{3,5}$ is a minor of G. Then it is easy to see that either G contains a subgraph K homeomorphic to $K_{3,5}$, or contains as a minor the graph $K'_{3,5}$ which is obtained from $K_{3,5}$ by splitting a vertex of degree 5 into two adjacent vertices x, y of degrees 3 and 4, respectively (see Fig. 5). In the latter case we will prove that $K'_{3,5}$ itself (and therefore also G) contains disjoint essential cycles in every embedding. We will use the notation of Fig. 5. The cycle $1\,a\,2\,x$ is disjoint from the complementary $K_{2,3}$-graph. Assuming that K has no two disjoint essential cycles under the given embedding, we see by Lemma 2.1 that $1\,a\,2\,x$ is not essential. Similarly, $1\,b\,2\,x$ is not essential. We have two contractible cycles in the $K_{2,3}$-graph on vertices, 1, 2, a, b, x. By Lemma 2.1, this is not possible, and we are done.

Suppose now that G contains a subgraph K which is homeomorphic to $K_{3,5}$. We may assume that G does not contain $K'_{3,5}$ as a minor. We also suppose that G has an embedding without two disjoint essential cycles. Denote by 1, 2, 3, 4, 5 the vertices of degree 3 in K, and let $1'$, $2'$, $3'$ be the vertices of degree 5 in K. Denote by K' the graph obtained from K by adding the edge 12. This edge is contained in three nonseparating triangles. Therefore at least one of them is essential. But each triangle is disjoint from a $K_{2,3}$-graph in K'. Therefore we have two disjoint essential cycles. Consequently, G does not contain K' as a minor. Consider now a $\{1', 2', 3'\}$-component B of G. Since K' is not a minor of G, B contains at most one among vertices 1, 2, 3, 4, 5. If B contains an essential cycle, then by Lemma 2.5, B contains an essential cycle that uses at most one vertex among $1'$, $2'$, $3'$. But such a cycle is disjoint from a $K_{2,3}$-graph in G. By Lemma 2.1, we have disjoint essential cycles. Thus B contains no essential cycles, and B can be changed by an elementary reduction to a triad, an empty graph, or an edge between two of $1'$, $2'$, $3'$. This implies that G is a plane extension of a graph from $\mathscr{K}^*_{3,k}$ (for some $k \geqslant 5$).

It remains to consider forbidden minors for the projective plane in which every 3-separator has either a triad or a triangle in one of the parts. There are exactly 11 such forbidden minors. They are shown in Fig. 6.

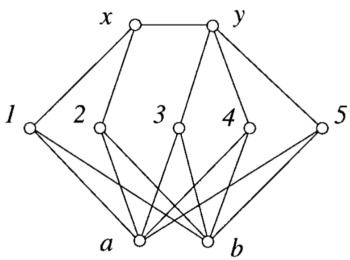

FIG. 5. The graph $K'_{3,5}$.

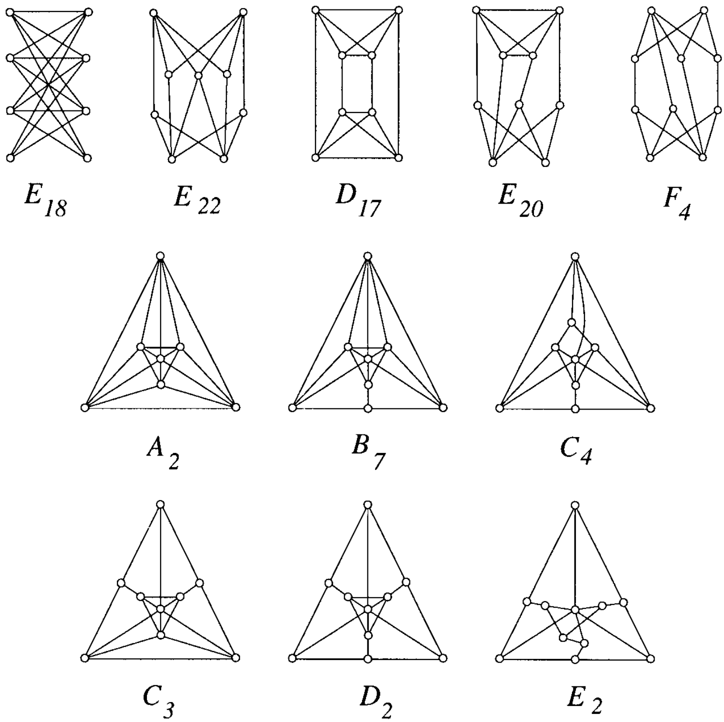

FIG. 6. Internally 4-connected forbidden minors.

Each of the graphs D_{17}, E_{20}, and F_4 contains two disjoint K-graphs, and we just apply Corollary 2.3.

The graph E_{18} is $K_{4,4}$ with a missing edge. Denote by 1, 2, 3, 4 and 1′, 2′, 3′, 4′ the vertices of E_{18} such that for all $i, j \in \{1, 2, 3, 4\}$, vertices i and j' are adjacent, except for $i=4$, $j=4$. The graph contains 9 pairs of disjoint cycles of the form $ij'k4'$ and $p'qr'4$ (where $\{i, k, q\} = \{1, 2, 3\}$ and $\{j', p', r'\} = \{1', 2', 3'\}$). One out of each pair must be contractible. Consequently, one of the vertices 4, or 4′ is contained in and least 5 contractible cycles of length 4. Since these cycles are induced and non-separating, they must be facial, and we have a contradiction since $\deg(4) = \deg(4') = 3$.

The graph E_{22} is obtained from $K_{5,4}$ by deleting a 4-matching. So, $V(E_{22}) = \{1, 2, 3, 4, 5, 1', 2', 3', 4'\}$ and the edges are ij' for $i=1, ..., 5$, $j=1, ..., 4$, and $i \neq j$. The subgraph on vertices 1′, 2′, 3, 4, 5 is a $K_{2,3}$-graph in E_{22}. By Lemma 2.1, we may assume that one of the cycles $1'32'5$ and $1'42'5$ is essential. This implies that the cycle $13'24'$ that is disjoint from it is contractible, and consequently it is also a facial cycle. Similarly we get that all other 4-cycles in the subgraph $E_{22}-5$ are facial. This already determines an embedding of this subgraph, giving no room to add the vertex 5 without changing any of the faces. A contradiction.

The next graph to consider is the graph A_2 which is obtained from the octahedron graph by adding a new vertex of degree 6. Any of the triangles of the octahedron is disjoint from the complementary K_4-graph. By Lemma 2.2, all these triangles must be facial if A_2 is embedded without two disjoint essential cycles. Clearly, this is a contradiction.

The remaining cases are easy. The graph B_7 is obtained from A_2 by a ΔY-exchange (replacing a triangle by a triad). Having an embedding of B_7 without two disjoint essential cycles we would therefore be able to get such an embedding of A_2 which is already excluded. Similarly we do the others: C_4 and C_3 are obtained by ΔY-exchange from B_7, D_2 is obtained from C_3, and E_2 from D_2. ∎

4. NON-FLAT GRAPHS

A piecewise-linear embedding of a graph in 3-space $\mathbf{R}^3$ is *flat* if every cycle of the graph bounds a 2-dimensional disk in $\mathbf{R}^3$ that is disjoint from the rest of the graph. A graph is *non-flat* if it does not have a flat embedding in $\mathbf{R}^3$. We will show in this section that non-flat graphs have two disjoint essential cycles in every embedding into a surface different from the projective plane. (Clearly, on the projective plane we cannot have two disjoint essential cycles.)

It is not that surprising that non-flat graphs usually have two disjoint essential cycles. The graphs that admit flat embeddings were recently characterized by Robertson, Seymour, and Thomas [7, 8] as the graphs which do not contain a minor isomorphic to one of seven graphs shown in Fig. 7. These graphs are known as *Petersen's family* since they can be obtained from Petersen's graph by means of $Y\Delta$- and ΔY-exchanges. It is also known [9] that Petersen's family contains precisely the minor minimal graphs that do not have an embedding into the projective plane with representativity $\leqslant 2$. The representation of the graphs in Fig. 7 is by means of embeddings in the projective plane. This includes the graph $K_{4,4}$ minus an edge in Fig. 7c which is not projective planar. Therefore two of its edges must cross.

THEOREM 4.1. *If a graph G contains one of the seven graphs in Petersen's family as a minor, then every embedding of G in a surface different from the projective plane has two disjoint essential cycles.*

Proof. It suffices to give a proof for the seven graphs in Petersen's family. Suppose that G is one of them, and that G is embedded in a surface

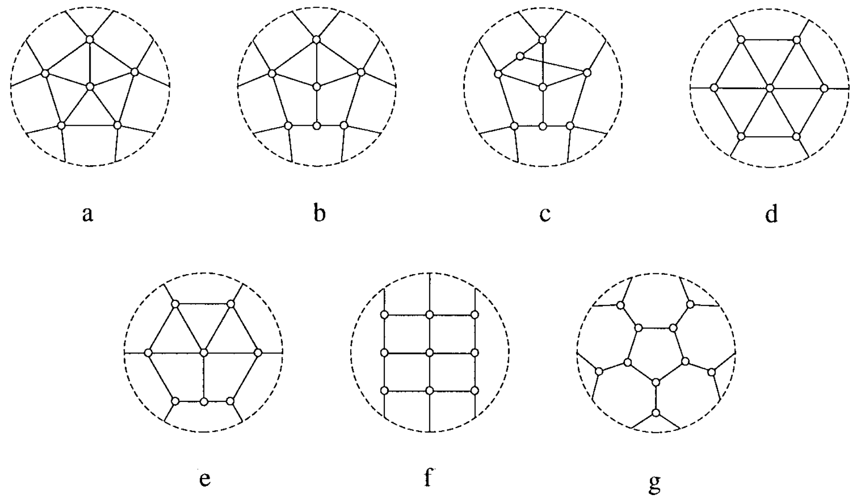

FIG. 7. Petersen's family.

Σ different from the projective plane, and that G does not contain two disjoint essential cycles.

Consider first the case $G = K_6$. There are $\frac{1}{2}\binom{6}{3} = 10$ different pairs of disjoint triangles in G. One in each pair is not essential, and thus facial. Therefore G has at least 10 faces. By Euler's formula we have:

$$\chi(\Sigma) \geqslant |V(G)| - |E(G)| + 10 = 1. \tag{1}$$

Since Σ is not the projective plane and not the 2-sphere (into which K_6 cannot be embedded), we have $\chi(\Sigma) \leqslant 0$, and this contradicts (1).

Since K_6 has no embedding without two disjoint essential cycles in a surface different from the projective plane, the same holds for the graphs obtained from K_6 by ΔY-exchanges. But this covers all the graphs of Petersen's family except the graph $K_{3,3} + v$ ($K_{3,3}$ plus a vertex adjacent to all six vertices of $K_{3,3}$), which is shown on Fig. 7d. This graph contains 9 pairs of disjoint cycles (a triangle determined by v and an edge of $K_{3,3}$ plus the complementary quadrilateral). In each of these pairs, one of the cycles is not essential. Since all the considered cycles are induced and nonseparating, they are facial if they are not essential. So we have a graph with 7 vertices, 15 edges, and at least 9 faces. By Euler's formula we see that $\chi(\Sigma) \geqslant 1$, and so Σ is the projective plane. ∎

THEOREM 4.2. *Let G be a graph which is not a planar extension of a graph from $\mathscr{K}^*_{3,k}$ $(k \geqslant 5)$. Then the following assertions are equivalent*:

(a) *G has two disjoint essential cycles in every embedding into any orientable surface.*

(b) *G has two disjoint essential cycles in every embedding into any surface different from the projective plane.*

(c) *G cannot be embedded in the projective plane with representativity at most* 2.

(d) *G contains one of the* 35 *forbidden minors for the projective plane or one of the graphs from Petersen's family as a minor.*

Proof. Equivalence of (c) and (d) was proved by Vitray [10]. By Theorems 3.1 and 4.1, (d) implies (b). Clearly, (b) implies (a). Hence, it suffices to show that (a) implies (c).

Suppose now that G can be embedded in the projective plane with representativity at most 2. If the representativity is 0 or 1, then G is a planar graph, and G does not satisfy (a) because of its embedding in the 2-sphere. If the representativity is 2, let γ be an essential curve meeting the graph G exactly in two vertices, x, and y. If we cut the projective plane along γ, we get an open disk D with x and y on its boundary appearing interlaced. It is easy to see that D can be embedded in the torus so that the closure of D is a disk with two pairs of opposite points (corresponding to x and y, respectively) identified. This determines an embedding of G in the torus that does not have disjoint essential cycles. ∎

5. BASIC EXAMPLES

In the rest of the paper we shall consider the structure of maps without disjoint essential cycles. There are embedded graphs which "obviously" do not contain disjoint essential cycles. Examples of such maps are:

(a) Any graph embedded graph in the 2-sphere or the projective plane.

(b) Any plane embedded graph. Such an embedded graph G contains no essential cycles at all, and there is an open disk in the surface containing G.

(c) A graph G embedded in such a way that for some vertex x of G, the subgraph $G-x$ is plane embedded.

(d) A map G in Σ has the *projective wheel structure* if there are vertices x, y_1, y_2, ..., y_t $(t \geqslant 1)$ in G such that the following holds:

(d1) $G-x-y_i$ is plane embedded for every i, $1 \leqslant i \leqslant t$.

(d2) There are disjoint open disks $D_1, D_2, ..., D_t$ in Σ such that for each i, $1 \leqslant i \leqslant t$, the boundary $\partial\overline{D_i}$ of the closure of D_i intersects G in vertices x, y_i, y_{i+1} (index modulo t), and these vertices appear on $\partial\overline{D_i}$ in order x, y_i, x, y_{i+1}. If $t > 1$, $\overline{D_i}$ is homeomorphic to a closed unit disk with a pair of opposite vertices on the boundary identified. If $t = 1$, $\overline{D_1}$ is a disk with two pairs of opposite points identified.

(d3) Every $\{x, y_1, y_2, ..., y_t\}$-component Q of G is either contained in some $\overline{D_i}$ $(1 \leqslant i \leqslant t)$, or it is a plane embedded $\{x, y_i\}$-component for some i. In the latter case, Q contains no essential cycles, and after an elementary reduction, Q will either disappear or it will be replaced by an edge between x and y_i.

Although the closures $\overline{D_i}$ of the disks D_i from (d2) are not disks, we shall call them *patches* of the embedding. Every map with the projective wheel structure can be easily transformed into a map in the projective plane. See Fig. 8 where shaded patches represent disks D_i and dotted lines represent the $\{x, y_i\}$-components that are not in the disks D_i $(i = 1, ..., t)$. The obtained map in the projective plane also has the projective wheel structure. Its representativity is at most two.

By (d1), every essential cycle in G either contains the vertex x, or it contains all of $y_1, ..., y_t$. Suppose now that there is an essential cycle C_1 through $y_1, ..., y_t$ and an essential cycle C_2 containing x, where $C_1 \cap C_2 = \emptyset$. Then C_2 is contained in an $\{x, y_1, ..., y_t\}$-component Q. By (d3) and since C_1 contains all vertices y_j, Q lies in some D_i. By (d2), C_2 intersects every (y_i, y_{i+1})-path in Q. Since the only possibility for C_1 to come from y_i to y_{i+1} in two distinct ways is to cross D_i, we see that C_1 must intersect C_2. It follows that a map with the projective wheel structure contains no disjoint essential cycles.

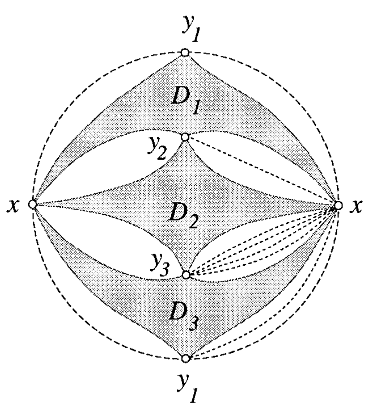

FIG. 8. The projective wheel structure.

We will use the following fact that can be easily verified. If G satisfies (d1)–(d3) and $\tilde{G}$ is a plane extension of G such that $x \in V(\tilde{G})$ (i.e., x is not a result of an elementary reduction of order 3), then $\tilde{G}$ has the projective wheel structure as well.

(e) *Maps of* $K_{3,k}$-*type* ($k \geqslant 0$): A map G is of $K_{3,k}$-*type* if it contains vertices x, y, z such that every $\{x, y, z\}$-component is plane embedded and the number of $\{x, y, z\}$-components containing all of x, y and z is equal to k. Then G is a plane extension of a graph from $\mathscr{K}^*_{3,k}$. (Let us observe that the converse does not hold.) Hence maps of $K_{3,k}$-type do not have disjoint essential cycles.

(f) K_5 and its plane extensions (K_5-*type*).

Above cases (a)–(f) correspond in a natural way to Dirac's graphs [2] without two disjoint cycles (see the introduction). Examples of type (a) and (b) are similar to forests, type (c) corresponds to graphs G with a vertex x such that $G - x$ is a forest, (d) to wheels (the vertex x corresponds to the centre of the wheel), $K_{3,k}$-type embeddings to graphs in which three vertices cover all the edges, and K_5-type is simply an analogue of K_5.

We will show that the above examples (together with some small variations) exhibit all maps without disjoint essential cycles. At this point, let us briefly discuss maps of $K_{3,k}$-type and their planar extensions.

Proposition 5.1. *Let G be a graph with an embedding of $K_{3,k}$-type in some surface Σ. Then G contains vertices x, y, z such that the following holds*:

(a) *There are exactly k well connected* 3-*patches $D_1, ..., D_k$ in Σ with x, y, z on their boundaries and such that $D_i \cap D_j = \{x, y, z\}$, $1 \leqslant i < j \leqslant k$.*

(b) *All $\{x, y, z\}$-components that are not contained in $D_1, ..., D_k$ are contained in $p \geqslant 0$* 1-*patches and* 2-*patches $D_{k+1}, ..., D_{k+p}$ where $\partial D_{k+i} \cap G \subset \{x, y, z\}$ $(1 \leqslant i \leqslant p)$ and such that $D_i \cap D_j \subseteq \{x, y, z\}$ $(1 \leqslant i < j \leqslant k+p)$.*

Proof. (a) and (b) hold by definition of $K_{3,k}$-type except that we need to prove that the patches D_i can be chosen such that they pairwise intersect only in x, y, z. Every $\{x, y, z\}$-component B is contained in a patch $D(B)$ such that $\partial D(B) \cap G$ is equal to $V(B) \cap \{x, y, z\}$. It is easy to see that for distinct $\{x, y, z\}$-components B, B', $D(B)$ and $D(B')$ can be replaced by their subsets $D_1(B)$ and $D_1(B')$ which contain B and B', respectively, and intersect as required. After doing this for all possible pairs of $\{x, y, z\}$-components, we get the required patch structure. ∎

Every map of $K_{3,k}$-type is a plane extension of an embedding of a graph from $\mathscr{K}^*_{3,k}$. The next proposition shows that the converse holds if $k \geqslant 4$.

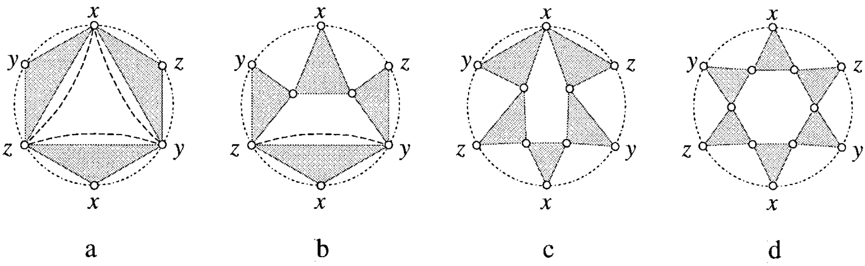

FIG. 9. Embeddings of planer extensions of $K_{3,3}$ without disjoint essential cycles.

Moreover, it classifies embeddings of planar extensions of $K_{3,k}$ that do not have disjoint essential cycles.

PROPOSITION 5.2. *Suppose that G is a planar extension of the graph $K_{3,k}$ where $k \geqslant 4$, and that G is not a planar extension of some $K_{3,l}$ $l < k$. Then an embedding of G does not have disjoint essential cycles if and only if the embedding is of $K_{3,k}$-type.*

Proof. Let $K = K_{3,k}$ with vertices x, y, z of degree k. Consider an embedding of G that is a plane extension of an embedding of K. Since $k \geqslant 4$, x, y, z are also vertices of G. The embedding of K has k well connected patches containing $\{x, y, z\}$-components, and since k is minimal, the same holds for G. Suppose now that G is embedded in some (other) surface and that an $\{x, y, z\}$-component B contains an essential cycle C. By Lemma 2.5, we may assume that $x \in V(C)$ but $y, z \notin V(C)$. Since $k \geqslant 4$, there are three $\{x, y, z\}$-components distinct from B that are attached to all of x, y, z. They determine a $K_{2,3}$-graph disjoint from C. By Lemma 2.1, we have an essential cycle disjoint from C. Consequently, no $\{x, y, z\}$-component contains an essential cycle. This implies that the embedding of G is of $K_{3,k}$-type. ∎

In contrast with Proposition 5.2, planar extensions of $K_{3,3}$ can have embeddings without disjoint essential cycles that are not of $K_{3,3}$-type. However, their structure cannot be too complicated. The general patch structure of such embeddings is shown in Fig. 9 (as embeddings in the projective plane) where the 3-patches are well connected and the broken lines represent an arbitrary number of additional 2-patches between the corresponding vertices.

6. LOW CONNECTIVITY CASES

The following lemma will help us discover elementary reductions.

LEMMA 6.1. *Let G be a graph embedded in a surface Σ, let K be a subgraph of G and let H be the union of all K-components in G. Suppose that*

H is connected, that no cutvertex of H belongs to K and that every cycle of H is contractible. Let A be the set of vertices of attachment of all K-components. Then there is a closed disc Δ in Σ containing H. If $\Delta \cap B = H$, then we can choose Δ so that $\partial\Delta \cap G = A$.

Proof. If H is 2-connected, then it is easy to show that there is a cycle C in H that bounds a closed disc Δ in Σ such that $H \subseteq \Delta$. We can, therefore, find a "tree-like" set of discs with disjoint interiors intersecting at cutvertices of H such that each disc contains some of the blocks of H and each block of H is contained in one of these discs. If two of these discs touch at a cutvertex v of H, we can add a small disk D_v around v such that the union of the two disks and D_v is a closed disc in Σ. After a finite number of steps we are left with a single disc Δ such that $H \subseteq \Delta$ and $\partial\Delta \cap G \subseteq H$. If $\Delta \cap G = H$, then $A \subseteq \partial\Delta$. Since there is an open disc in Σ that contains Δ, we can get another closed disc Δ' containing Δ such that $\partial\Delta' = A$. ∎

Our first reduction lemma enables us to restrict our attention to 2-connected graphs.

LEMMA 6.2. *Let G be a graph embedded in a surface Σ without two disjoint essential cycles. Then one of the following cases holds*:

(a) *G is a plane extension of a 2-connected graph H in Σ (with elementary reductions of order* 0 *and* 1 *only*).

(b) *G is a planar graph embedded in Σ in such a way that for some cutvertex v of G, $G - v$ contains no essential cycles.*

Proof. If G is disconnected, then at most one of its components contains an essential cycle. All other components can be removed by using elementary reductions of order 0. Thus we may assume that G is connected. If v is a cutvertex of G, let $B_1, ..., B_s$ be the $\{v\}$-components of G. If one of them, say B_1, does not contain essential cycles, then by Lemma 6.1, there is an elementary reduction of order 1 that removes B_1 (and possibly some other B_i) from G. By repeating such reductions, we get a graph H with all endblocks containing essential cycles. Consequently, every pair of blocks has a vertex in common, and so, if H is not 2-connected, it contains exactly one cutvertex v (or $|V(G)| \leqslant 2$ in which case we clearly have (b)). Every essential cycle must use v. Therefore $H - v$ contains no essential cycles. By Lemma 6.1, each component R of $H - v$ is contained in a closed disc Δ_R such that all vertices of R that are adjacent to v lie on $\partial\Delta_R$. This implies that H is a planar graph (although the embedding in Σ is not necessarily plane). It follows that the original graph G is also planar and that it satisfies (b). ∎

Our next result describes the geometric structure of maps in which all essential cycles are covered by two vertices.

LEMMA 6.3. *Let G be a graph in Σ that does not contain two disjoint essential cycles. Suppose that there are vertices x and y of G such that every essential cycle in G contains either x or y. Then one of the following cases holds*:

(a) *G is a planar graph embedded in Σ in such a way that for some vertex v of G, $G-v$ contains no essential cycles.*

(b) *G is a projective planar graph, and its embedding in Σ (as well as an embedding in the projective plane) has the projective wheel structure.*

(c) *The embedding of G is of $K_{3,k}$-type ($k \geqslant 0$).*

(d) *G is of K_5-type. In this case G is nonplanar.*

Proof. By using elementary reductions of order at most 2, we get a map on which no further nontrivial elementary reductions of order at most 2 are possible. The obtained map still has the properties stated in the lemma, and it satisfies (a), (b), (c), or (d) if and only if the original map does. By Lemma 6.2, we may assume that G is 2-connected. Excluding (a), we may also assume that there are essential cycles that do not use x, and there are some that do not use y.

Suppose that for some $u, v \in V(G)$ we have two or more $\{u, v\}$-components $B_1, \ldots, B_s$ where $s \geqslant 2$. If some B_i contains no essential cycles, it is just an edge from u to v. Suppose that B_1 and B_2 both contain essential cycles. By Lemma 2.5, each of them contains essential cycles that use at most one of u or v. Now it is easy to see that we have case (a). Hence, at most one of B_i is nontrivial. If some parallel edges remain, Lemma 6.1 implies that each pair of such edges forms an essential cycle. Thus, we may assume that G is "almost" 3-connected (up to some parallel edges). All parallel pairs in G are either covered by a single vertex, or there is a third vertex z and parallel edges appear only between x, y, and z. This is obvious since the parallel pairs determine essential cycles, and so the subgraph on the parallel edges does not contain a 2-matching.

The case when we have parallel edge pairs between any two of x, y, z is easy: If B is an $\{x, y, z\}$-component, it is plane embedded. If not, B contains an essential cycle using only one among x, y, z (Lemma 2.5), and this cycle is disjoint from one of the parallel pairs. Consequently, the map is of $K_{3,k}$-type for some $k \geqslant 0$.

Suppose that neither x nor y covers all parallel edges. Let z be the vertex which does. Since x, y cover all essential cycles, the only parallel edges are between z, x and between z, y. But then x and z cover all essential cycles. Then we can take z, x to play the role of vertices x, y covering all essential cycles. Therefore we may assume from now on that x covers all parallel edge pairs, and that G is 3-connected up to possible parallel edges at x.

Consider the graph $G' = G - x - y$. It is connected and plane embedded. If e, f are edges from y (or x) to G', there is a cycle C_{ef} determined by e, f and a path in G' joining the ends of e and f in G'. Although there are several possibilities for selecting this cycle, the homotopy class of C_{ef} in Σ is uniquely determined since G' is plane embedded. We say that e and f are *homotopic* if C_{ef} is contractible on Σ. The homotopy relation partitions the edges from y to G' (and similarly the edges from x to G') into disjoint homotopy classes. In the local rotation at y on Σ, the edges of the same homotopy class are consecutive. Therefore it is possible to split the vertex y into several new vertices, one for each homotopy class of edges from y to G', such that the edges at y of any homotopy class remain incident with the corresponding copy of y and the new graph is still embedded in Σ. The same operation can be done with x. By splitting x and y, as mentioned above, and removing the edges between x and y, we get a plane embedded graph H in Σ. See Fig. 10 for an example on the torus. We will refer to the new vertices obtained from x and y after splitting as *copies* of x and y, respectively. Note that all copies of x and y lie on the boundary of the outer face of H.

Since $G - x$ is not plane embedded, H contains at least two copies of y. Our aim is to show that there are exactly two. Similarly, there are at least two copies of x in H. Since G' is connected, no copy of x or y is a cutvertex of H. Note that H is connected.

Suppose first that H is 2-connected. Any path in H between two copies of y (or x, respectively) determines an essential cycle in G. Therefore every path in H between copies of y intersects any path between copies of x. Since H is 2-connected, this implies that the copies of y and x alternate on the cycle bounding the outer face of H, and there are exactly two copies of each of them. This gives the projective wheel structure of G (with $t = 1$).

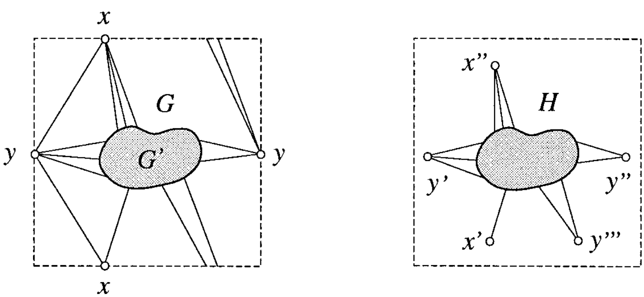

FIG. 10. The graph H.

Suppose now that H is not 2-connected, and let z be a cutvertex of H. Since all copies of x and y lie on the boundary of the outer face of H, we may assume that z is also on its boundary. (Otherwise we have an elementary reduction at z in G.) Let B be one of the components of $H-z$. Since G is "almost" 3-connected, B is either just one of the copies of x or y, or there are at least two of these vertices in B. Let us say that B is *trivial* if the former possibility occurs, and *nontrivial* otherwise. If B is nontrivial, it contains copies of x and of y. Otherwise a path between two copies of x (or y) would exist in B, and this path would be disjoint from any path between copies of the other vertex. It may happen that in B there is just one copy of x and one copy of y. In this case, $B+z$ can be changed into a triad (attached to x, y, z in the graph G) by applying an elementary reduction of order 3. We call such a component B of $H-z$ a *triad component*. Note that a triad component is well connected in G.

Let us first consider the case when there are at least three nontrivial components of $H-z$. If one of them contains two copies of x or two copies of y, it is easy to see that there are disjoint essential cycles in G. Otherwise, all of them are triad components. It follows that the embedding of G is of $K_{3,k}$-type.

From now on we may assume that for every cutvertex z of H, we get at most two nontrivial components. If for some z, all the components are either trivial or triad, then we have a plane extension of an embedding of some $K_{3,k}$. Therefore we may assume that for every choice of z we have at least one nontrivial, non-triad component B of $H-z$.

Suppose that a block Q of H contains at least three cutvertices z such that $H-z$ contains two nontrivial components. If there are four or more such cutvertices in Q, it is easy to see that we have disjoint essential cycles. Hence, there are exactly three. None of them is a copy of x or y. If there is another cutvertex in Q (separating trivial components), or if Q contains a copy of x or y, then we get disjoint essential cycles. The same happens if at least one of the three cutvertices of Q has a non-triad component (disjoint from Q). Therefore all three cutvertices in Q have triad components. Consequently, G is of K_5-type.

From now on we will assume that every cutvertex z of H gives rise to at most two nontrivial components of $H-z$, and that every block of H contains at most two cutvertices which give rise to two nontrivial components. Therefore the block structure of H is "caterpillar-like". Blocks are arranged in linear order with possible pendant edges (corresponding to trivial components) attached to any of the vertices (see Fig. 11). Let $B_1, B_2, ..., B_q$ be the consecutive blocks of H in this chain structure. (We do not count the pendant edges.)

Consider first the case when $q=1$. The situation is equivalent to the case when H is 2-connected if at every cutvertex of H, there is exactly one

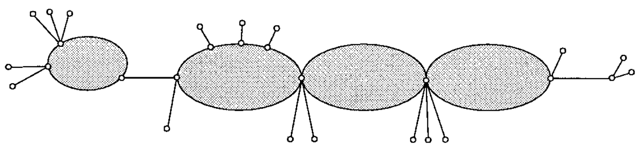

FIG. 11. The caterpillar structure of H.

pendant edge (trivial component). So we may assume that a cutvertex z has two or more trivial components. If both of them are copies of x, the two copies of y can be joined in $H-z$ (giving rise to the second essential cycle), unless a copy of y is also one of the trivial components at z. At most one copy of y is found elsewhere in H. Since x covers all parallel edges, there is just one copy of y pendant at z. There is exactly one more copy of y in H. It is either pendant at a cutvertex z', or it is contained in B_1. In each case, it is easy to see that we must have the projective wheel structure with $t=3$ or $t=2$, and $y_1=y$, $y_2=z$, $y_3=z'$ (whenever $t=3$). This completes the analysis in the case when two copies of x are pendant at z. Since x covers all parallel edges, we do not have two copies of y pendant at z. Now we may assume that at z we have a copy of x and a copy of y. If there is just one more copy of x and just one more copy of y, it is easy to discover the projective wheel structure. Suppose now that we have two or more additional copies of x. If there are also two more copies of y, the only possibility is that we have three cutvertices of H, each with a pendant pair x, y. In this case, G is of K_5-type (if x and y are adjacent), of $K_{3,3}$-type, or of $K_{3,2}$-type (x, y nonadjacent). Otherwise, there is just one more copy of y, and it can be seen that we have the projective wheel structure.

Suppose now that $q \geqslant 2$. For $i=2, 3, ..., q$, denote by y_i the cutvertex shared by B_{i-1} and B_i. We will say that y_i is *simple* if one of the parts of $H-y_i$ is a triad component. Let us first assume that y_2 and y_q are not simple. In this case B_1 contains two copies of x, say, and one copy of the other vertex y. Similarly, B_q contains two copies of x and one copy of y. Since there are paths between the two copies of x in $B_1 - y_2$ (and the same in $B_q - y_q$), there are no other copies of y in H. But there may be other copies of x in $B_1, B_2, ..., B-q$, or pendant at these blocks. However, for each i, there are at most two copies of x that belong to $B_i \backslash \{y_i, y_{i+1}\}$ or are pendant at vertices of B_i different from y_i and y_{i+1}. If there are two such copies x', x'' for some i, then every path in B_i joining them must cross every (y_i, y_{i+1})-path in B_i. It follows that x', x'', y_i, and y_{i+1} appear on the outer boundary of B_i in the interlaced order x', y_i, x'', y_{i+1}. The same holds in B_1 if y_1 is considered as a copy of y in this part. Similarly in B_q. It is now evident that we have the projective wheel structure.

The next case is when y_2 is simple and y_q is not simple. $B_q - y_q$ contains two copies of the same vertex, say x, and a copy of y. No other block B_i $(1 < i < q)$ has a copy of y attached to it. Suppose that $B_q - y_q$ contains two copies of y. Then $q = 2$ and no copy of x or y is pendant at y_2. Now it is easy to see that x and y are pendant in two pairs at B_q and we have a K_5-type embedding. This case is complete.

Suppose now that B_q contains just one copy of y. Then one finds the projective wheel structure as before.

We argue as above if y_q is simple and y_2 is not simple.

The remaining possibility is when y_2 and y_q are both simple. We have a copy of x and a copy of y at each of the parts corresponding to B_1 and B_q, respectively. If there is no other copy of y, we get the projective wheel structure (as above). Similarly, if there are no other copies of x. Suppose now that we have additional copies of x and of y. Let K be the graph obtained from H by deleting $B_1 - y_2$ and $B_q - y_q$ (and pendant copies of x and y) and adding vertices x^*, y^*. Join x^* to all (remaining) copies of x and y^* to all copies of y. Any two disjoint paths in K from $\{x^*, y^*\}$ to $\{y_2, y_q\}$ give rise to disjoint essential cycles in G. By Menger's Theorem we have a vertex $z \in V(K)$ blocking all paths between these two sets. It is easy to see that z is a cutvertex of H. Since any cutvertex of H gives rise to at most two nontrivial components, we have just copies of x and y pendant at z. If z lies on every path in H from y_2 to y_q, then the $\{z\}$-component of H containing B_1 does not contain disjoint essential cycles. Similarly for the $\{z\}$-component of B_q. Thus we have $K_{3,2}$-type (determined by x, y, z). On the other hand, if there is a path P in $H - z$ from y_2 to y_q, then we have K_5-type with respect to vertices x, y, z, y_2, y_q. ∎

Now we are able to show that we can restrict our main attention to 3-connected graphs.

LEMMA 6.4. *Let G be a graph in Σ that does not contain two disjoint essential cycles. Then either*:

(a) *G is a plane extension of a 3-connected graph H in Σ (with elementary reductions of orders at most 2).*

(b) *G is a planar graph embedded in Σ in such a way that for some vertex v of G, $G - v$ contains no essential cycles.*

(c) *G is a projective planar graph, and its embedding in Σ (as well as an embedding in the projective plane) has the projective wheel structure.*

(d) *The embedding of G is of $K_{3,k}$-type, $k \geqslant 0$.*

(e) *G is of K_5-type.*

Proof. All choices (a)–(e) resist elementary reductions of order at most 2. Therefore we can assume by Lemma 6.2 that G is 2-connected. Suppose that x, y is a 2-separator of G. If some $\{x, y\}$-component of G does not contain essential cycles, then it can be replaced by a single edge xy using an elementary reduction guaranteed by Lemma 6.1. Therefore we may assume that every $\{x, y\}$-component, that is not just an edge from x to y, contains an essential cycle. Note that we may have several edges between x and y.

Let B be an $\{x, y\}$-component, and let C be an essential cycle in B. By Lemma 2.5, B contains an essential cycle that does not use one of x or y. Thus, every $\{x, y\}$-component is either just an edge, or it contains an essential cycle that uses at most one of x and y. Suppose that we have two or more $\{x, y\}$-components that are not just edges. Let C_1 be an essential cycle in one of them that does not use y, say. Let C_2 be such a cycle in another $\{x, y\}$-component. Since $C_1 \cap C_2 \neq \emptyset$, we have $x \in V(C_1) \cap V(C_2)$. It follows that any other essential cycle in G passes through x. Therefore $G - x$ is plane embedded. In the same way as in the proof of Lemma 6.2, we conclude that G is a planar graph.

It remains to consider the case when G has only one nontrivial $\{x, y\}$-component B. (If there are none, we clearly have case (b).) Since x, y is a 2-separator, there is more than just one edge between x and y. By making appropriate elementary reductions, we may assume that no two such edges are homotopic, i.e., every pair determines an essential cycle in G. Thus, every essential cycle in G uses either x or y (or both). By Lemma 6.3 we have (b), (c), (d), or (e).

If there is another 2-separator u, v, we do the same, and we either get one of the cases (b)–(e), or reduce $\{u, y\}$-components so that u, v is no longer a 2-separator. This gives rise to a 3-connected graph. ∎

7. PLANAR GRAPHS

We have shown that graphs that are not embeddable in the projective plane always contain disjoint essential cycles, except when they are planar extensions of graphs from $\mathscr{K}^*_{3,k}$, $k \geqslant 5$. In this section we give for a large class of the remaining graphs a complete description of which of their embeddings have disjoint essential cycles and which do not. First we prove a lemma.

LEMMA 7.1. *Let G be a 2-connected plane graph and let* $\mathbf{C}$ *be a family of cycles of G satisfying the following conditions*:

(a) *No two cycles in* $\mathbf{C}$ *are disjoint.*

(b) *If* $C \in \mathbf{C}$, *let* D *be the disk bounded by* C, *and let* $D' = \bar{D}$, $D'' = \mathbf{R}^2 \backslash D$. *Then there are facial cycles* $C^{(1)}, C^{(2)} \in \mathbf{C}$ *such that* $C^{(1)} \subseteq D'$, $C^{(2)} \subseteq D''$.

Then either

(i) *G contains two vertices such that every cycle in* $\mathbf{C}$ *contains at least one of them, or*

(ii) *G has four faces bounded by cycles* $C_1, C_2, C_3, C_4 \in \mathbf{C}$ *and distinct vertices* $x_{ij} \in V(C_i) \cap V(C_j)$ $(1 \leqslant i < j \leqslant 4)$ *such that G is the union of four patches* D_1, D_2, D_3, D_4 *as shown in Fig.* 12. *Every cycle* $C \in \mathbf{C}$ *contains edges from at least three of the patches* D_1, D_2, D_3, D_4, *and none of these patches contains just a triad.*

Proof. Suppose that (i) is not the case. We will pay special attention to those cycles in $\mathbf{C}$ which are facial in the corresponding plane embedding of G. Denote the set of such cycles by $\mathbf{C}_0 \subset \mathbf{C}$.

Suppose first that there is a vertex $x \in V(G)$ which is contained in at least 3 cycles from $\mathbf{C}_0$, say C_2, C_3, C_4. Since x does not cover $\mathbf{C}$, there is a cycle $C_1 \in \mathbf{C}$ which does not contain x. By (b) we may assume that $C_1 \in \mathbf{C}_0$. Denote by z_i a vertex in $C_1 \cap C_i$, $i = 2, 3, 4$. Let $C' \in \mathbf{C}$ be a cycle that contains neither x nor z_2. Either the disk D' bounded by C' or its complement does not contain x. The same component of $\mathbf{R}^2 \backslash C'$ does not include z_2 since x, z_2 do not lie on C' but they lie on the boundary of a common face bounded by C_1. By (b), we may thus assume that $C' \in \mathbf{C}_0$. It is easy to see that if $z_3 \neq z_4$, then C' is disjoint from one among the cycles $C_2, C_3, C_4 \in \mathbf{C}$. Therefore, $z_3 = z_4$. Now we repeat the same procedure with z_3 instead of z_2, and we get a contradiction with our assumptions.

We may assume from now on that no three cycles from $\mathbf{C}_0$ meet a vertex. Excluding (i), $\mathbf{C} \neq \emptyset$ and G is not just a cycle. Thus, there are at least two

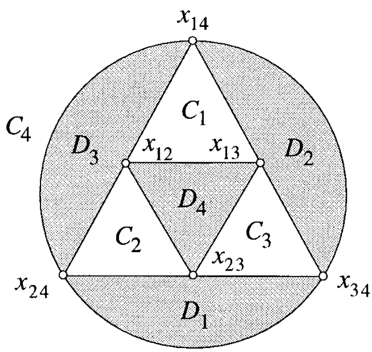

FIG. 12. The octahedron patch structure.

cycles in $\mathbf{C}_0$ because of (b). Denote them by C_1 and C_2 and let $x_{12} \in V(C_1 \cap C_2)$. Since x_{12} does not cover $\mathbf{C}$, we see as above that there is a cycle $C_3 \in \mathbf{C}_0$ that does not contain x_{12}. Denote by x_{ij} a vertex shared by C_i and C_j, $1 \leqslant i < j \leqslant 3$. Consider a cycle $C_4 \in \mathbf{C}$ not containing x_{12} and x_{13}. Since these two vertices are in the same face, we may assume by (b) that $C_4 \in \mathbf{C}_0$. Define vertices x_{ij} as above also in cases when $1 \leqslant i < j = 4$. All vertices x_{ij}, $1 \leqslant i < j \leqslant 4$, are pairwise distinct since no three cycles from $\mathbf{C}_0$ meet at a vertex. Then the situation is as Fig. 12 where C_4 is assumed to bound the infinite face.

By adding a vertex in each of the faces bounded by $C_1, ..., C_4$ and joining these vertices through vertices x_{ij} of G, we get an embedding of K_4. This uniquely determines the 3-patches of Fig. 12 as subgraphs of G. If one of the shaded patches (say the one in the centre) contains a cycle from $\mathbf{C}$, we get a cycle disjoint from one of the C_i (C_4 in our case). Consequently, we have (ii). Since no three cycles from $\mathbf{C}_0$ meet at a vertex, not patch D_i $(i = 1, 2, 3, 4)$ contains just a triad. ∎

Lemma 7.1 will be used in case when $\mathbf{C}$ is the collection of essential cycles of a planar graph embedded in some surface. If the patch structure of an embedded graph G is as shown in Fig. 12, we say that the map is of *octahedron type.*

COROLLARY 7.2. *Let G be a planar graph embedded in a surface Σ. Then G has no disjoint essential cycles if and only if one of the following cases holds*:

(a) *For some vertex $v \in V(G)$, $G - v$ contains no essential cycles.*

(b) *The embedding of G has the projective wheel structure.*

(c) *The embedding is a plane extension of an embedding of $K_{3,k}$ where $k \in \{0, 1, 2\}$.*

(d) *The embedding of G is of octahedron type.*

Proof. Sufficiency of (a)–(d) is clear. Suppose now that G does not contain disjoint essential cycles. By Lemma 6.2 we may assume that G is 2-connected since the only elementary reductions used to make G 2-connected are of order 0 or 1. Let $\mathbf{C}$ be the set of essential cycles of G on Σ. We claim that $\mathbf{C}$ satisfies conditions (a) and (b) of Lemma 7.1. Property (a) holds by assumptions. Let $C \in \mathbf{C}$ and let D, D' be as in (b). Suppose that no facial cycle in D' belongs to $\mathbf{C}$ and that C is such an example with the smallest number of faces in D'. Clearly, C is not facial. It is easy to see that D contains a face F of G such that $D \backslash F$ is homeomorphic to a disk D' in the plane. Let C_1 be the facial walk of F and let C' be the cycle bounding D'. By our assumptions, $C_1 \notin \mathbf{C}$. By elementary properties of homotopy we see that $C' \in \mathbf{C}$. If C' is facial, this contradicts our choice of C. Otherwise, it contradicts minimality of C. Existence of $C^{(2)}$ is proved in the same way.

Having verified (a) and (b), we have case (i) or (ii) of Lemma 7.1. Suppose that (i) is the case. By Lemma 6.3 we either have (a), (b) or (c). Since G is planar, $K_{3,k}$-type with $k \geqslant 3$ is excluded.

If we have (ii), then there are four closed disks $D_1, ..., D_4 \subset \Sigma$ which intersect only in points x_{ij}, $1 \leqslant i < j \leqslant 4$, as shown in Fig. 12, and such that $G \subset D_1 \cup D_2 \cup D_3 \cup D_4$. None of the disks D_i contains essential cycles. Thus we have (d). ∎

8. CONCLUSION

Our results present almost a complete solution of the following *Disjoint Essential Cycles Problem*:

(DECP) Given a graph G, characterize all embeddings of G in closed surfaces such that no two essential cycles are disjoint.

First of all, if G cannot be embedded in the projective plane, then the following two cases occur. If G contains three vertices, say x, y, z, such that for each $\{x, y, z\}$-component B the graph B together with edges xy, xz, yz is planar, then an embedding of G does not have two disjoint essential cycles if and only if the embedding is of $K_{3,k}$-type for some $k \geqslant 5$ (Theorem 3.1(b)). Otherwise, every embedding of G contains disjoint essential cycles (Theorem 3.1(a)).

Similarly, if G admits an embedding in the projective plane with representativity 3 or more, then an arbitrary embedding of G has disjoint essential cycles if and only if it is not an embedding in the projective plane (Theorem 4.1).

At the other extreme, embeddings of planar graphs without disjoint essential cycles are characterized by Corollary 7.2. The only remaining case is a nonplanar graph with an embedding of representativity 2 in the projective plane. (Note that graphs with an embedding of representativity 1 in the projective plane are planar.) Suppose that G is such a graph. By the results of Section 6 it suffices to consider the case when G is 3-connected. Its embedding with two vertices covering all the essential cycles are further characterized by Lemma 6.3. Excluding this possibility, we then use the fact that there is a vertex x and a contiguous subset (with respect to the local rotation at x) of edges incident to x whose removal results in a 2-connected planar graph G'. Moreover, the projective embedding of G and the planar embedding of G' are closely related. Now, Corollary 7.2 can be applied. A long and tedious case analysis then yields a characterization of embeddings of G without disjoint essential cycles. However, the case analysis is long, and we decided not to include it into this paper.

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