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# Planar Graphs on Nonplanar Surfaces

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It is shown that embeddings of planar graphs in arbitrary surfaces other than the 2-sphere have a special structure. It turns out that these embeddings can be described in terms of noncontractible curves in the surface, meeting the graph in at most two points (which may taken to be vertices of the graph). The close connection between the homology group of the surface and the planar graph embeddings is perhaps the most interesting aspect of this study. Some important consequences follow from these results. For example, any two embeddings of a planar graph in the same surface can be obtained from each other by means of simple local reembeddings very similar to Whitney's switchings. 

## 1. INTRODUCTION

Let $\Pi$ be a (2-cell) embedding of a graph $G$ into a *nonplanar surface* $S$, i.e., a closed surface distinct from the 2-sphere. Then we define the *face-width* $\mathrm{fw}(\Pi)$ (also called the *representativity*) of the embedding $\Pi$ as the smallest number of (closed) faces of $G$ in $S$ whose union contains a noncontractible curve.

One of the first results about the face-width, due to Robertson and Vitray [5] (cf. also [6]), considers the face-width of nonplanar embeddings of planar graphs. They proved that a planar graph embedded in a nonplanar surface has face-width at most two. Our main concern is to strengthen this result to obtain, essentially, a simple description of the

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structure of planar graphs embedded in nonplanar surfaces. Indirectly, this characterizes embedded graphs whose dual graphs are planar. It turns out that these embeddings can be described in terms of noncontractible curves in the surface, meeting the graph in at most two points (which may be taken to be vertices of the graph). We prove, in particular, that such curves pass through all vertices whose local rotation differs from that in a planar embedding of the graph; see Theorems 4.4 and 4.9. The close connection between the first homology group of the surface and the planar graph embeddings is perhaps the most interesting aspect of this study. Several important consequences follow from these results. For example, any two embeddings of a 2-connected planar graph in the same surface can be obtained from each other by using very simple elementary changes, called generalized Whitney switchings (see Theorem 7.1 and Corollary 7.3). This generalizes Whitney's Theorem [7] stating that any two embeddings of a 2-connected graph in the plane can be obtained from each other using a sequence of Whitney's 2-switchings.

The structure of embeddings of planar graphs in the projective plane is analysed in more details in [4]. In this paper we give such a description for the case of the torus and the Klein bottle under some additional restrictions on the embedding; see Corollary 6.3. It turns out that planar graphs embed in the torus in a particularly simple way. The local switches necessary for the generalized Whitney's theorem are explicitly described.

In the rest of the paper we assume that $G$ is a 2-connected planar graph. Most of the results can easily be extended to the 1-connected case but sometimes some technical conditions should be added. We shall repeat this assumption only in the statements of the main results to warn the readers that have skipped reading this part.

## 2. EMBEDDINGS

Let $G$ be a connected graph. 2-cell embeddings of $G$ in closed surfaces can be described in a purely combinatorial way by specifying:

(1) A *rotation system* $\pi = (\pi_v;\, v \in V(G))$; for each vertex $v$ of $G$ we have a cyclic permutation $\pi_v$ of edges incident with $v$, representing their circular order around $v$ on the surface. The cyclic sequence $e$, $\pi_v(e)$, $\pi_v^2(e)$, $\pi_v^3(e), \ldots$ is called *Π-clockwise ordering* around $v$.

(2) A *signature* $\lambda\colon E(G) \to \{-1, 1\}$. Suppose that $e = uv$. Following the edge $e$ on the surface, we see if the local rotations $\pi_v$ and $\pi_u$ are chosen consistently or not. If yes, then we have $\lambda(e) = 1$, otherwise we have $\lambda(e) = -1$.

The reader is referred to [1] for more details. We will use this description as a definition: An *embedding* of a connected graph $G$ is a pair $\Pi=(\pi, \lambda)$ where $\pi$ is a rotation system and $\lambda$ is a signature. Having an embedding $\Pi$ of $G$, we say that $G$ is *$\Pi$-embedded.* The embedding $\Pi$ is *nonorientable* if there is a cycle with an odd number of edges $e$ having $\lambda(e)=-1$. Such a cycle is *$\Pi$-onesided.* Other cycles are *$\Pi$-twosided.* We define *$\Pi$-facial walks* as closed walks in the graph that correspond to face boundaries of the corresponding topological embedding. If $W$ is a walk, any subwalk of $W$ is a *segment* of $W$. If $e=uv \in E(G)$, the pair $\{e, \pi_v(e)\}$ forms a *$\Pi$-angle.* A pair of edges that forms a $\Pi$-angle is *$\Pi$-consecutive.* We define the *genus* and the *Euler characteristic* $\chi(\Pi)$ of $\Pi$ as the genus and the Euler characteristic of the corresponding topological embedding, respectively.

If $\Pi=(\pi, \lambda)$ is an embedding of a graph $G$ and $H$ is a subgraph of $G$, then the *restriction* of $\Pi$ to $H$ is the embedding of $H$ whose rotation system is obtained from $\pi$ by ignoring all edges in $E(G)\backslash E(H)$ and whose signature is the restriction of $\lambda$ to $E(H)$.

If $G$ is a $\Pi$-embedded graph and $C$ is a $\Pi$-twosided cycle of $G$, then we define the *left graph* and the *right graph* of $C$ as follows. Select a vertex $v \in V(C)$, and let $e$ and $e'$ be the edges of $C$ incident with $v$. If $e'=\pi_v^k(e)$, then all edges $e$, $\pi_v(e)$, $\pi_v^2(e)$, ..., $\pi_v^k(e)$ are said to be *on the left side* of $C$. Then we traverse $C$ and determine left edges at each vertex of $C$ in the same way as at $v$. However, after traversing an edge $f$ of $C$ with $\lambda(f)=-1$, we change clockwise orientation to anticlockwise, and vice versa. In particular, traversing the edge $e'=vu$ from $v$ to $u$, the left edges at $u$ are $e'$, $\pi_u(e')$, $\pi_u^2(e')$, ..., $\pi_u^l(e')$ (where $\pi_u^l(e') \in E(C)$) if we have the clockwise orientation. Having the anticlockwise orientation, the left edges are $\pi_u^l(e')$, $\pi_u^{l+1}(e')$, ..., $e'$. Since $C$ is $\Pi$-twosided, the clockwise orientation is the same as at the beginning when we come back to the initial vertex $v$ after traversing the entire cycle $C$. An arbitrary edge $e$ (possibly not incident with $C$) is also said to be *on the left side* of $C$ if one of its ends is connected by a path in $G-C$ to an end of an edge on the left side of $C$ (and incident with $C$). Now the left graph $G_l=G_l(\Pi, C)$ is defined as the graph induced by all edges on the left side of $C$. The right graph $G_r=G_r(\Pi, C)$ is defined analogously. Note that $C \subseteq G_l \cap G_r$.

Let $C$ be a $\Pi$-twosided cycle and $G_l$ and $G_r$ its left and right graph. If $G_l \cap G_r=C$, then $C$ is said to be *$\Pi$-bounding.* A $\Pi$-bounding cycle $C$ is *$\Pi$-contractible* if the embedding of $G$ restricted to $G_l(\Pi, C)$ (or to $G_r(\Pi, C)$) is an embedding of genus 0. In this case, $G_l(\Pi, C)$ (or $G_r(\Pi, C)$, respectively) is called the *$\Pi$-interior* of $C$, and the rest of $G$, $G_r(\Pi, C)\backslash E(C)(G_l(\Pi, C)\backslash E(C)$, respectively) is the *$\Pi$-exterior* of $C$. By definition, $\Pi$-onesided cycles are $\Pi$-nonbounding.

## 3. PATCHES

Let $G$ be a $\Pi$-embedded 2-connected planar graph. Suppose that $G$ contains a $\Pi$-contractible cycle $C$ such that only two vertices $u, v$ of $C$ have incident edges that are in the $\Pi$-exterior of $C$. The replacement of the $\Pi$-interior of $C$ by the edge $uv$ is called a 2-*reduction*. Note that $G$ admits an embedding $\Pi'$ in the 2-sphere such that the $\Pi'$-interior of the cycle $C$ is the same as the $\Pi$-interior of $C$ and hence the same 2-reduction can be performed to $\Pi'$. If $v$ is a vertex of degree 2 in $G$ and the neighbors of $v$ are $u$ and $w$, then the replacement of $v$ and its incident edges by the edge $uw$ is also called a 2-*reduction*.

Let us now suppose that no 2-reductions are possible. Let $\Pi'$ be an embedding of $G$ in the 2-sphere. Then we define $\mathrm{CS}_0(\Pi, \Pi')$ to be the set of all $\Pi$-facial walks, that are also $\Pi'$-facial, together with all paths $P$ in $G$ such that $P$ is simultaneously a segment of a $\Pi$-facial walk and a segment of a $\Pi'$-facial walk. Since $G$ is 2-connected, $\Pi'$-facial walks are cycles and hence $\mathrm{CS}_0(\Pi, \Pi')$ contains only paths and cycles. Denote by $\mathrm{CS}(\Pi, \Pi')$ the subset of $\mathrm{CS}_0(\Pi, \Pi')$ consisting of all cycles and paths that are not contained in another element of $\mathrm{CS}_0(\Pi, \Pi')$. If $W$ is a $\Pi$-facial walk from $\mathrm{CS}(\Pi, \Pi')$, then we replace $W$ by a graph $\tilde{W}$ as shown in Fig. 1. Similarly, if $W \in \mathrm{CS}(\Pi, \Pi')$ is a maximal common segment of a $\Pi$-facial walk and a $\Pi'$-facial walk, then we replace $W$ by $\tilde{W}$ as shown in Fig. 2. (As a special case, when $W$ is just a path consisting of a single edge $e$ of $G$, this operation is just a subdivision of $e$ obtained by inserting five vertices of degree 2 on $e$.) When we do such replacements for all cycles and paths from $\mathrm{CS}(\Pi, \Pi')$, we obtain a 2-connected planar graph $\bar{G}$ containing the (subdivided) graph $G$ as a subgraph. The embeddings $\Pi$ and $\Pi'$ can be naturally extended to embeddings of $\bar{G}$.

In general, starting with $G$ we first perform all possible 2-reductions and then we construct, from the obtained graph $G'$, the graph $\overline{G'}$ as described above. If the graph $\overline{G'}$ contains a path whose interior vertices are all of degree 2, then we replace such a path by a single edge. After all such

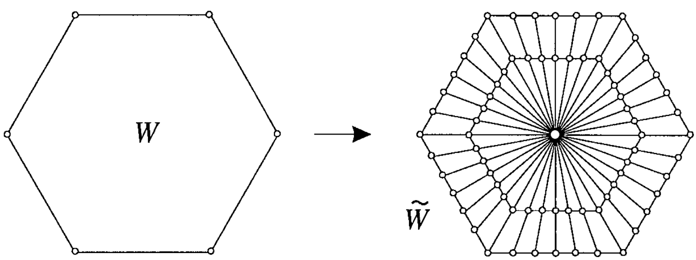


FIG. 1. Filling up common faces.

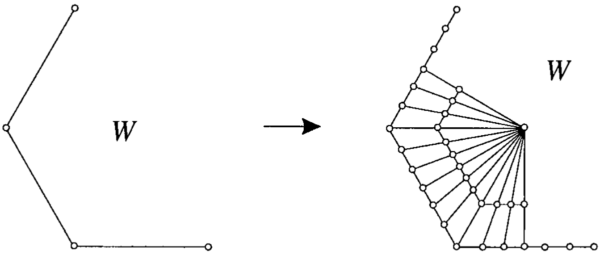


FIG. 2. Filling up maximal common facial segments.

replacements, if one half of an original edge $e$ of $G'$ becomes just an edge $\bar{e}$ in the resulting graph (and the other half of $e$ contains vertices of degrees 3 and 4), then we contract $\bar{e}$ to a point so that the "middle vertex" of $e$ is identified with an end of $e$. After doing all such changes, we obtain an embedded graph $\tilde{G}$ that is called a *patch extension* of the $\Pi$-embedded graph $G$. The patch extension $\tilde{G}$ contains a subdivision of $G$ as a subgraph.

It is clear that the embeddings $\Pi'$ and $\Pi$ can be extended to embeddings of $\tilde{G}$ in the same surfaces such that all triangles and quadrangles shown in Figs. 1 and 2, respectively, are facial. In particular, $\tilde{G}$ is a planar graph. It is also easy to see that if $G$ is 2-connected (3-connected, respectively), then so is $\tilde{G}$.

Let $\tilde{G}$ be the patch extension of a 2-connected $\Pi$-embedded planar graph $G$. Denote by $\tilde{\Pi}$ the corresponding embedding of $\tilde{G}$. The $\tilde{\Pi}$-facial walks that are not facial walks of the plane embedding of $\tilde{G}$ are the *patch facial walks* and the corresponding faces are the *patch faces.* Vertices of $\tilde{G}$ that belong to two or more patch facial walks are *patch vertices.* Segments of patch facial walks joining patch vertices are also segments of facial walks of $\tilde{G}$ embedded in the plane. They are called *patch edges.* Two $\tilde{\Pi}$-consecutive patch edges incident with the same patch vertex $v$ form a *patch angle* at $v$. The *patch degree* of a patch vertex $v$ is the number of patch angles at $v$. Let us remark that each patch vertex $v$ is either a vertex of $G$ or the middle vertex of a subdivided edge of $G$. In the latter case, the patch degree of $v$ is equal to two.

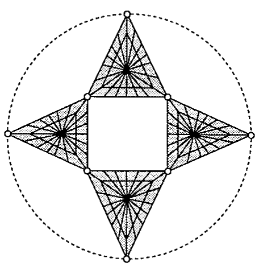

FIG. 3. Patch structure of the octahedron in the projective plane.

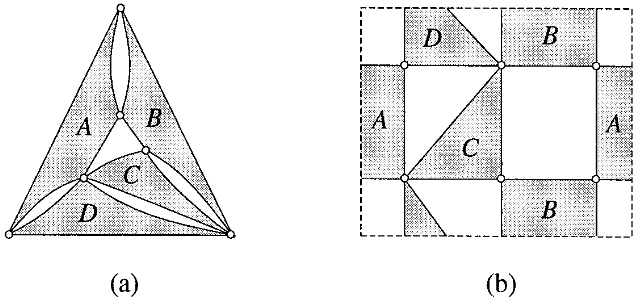


FIG. 4. Patch structure of an embedding in the torus.

Edges $e$ and $f$ of $\tilde{G}$ are *similar* of they both lie on the same patch edge or if they both lie on the same $\tilde{\Pi}$-facial walk that is not a patch facial walk. The smallest equivalence relation on $E(\tilde{G})$ containing the similarity relation determines a partition of edges of $\tilde{G}$ into subgraphs of $\tilde{G}$. They are called *patches* of $G$ (with respect to the embedding $\Pi$). It is convenient to consider the patches as being subsets of the surface of the embedding $\Pi$ consisting of the corresponding subgraph of $\tilde{G}$ together with all non-patch $\tilde{\Pi}$-faces that they contain. As such, distinct patches have disjoint interiors and they meet only in common patch vertices. They partition the complement of the interiors of patch faces in $S$. The interior (in $S$) of every patch is homeomorphic to an open disk in the plane with $p \geqslant 0$ holes.

Geometrically we will represent patches as shaded areas on the surface and will refer to the combinatorial structure of patches of an embedding $\Pi$ as the *patch structure* of $\Pi$. For example, Fig. 3 shows the patch extension of the octahedron embedded in the projective plane. The shaded areas in Fig. 3 are the patches. Another example of a planar graph (Fig. 4(a)) and its embedding in the torus (Fig. 4b) shows a more complicated patch structure.

## 4. PATCH ANGLES AND 2-CURVES

Suppose that $\Pi$ is an embedding of a 2-connected planar graph $G$ in a nonplanar surface $S$. Let $k$ be an integer. A simple closed curve $\gamma$ in $S$ is a *$k$-curve* if it satisfies the following conditions:

(C1) $\gamma$ intersects the graph $G$ in exactly $k$ points and all these points are patch vertices of $G$.

(C2) $\gamma$ uses only patch vertices and patch faces of $G$.

(C3) $\gamma$ is $\Pi$-noncontractible.

In (C3) we have used the term $\Pi$-noncontractible for a curve $\gamma$ that is not a cycle of $G$. However, if we consider the segments of $\gamma$ in the $\Pi$-faces of $G$ as edges between the corresponding patch vertices, the embedding $\Pi$ can be naturally extended to the union of $G$ and these edges. Then $\gamma$ corresponds to a cycle in the new graph, and contractibility of $\gamma$ refers to contractibility of this cycle. Similarly, we can use other concepts that were introduced for cycles in embedded graphs also for $k$-curves.

We will be interested mainly in 1-curves and 2-curves. Clearly, these curves coincide in $G$ and in the patch extension $\tilde{G}$ of $G$. In the case of the patch extension $\tilde{G}$, condition (C2) is automatically satisfied for any curve $\gamma$ for which (C1) and (C3) hold. Therefore it is more convenient to work wit $\tilde{G}$ instead of the original graph $G$. Since $\tilde{G}$ has the same patch structure as $G$, we can assume from now on that our planar graph $G$ is the patch extension graph of some planar graph. In particular, we assume that no 2-reductions are possible.

Two 1-curves are *equivalent* if they use the same patch face $\Phi$, the same patch vertex $v$ and the same pair of patch angles of $\Phi$ at $v$. Two 2-curves are *equivalent* if they pass through the same pair of patch faces and patch vertices and use the same patch angles. We will distinguish 1-curves and 2-curves only up to equivalence.

By a 1/2-*curve* we shall refer to 1-curves and 2-curves. In this section we shall prove that there are 1/2-curves "everywhere in the surface" except inside the patches where the embedding of the graph $G$ locally matches planar embeddings of $G$.

LEMMA 4.1. *Let $\{\alpha, \beta\}$ be a patch angle at a patch vertex $v$ in a patch facial walk $\Phi$. If the patch edge $\alpha$ does not occur on $\Phi$ twice, then there is a cycle $C(\alpha, \beta) \subseteq \Phi$ that contains $\alpha$. If $v$ appears on $\Phi$ just once, then $C(\alpha, \beta)$ also contains $\beta$.*

*Proof.* Those patch edges of $\Phi$ that occur on $\Phi$ just once form an Eulerian graph. Hence the claim. ∎

Disjoint pairs $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ of (patch) edges incident to the same patch vertex $v$ $\Pi'$-*interlace* if $\alpha$ and $\beta$ (and hence also $\gamma$ and $\delta$) are not consecutive under the rotation system of $\Pi'$ restricted to $\{\alpha, \beta, \gamma, \delta\}$.

LEMMA 4.2. *Suppose that $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ are $\Pi'$-interlacing patch angles at a patch vertex $v$. Then either there is a 1-curve through $\{\alpha, \beta\}$ or through $\{\gamma, \delta\}$, or there is a 2-curve through $v$ that uses both patch angles $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ at $v$.*

*Proof.* If $v$ appears more than once on the patch facial walk containing the angle $\{\alpha, \beta\}$, then there is a simple closed curve $\psi$ in the corresponding patch face through the angle $\{\alpha, \beta\}$. Since $G$ is 2-connected, $\psi$ is $\Pi$-nonbounding and hence a 1-curve. Similarly for the angle $\{\gamma, \delta\}$. Otherwise, let $C(\alpha, \beta)$ and $C(\gamma, \delta)$ be cycles from Lemma 4.1. In the plane, these two cycles cross at $v$, and hence they have another point $\tau$ in common. We may assume that $\tau$ is a patch vertex. Let $\psi$ be a simple closed curve through the angles $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$ that intersects $G$ only at $v$ and $\tau$. If $\psi$ is contractible, then it bounds a disk containing more than just one edge in its interior. Because of our initial 2-reductions, this is not possible, and hence $\psi$ is a 2-curve. ∎

Let $\gamma$ and $\gamma'$ be simple closed curves on a surface $S$ that intersect in finitely many points. Suppose that $z \in \gamma \cap \gamma'$. Then $\gamma$ and $\gamma'$ *cross* at $z$ if $z$ has an open neighborhood $U$ homeomorphic to the plane such that the homeomorphism maps $U \cap \gamma$ onto the $x$-axis and $U \cap \gamma'$ onto the $y$-axis in the plane. Otherwise they *touch* at $z$. The curves are *noncrossing* if they touch at each of their points of intersection.

A similar proof as above yields the following result:

LEMMA 4.3. *Let $F$ and $F'$ be $\Pi$-facial walks with an edge $e = uv$ in common. Suppose that $F = \alpha u e v \beta \ldots$ and that $F' = \delta u e v \gamma \ldots$. Let $H$ be the subgraph of $G$ consisting of edges $e$, $\alpha$, $\beta$, $\gamma$, and $\delta$. Suppose that in the rotation system of $\Pi'$ restricted to $H$, the local rotation at $u$ is $(\alpha e \delta)$ and the local rotation at $v$ is $(\beta e \gamma)$, and that the signature of $e$ is* 1. *If there is no* 1*-curve in $F$ or in $F'$ through $v$ or through $u$, then there is a* 2*-curve through $F$ and $F'$ that uses the $\Pi$-angles $\{\alpha, e\}$ and $\{\delta, e\}$*

Lemma 4.2 yields an important result about existence of 1/2-curves.

THEOREM 4.4. *Let $B$ be a* 2*-connected planar graph that is $\Pi$-embedded in a nonplanar surface. Then for every patch face $\Phi$ and every patch vertex $v$ of $\Phi$, there is either a* 1*-curve through $v$ or there is a* 2*-curve through $\Phi$ and $v$.*

*Proof.* Let $v$ be a patch vertex at a patch angle $\{\alpha, \beta\}$ of $\Phi$. Split the patch edges that are incident to $v$ and different from $\alpha$ and $\beta$ into two classes, depending in which $\Pi'$-subinterval from $\alpha$ to $\beta$ they are. Since $\alpha$ and $\beta$ are not $\Pi'$-consecutive, there are $\Pi$-consecutive patch edges $\gamma, \delta$ that are in different classes. They determine a patch angle that $\Pi'$-interlaces with $\{\alpha, \beta\}$, and Lemma 4.2 can be applied. ∎

The following result is a simple corollary of Theorem 4.4.

COROLLARY 4.5. *Suppose that $G$ has $k$ patch edges. If $\mathrm{fw}(\Pi) = 2$, then there is a set of at least $k/4$ nonequivalent* 2*-curves such that each of these*

*2-curves passes through a patch angle that is not used by any of the other 2-curves in the set.*

*Proof.* Denote by $v_1, ..., v_s$ the patch vertices of $G$. For $i=1, ..., s$, let $d_i$ be the number of patch edges incident with $v_i$. The number of patch angles is at least $\sum_{i=1}^{s} d_i/2 = k$. Since every 2-curve uses four patch angles, Theorem 4.4 implies that the number of nonequivalent 2-curves satisfying the "minimality" condition of the corollary is at least $k/4$. ∎

We cannot argue in the same way as above if $\mathrm{fw}(\Pi)=1$. The result that we get is slightly weaker.

COROLLARY 4.6. *If G has k patch edges, then there is a set of at least $k/(76-48\chi(S))$ nonequivalent 1/2-curves, where $\chi(S)$ denotes the Euler characteristic of $\Pi$. Each of these curves passes through a patch angle that is not used by any of the other curves in the set.*

*Proof.* A patch angle is said to be *nonsimple* if it is used by a 1-curve. It is *bad* if no 1/2-curve uses it. For every patch angle $\{\alpha, \beta\}$ there is an angle $\{\gamma, \beta\}$ that $\Pi'$-interlaces with it (cf. the proof of Theorem 4.4.). If an angle $\{\alpha, \beta\}$ at a patch vertex $v$ is bad, the $\{\gamma, \delta\}$ is nonsimple. Let $W$ be the patch facial walk containing the angle $\{\gamma, \delta\}$. Write $W$: $\gamma_1 Q_1 \delta_2 \gamma_2 Q_2 \delta_3 \cdots \gamma_s Q_s \delta_1$ where subwalks $Q_1, ..., Q_s$ do not contain $v$ and each angle $\{\gamma_i, \delta_i\}$ contains $v$. Suppose that the two angles $\{\rho, \alpha\}$ and $\{\beta, \sigma\}$ adjacent to the bad angle $\{\alpha, \beta\}$ are not nonsimple. Then $\gamma_i, \delta_i \notin \{\alpha, \beta\}$ for $i=1, ..., s$. It is easy to see that the number of angles $\{\gamma_i, \delta_i\}$ that $\Pi'$-interlace with $\{\alpha, \beta\}$ is even. In the same way as we proved Lemma 4.2 we see that $C(\alpha, \beta)$ and the cycle $\gamma_i Q_i \delta_{i+1}$ intersect only once, and therefore no consecutive pair $\gamma_i, \delta_{i+1}$ on $W$ $\Pi'$-interlaces with $\{\alpha, \beta\}$ ($i=1, ..., s$; the index $i+1$ taken modulo $s$). These properties imply that the total number of angles at $v$ that $\Pi'$-interlace with $\{\alpha, \beta\}$ is even. Hence we have:

(P1) If the two angles $\{\rho, \alpha\}$ and $\{\beta, \sigma\}$ adjacent to the bad angle $\{\alpha, \beta\}$ are not nonsimple, then $\rho, \sigma$ do not $\Pi'$-interlace with $\alpha, \beta$.

Let us now consider the patch angles (of $\Pi$) at a patch vertex $v$. Let $p$ be the number of nonsimple ones. Since $G$ is 2-connected, no two 1-curves through $v$ are homotopic. Therefore the number of 1-curves through $v$ is at most $4-3\chi(S)$ (see, e.g. [2, Proposition 3.6]). This implies that

$$p \leqslant 8-6\chi(S). \tag{1}$$

Let $r$ be the number of bad angles at $v$ and let $q$ be the number of the remaining angles at $v$ (simple and not bad). Denote by $\{\alpha_1, \beta_1\}, ..., \{\alpha_r, \beta_r\}$ the bad angles in the order determined by $\Pi$.

Suppose that these are $s$ consecutive bad angles, say $\{\alpha_1, \beta_1\}$, ..., $\{\alpha_s, \beta_s\}$, such that for $i=1, ..., s-1$, there is no patch angle between $\{\alpha_i, \beta_i\}$ and $\{\alpha_{i+1}, \beta_{i+1}\}$. Property (P1) implies that for each $i=2, ..., s-1$ all edges $\alpha_1, \beta_1, ..., \alpha_{i-1}, \beta_{i-1}, \alpha_{i+1}, \beta_{i+1}, ..., \alpha_s, \beta_s$ are in the same $\Pi'$-part between $\alpha_i$ and $\beta_i$. A nonsimple angle $\{\gamma_i, \delta_i\}$ that $\Pi'$-overlaps with $\{\alpha_i, \beta_i\}$ has one of its edges, say $\gamma_i$, in the other $\Pi'$-part between $\beta_i$ and $\alpha_i$. Then the edges $\gamma_2, ..., \gamma_{s-1}$ are all distinct and hence $s-2 \leqslant 2p$. By (1), $s \leqslant 18 - 12\chi(S)$. This implies that

$$r \leqslant (18 - 12\chi(S))(p+q). \tag{2}$$

Now, (2) implies a bound on the number of patch angles at $v$

$$p+q+r \leqslant (19 - 12\chi(S))(p+q)$$

in terms of the number of 1-curves and 2-curves through $v$. Now the same conclusion as used in the proof of Corollary 4.5 yields the bound of the corollary. ∎

A more careful application of methods in the above proof yields a better bound than presented above. However, this bound still depends on the genus of $S$ and we do not see a way how to improve it to a similar bound as obtained in Corollary 4.5 in the case of face-width two.

Let $P$ be path in $G$ and let $v_1, ..., v_k$ be interior vertices of $P$. The edges incident with $v_1, ..., v_k$ can be classified as edges on the left side of $P$ (or on the right side of $P$) in the same way as in the definition of the left (and the right) graph of a $\Pi$-twosided cycle. Denote by $E'_P$ the set of all edges of $G$ incident to $v_1, ..., v_k$ that are distinct from edges on $P$. By splitting $E'_P$ into the set of edges on the left side and the set of edges on the right side of $P$, respectively, we obtain the *$\Pi$-splitting* at $v_1, ..., v_k$ with respect to $P$. (It may happen that the splitting is not a partition of $E'_P$.)

LEMMA 4.7. *Let $v$ be a patch vertex and $e, f$ edges (possibly inside patches) incident with $v$. If no 1-curve contains $v$ and no 2-curve crosses the path $P = evf$ at $v$, then the $\Pi$-splitting and the $\Pi'$-splitting at $v$ with respect to $P$ coincide.*

*Proof.* Suppose that no 1-curve passes through $v$. Let $C_1, C_2$ and $C'_1, C'_2$ be the $\Pi$-splitting and the $\Pi'$-splitting, respectively. If they are not the same, there is a pair $\alpha, \beta$ of $\Pi$-consecutive patch edges in $C_1$ (say) such that $\alpha \in C'_1$ and $\beta \in C'_2$. Since $e$ and $f$ $\Pi'$-interlace with $\{\alpha, \beta\}$, there is a pair of $\Pi$-consecutive edges $\gamma, \delta \in C_2 \cup \{e, f\}$ that $\Pi'$-interlace with $\{\alpha, \beta\}$. We are done by applying Lemma 4.2. ∎

The condition of Lemma 4.7 that there are no 1-curves through $v$ cannot be omitted. It may happen that the $\Pi$ and $\Pi'$-splittings at $v$ with respect to $P$ do not coincide and that there are neither 1-curves nor 2-curves crossing $P$ at $v$. An example in the torus is shown in Fig. 5 where $P$ is the vertical path at $v$.

LEMMA 4.8. *Suppose that $P = eu\sigma vf$ is a path in $G$ where $\sigma$ is a patch edge joining patch vertices $u$ and $v$. Suppose that no 1-curve passes through $u$ or $v$ and that no 2-curve crosses $P$ at $u$ or $v$. Then the $\Pi$-splitting and the $\Pi'$-splitting at $u$ and $v$ with respect to $P$ are the same.*

*Proof.* By Lemma 4.7, the $\Pi$-splitting at $u$ coincides with the $\Pi'$-splitting at $u$ with respect to the path $eu\sigma$. Similarly at $v$. If the patch containing $\sigma$ is not just an edge, the two pairs of splittings are clearly the same as the splittings with respect to $P$. Otherwise, we simply apply Lemma 4.3. ∎

Above results imply the following.

THEOREM 4.9. *Let $G$ be a $\Pi$-embedded 2-connected planar graph. Suppose that $C$ is a $\Pi$-nonbounding cycle of $G$. If no 1-curve passes through a vertex of $C$, then $C$ contains at least two vertices at which some 2-curve crosses $C$.*

*Proof.* Since $C$ is $\Pi'$-bounding, the edges of $E(G)\backslash E(C)$ can be classified as *interior* or *exterior edges*, depending on whether they are in the $\Pi'$-interior or in the $\Pi'$-exterior of $C$, respectively.

Suppose that there is a vertex $u_0 \in V(C)$ and that no 2-curve crosses $C$ at a vertex distinct from $u_0$. Consider $C\backslash\{u_0\}$ as an open path $P$ in $G$. Lemmas 4.7 and 4.8 imply that the $\Pi$-splitting and the $\Pi'$-splitting at the internal vertices of $P$ are the same. Suppose that $C$ is $\Pi$-twosided. Since there is no 1-curve through $u_0$, $u_0$ cannot be the only vertex of $C$ that has an incident edge $e \notin E(C)$ which is on the left side of $C$. Similarly on the right. Since $C$ is $\Pi$-nonbounding, $G_l(\Pi, C) \cap G_r(\Pi, C) \neq C$. This implies

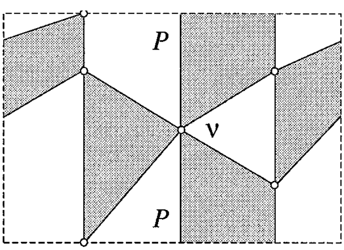


FIG. 5. No 1-curve or 2-curve crosses $P$.

that there is a $\Pi$-facial walk $W$ that uses interior as well as exterior edges. The same conclusion holds also when $C$ is $\Pi$-onesided.

The change from an interior to an exterior edge (or vice versa) in the facial walk $W$ can only be achieved at $u_0$. Consequently, $W$ meets $u_0$ at least twice. Since $G$ is 2-connected, the curve in the face of $W$ connecting the two appearances of $u_0$ on $W$ is $\Pi$-noncontractible, hence a 1-curve. A contradiction. ∎

It is worth mentioning that for any $k$ there are examples of embeddings $\Pi$ of planar graphs with $\Pi$-noncontractible (but necessarily $\Pi$-bounding) cycles that are neither crossed nor touched by an $s$-curve for $s \leqslant k$.

A simple corollary of Theorem 4.9 is:

COROLLARY 4.10. *Let $C$ be a $\Pi$-nonbounding cycle of $G$. Then there are patch faces $\Phi_1, \Phi_2$ that intersect at a vertex $v$ of $C$ such that $\Phi_1$ is on the left side of $C$ at $v$, and $\Phi_2$ is on the right side of $C$ at $v$.*

*Proof.* Let us first remark that the left and the right side of $C$ are defined locally at each vertex of $C$ also when $C$ is $\Pi$-onesided. Suppose now that at each vertex of $C$, patch faces are only on one side of $C$. Then there is a cycle $\tilde{C}$ homotopic to $C$ that contains only vertices in the interiors of patches. In particular, no 1/2-curve intersects $C$. Theorem 4.9 now gives a contradiction. ∎

Let $\Gamma = \{C_1, ..., C_k\}$ be a set of cycles of $G$. If there is a set $D$ of $\Pi$-facial walks such that any edge $e \in E(G)$ appears exactly once in facial walks from $D$ exactly when $e$ is contained in an odd number of cycles from $\Gamma$, then $\Gamma$ is *$\Pi$-bounding*. We also say that $\Gamma$ bounds $D$. If no nonempty subset of $\Gamma$ is $\Pi$-bounding, then $\Gamma$ is *homologically independent*. If $\{C_1, C_2\}$ is $\Pi$-bounding, then $C_1$ and $C_2$ are *$\Pi$-homologic*. The same definitions apply for sets of noncrossing 1/2-curves.

Theorems 4.4 and 4.9 show that there are many 1/2-curves. Based on these results we formulate the following conjecture that is, in a sense, a claim dual to Theorem 4.9.

*Conjecture* 4.11. Suppose that $G$ is a 2-connected planar graph that is $\Pi$-embedded in an orientable surface of genus $g$ with face-width 2. Then there is a set $\{\gamma_1, ..., \gamma_g\}$ of pairwise noncrossing homologically independent 2-curves.

A corresponding conjecture for nonorientable surfaces $S$ claims that there is a set $\Gamma = \{\gamma_1, ..., \gamma_k\}$ of homologically independent 2-curves such that twice the number of twosided 2-curves plus the number of onesided 2-curves in $\Gamma$ equals the nonorientable genus of $S$, i.e., $2 - \chi(S)$.

It may be true that even the following stronger property holds: If $\Gamma$ is any maximal set of pairwise noncrossing 2-curves (i.e., any 2-curve that is equivalent to no curve in $\Gamma$ crosses some curve from $\Gamma$), then $\Gamma$ contains a set of 2-curves satisfying Conjecture 4.11.

## 5. CHOICE OF $\Pi'$

Let $G$ be a $\Pi$-embedded planar graph, and let $\Pi'$ be an embedding of $G$ in the 2-sphere. The embedding $\Pi'$ *maximally coincides* with $\Pi$ if for every embedding $\Pi''$ of $G$ in the 2-sphere, $\mathrm{CS}_0(\Pi, \Pi') \subseteq \mathrm{CS}_0(\Pi, \Pi'')$ implies that $\mathrm{CS}_0(\Pi, \Pi') = \mathrm{CS}_0(\Pi, \Pi'')$.

Although the results of the previous sections hold for an arbitrary embedding $\Pi'$, an additional assumption that $\Pi'$ maximally coincides with $\Pi$ makes some of the results "stronger" since if $\mathrm{CS}_0(\Pi, \Pi'') \subset \mathrm{CS}(\Pi, \Pi'')$, then every patch angle with respect to $\Pi''$ is also a patch angle with respect to $\Pi'$, but there is a patch angle with respect to $\Pi''$ that disappears if we take $\Pi'$ instead of $\Pi''$.

Given a planar embedding $\Pi''$ of $G$, it is easy to find an embedding $\Pi'$ that maximally coincides with $\Pi$ and such that $\mathrm{CS}_0(\Pi, \Pi'') \subseteq \mathrm{CS}_0(\Pi, \Pi')$. The procedure is as follows. Take an arbitrary patch angle of $\Pi''$, subdivide the edges of this angle and connect the inserted vertices by a new edge. If the resulting supergraph of the patch extension $\tilde{G}$ of $G$ is planar, it determines a planar embedding $\Pi_1$ of $G$ such that $\mathrm{CS}_0(\Pi, \Pi'') \subset \mathrm{CS}_0(\Pi, \Pi_1)$. By repeating the same with other angles, we eventually stop with an embedding that maximally coincides with $\Pi$. Note that this gives a good characterization of embeddings $\Pi'$ that maximally coincide with $\Pi$.

The choice of $\Pi'$ that maximally coincides with $\Pi$ has another advantage: our results easily carry over to graphs that are not 2-connected by applying the following proposition.

PROPOSITION 5.1. *Let $G$ be a connected $\Pi$-embedded planar graph and let $\Pi'$ be an embedding of $G$ in the* 2*-sphere that maximally coincides with $\Pi$. Then the patch extension $\tilde{G}$ of $G$ with respect to $\Pi'$ is* 2*-connected.*

*Proof.* Suppose that $v$ is a cutvertex of $\tilde{G}$. Then $v$ is also a cutvertex of $G$ (viewed as a subgraph of $\tilde{G}$). There are $\Pi$-consecutive edges $e_1, e_2$ belonging to distinct $\{v\}$-bridges $B_1, B_2$ in $\tilde{G}$. It is easy to see that $\Pi'$ can be changed so that $e_1$ and $e_2$ become $\Pi'$-consecutive. The embedding in the plane cannot change the triangular and quadrangular faces of $\tilde{G}$ inside the patches. Hence the new embedding contradicts the assumption that $\Pi'$ maximally coincides with $\Pi$. ∎

## 6. A CHESSBOARD PATTERN

Let $\gamma_1, \gamma_2$ be noncrossing nonbounding twosided 2-curves. Suppose that $\gamma_1$ and $\gamma_2$ are homologic and let $D$ be the subset of the surface that they bound. For $i=1,2$ we denote by $x_i$ and $y_i$ the vertices of $\gamma_i \cap G$, and by $f_i, g_i$ the $\Pi$-faces used by $\gamma_i$. We have the following corollary of Theorem 4.9.

PROPOSITION 6.1. *Suppose that* $\mathrm{fw}(\Pi)=2$. *Let* $\gamma_1$ *and* $\gamma_2$ *be homologic* 2-*curves as introduced above. If there is no* 2-*curve crossing both* $\gamma_1$ *and* $\gamma_2$ *and if every* 2-*curve in D is equivalent either to* $\gamma_1$ *or to* $\gamma_2$, *then* $\{f_1, g_1\} \cap \{f_2, g_2\} \neq \emptyset$, *say* $f_1 = f_2$, *and* $\partial f_1 \cap D$ *consists of two segments joining* $x_1, x_2$ (*say*) *and* $y_1, y_2$, *respectively.*

*Proof.* Suppose that $\varphi = \partial f_1 \cap D$ is connected, and so is $\psi = \partial g_1 \cap D$. The closed walk $W$ composed of $\varphi$ and $\psi$ is homotopic to $\gamma_1$. Thus $W$ contains a $\Pi$-nonbounding cycle $C$. By Theorem 4.9, $C$ is crossed by a 2-curve, say $\gamma$. Since $\gamma$ crosses $C$, it is equivalent neither to $\gamma_1$ nor to $\gamma_2$. Thus $\gamma$ is not entirely in $D$, and hence it crosses exactly one of $\gamma_1, \gamma_2$, say $\gamma_1$. Since $C$ is in $D$, $\gamma$ crosses $\gamma_1$ (at least) twice. Let $x, y$ and $f, g$ be the vertices and $\Pi$-faces, respectively, used by $\gamma$. Then one of the following two cases occurs.

*Case* 1: $x = x_1$ and $y = y_1$. Let $\gamma'$ be the curve consisting of the segment of $\gamma$ in $D$ and of the segment of $\gamma_1$ in $f_1$, and let $\gamma''$ be the curve consisting of the segment of $\gamma$ in $D$ and the segment of $\gamma_1$ in $g_1$. Then either $\gamma'$ or $\gamma''$ is a 2-curve in $D$ that crosses $C$. As we have proved above, this is a contradiction.

*Case* 2: $x, y \neq x_1$ (say). Then $\gamma$ crosses $\gamma_1$ in the interior of $f_1$, say, so we may assume that $f = f_1$. If $x, y$ are both in $D$, the part of $\gamma$ that is in $f$ can be redrawn inside $f$ so that $\gamma \subseteq D$, a contradiction. (If $\gamma$ would become contractible after this change, $f_1$ would contain a 1-curve, a contradiction with $\mathrm{fw}(\Pi)=2$.) So $y \notin D$, say. Since $\gamma$ crosses $C$, we have $x \in D$ and $x \neq x_1, y_1$. It follows that $g = g_1$. We now conclude as in Case 1. ∎

Let $K$ be a subgraph of $G$. A *K-bridge* in $G$ is a subgraph of $G$ which is either an edge of $E(G) \backslash E(K)$ with its endpoints in $K$, or it is a connected component of $G - V(K)$ together with all edges (and their endpoints) between this component and $K$. We say that a $K$-bridge $B$ is *attached* to a vertex $x$ of $K$ if $x \in V(B \cap K)$. For $X \subseteq V(G)$, an *X-bridge* is a $K$-bridge where $K$ is the edgeless graph with vertex set equal to $X$.

Suppose that $C_1$, $C_2$ are disjoint $\Pi$-homologic cycles that bound a subset $D$ of the faces of $\Pi$. If $D$ is a cylinder (i.e., its Euler characteristic is 0), we say that $C_1$ and $C_2$ are *$\Pi$-homotopic*. Similar definition applies if $C_1$ and $C_2$

touch. In that case they bound a *degenerate cylinder*. Instead of cycles we can also use simple closed curves $\gamma_1, \gamma_2$ on the surface of $\Pi$. Then we say that $D$ is a (*degenerate*) *cylinder* between $\gamma_1$ and $\gamma_2$ and that $\gamma_1, \gamma_2$ are *homotopic*.

If $\gamma_1, ..., \gamma_k$ are disjoint homotopic simple closed curves in $S$, then any two of them bound a cylinder, and they can be enumerated such that for $i = 1, ..., k-1$, the cylinder between $\gamma_1$ and $\gamma_{i+1}$ contains none of the other curves $\gamma_j$, $j \neq i, i+1$. Such enumeration is *natural*. The same definition can be used when $\gamma_1, ..., \gamma_k$ intersect but none of their intersection is a crossing.

Let $D$ be a cylinder between two homotopic nonbounding curves in the surface of $\Pi$. Suppose that $\gamma'$ and $\gamma''$ are noncrossing 1/2-curves in the interior of $D$. Then $\gamma', \gamma''$ are homotopic and they bound a cylinder $D' \subseteq D$ (possibly degenerate if they touch). Let $\gamma_1, ..., \gamma_k$ be a maximal family of pairwise noncrossing 1/2-curves in $D'$ such that $\gamma_1 = \gamma'$ and $\gamma_k = \gamma''$. Then we have:

THEOREM 6.2. *Suppose that the planar embedding $\Pi'$ of the 2-connected graph $G$ maximally coincides with $\Pi$. Let $\gamma_1, ..., \gamma_k$ be as above and suppose that they are naturally enumerated. Let $D_i$ be the (degenerate) cylinder between $\gamma_i$ and $\gamma_{i+1}$, $i = 1, ..., k-1$. If every 1/2-curve that intersects $V(G) \cap D'$ is contained entirely in $D$, then there are patch faces $F_1, ..., F_{k-1}$ such that for $i = 1, ..., k-1$, $F_i$ contains a segment of $\gamma_i$ and a segment of $\gamma_{i+1}$ (where in case of 1-curves the segment could be just the vertex of $G$ crossed by the curve) and such that $\partial F_i \cap D_i$ consists of two $\Pi$-facial segments joining vertices of $\gamma_i \cap G$ with vertices of $\gamma_{i+1} \cap G$.*

*Proof.* Let $\Gamma = \{\gamma_1, ..., \gamma_k\}$. Consider an arbitrary consecutive pair of curves in $\Gamma$, say $\gamma_1, \gamma_2$. We shall use the notation $f_i, g_i, x_i, y_i$ $(i = 1, 2)$ introduced for Lemma 6.1. In case when $\gamma_1$ (or $\gamma_2$) is a 1-curve, we have $x_1 = y_1$ and in that case we can take as $g_1$ (say) any face containing $x_1$.

First, we claim that the only patch vertices in $D_1$ are $x_1, y_1, x_2, y_2$. If not, let $x$ be another one. By Theorem 4.4, there is a 1/2-curve $\gamma$ through $x$. By maximality of $\Gamma$, $\gamma$ is not entirely contained in $D_1$ and hence it is a 2-curve. Since $\gamma$ cannot escape out of $D$, it is homotopic to $\gamma_1$ and it must intersect $\gamma_1$ (or $\gamma_2$) twice. Let $x'$ be the other patch vertex used by $\gamma$. If $x' \notin \{x_1, y_1\}$, then a segment of $\gamma_1$ between the points of $\gamma_1 \cap \gamma$ can be replaced by a segment of $\gamma$ in $D_1$, yielding a 2-curve in $D_1$ through $x$. A contradiction. If $x' = x_1$ (say), let $f_1$ and $g$ be the faces used by $\gamma$. As above in the case when $x' \notin \{x_1, y_1\}$ we can try to change a segment of $\gamma$ by a segment of $\gamma_1$ and get a contradiction. If this is not possible, then one of the possible replacements yields a 3-curve (and so $y_1 \neq x_1$), the other one a contractible curve $\gamma''$ through $x$ and $x_1$. Because of 2-reductions that we have performed prior to defining the patches, the interior of $\gamma''$ contains just an edge $a = xx_1$ of $G$. By the above we may assume that no 1-curve intersects $G$ in $x$. Therefore $g \neq f_1$. Let $\{a, b\}$ be the patch angle in $g$ and $\{a, d''\}$ the patch angle in $f_1$ used by $\gamma$ at $x$. Since these are patch angles, $a$ and $b$ are not $\Pi'$-consecutive and neither are $a$ and $d'$. This implies that there is a patch angle $\{c, d\}$ such that $\{a, b\}$ and $\{c, d\}$ $\Pi'$-interlace and such that $\{a, b\} \cap \{c, d\} = \emptyset$. By Lemma 4.2, there is a 2-curve $\gamma_0$ through the patch angles $\{a, b\}$ and $\{c, d\}$. By using this 2-curve instead of $\gamma$, we obtain a contradiction since both patch faces of $\gamma_0$ are distinct from $f_1, g_1, f_2, g_2$.

Suppose now that there is no face $F_1$ as claimed. Then $C = (\partial f_1 \cap D_1) \cup (\partial g_1 \cap D_1)$ (or $C = \partial f_1 \cap D_1$ if $x_1 = y_1$) is a $\Pi$-noncontractible cycle in $D_1$ that is composed of one or two patch edges between $x_1$ and $y_1$. Similarly we have a $\Pi$-noncontractible cycle $C' = (\partial f_2 \cap D_1) \cup (\partial g_2 \cap D_1)$ (or $C' = \partial f_2 \cap D_1$) on the other side.

Let us first assume that $C \cap C' = \emptyset$. By Corollary 4.10, there exists a patch face $\Phi$ between $C$ and $C'$. If $x_1 \neq y_1$, $\Phi$ cannot contain $x_1$ and $y_1$ since otherwise a segment of $\gamma_1$ could be replaced by an arc in $\Phi$, contradicting maximality of $\Gamma$. Similarly for $x_2, y_2$. Since $G$ is 2-connected, $\Phi$ contains at least two patch angles. By maximality of $\Gamma$, these angles are at distinct vertices. Thus we may assume that $x_1, x_2 \in \Phi$, and these are the only patch vertices of $\Phi$. Let $\alpha_i$ be the patch angle of $\Phi$ at $x_i$, $i = 1, 2$. If there is a 1/2-curve $\gamma$ through $\alpha_1$, it must exit $\Phi$ through $\alpha_2$. Since $\gamma$ cannot escape $D$, it is entirely in $D_1$, a contradiction. Consequently, Theorem 4.4 implies that for $i = 1, 2$, there is a 1-curve $\gamma'_i$ through $x_i$. The cylinder $D'_1$ between $\gamma'_1, \gamma'_2$ contains a 3-connected block of $G$. There is a planar embedding of $G$ that coincides with $\Pi$ in $D'_1$, and coincides with $\Pi'$ elsewhere. Since $D_1 \subseteq D'_1$, this contradicts the assumption that $\Pi'$ maximally coincides with $\Pi$ if $y_1 \neq x_1$ or $y_2 \neq x_2$. Otherwise, $F_1 = \Phi$ is the required face.

The other case is when $C$ and $C'$ intersect. By symmetry we may assume that $x_2 \in V(C \cap C')$. Then $f_1$ (say) contains $x_2$. Suppose first that $\{x_1, y_1\} \cap \{x_2, y_2\} = \emptyset$. If $x_2 = y_2$, we can take $f_1$ for $F_1$. Otherwise, denote by $\alpha$ the patch angle of $f_1$ at $x_2$. A 1/2-curve $\gamma$ through $\alpha$ crosses $C$ at $x_2$ and thus it crosses $C$ in another patch vertex. Let $f_1$ and $f$ be the patch faces used by $\gamma$. If $f = f_2$ or $f = g_2$ or $f$ is between $C$ and $C'$, then $\gamma$ can be taken to be entirely in $D_1$, a contradiction with maximality of $\Gamma$. Otherwise, $\gamma$ intersects $C$ at $y_2$ and its re-routing through $f_2$ or $g_2$ gives rise to the previous case. The conclusion is that no 1/2-curve uses $\alpha$. By Theorem 4.4, there is a 1-curve $\gamma'_2$ through $x_2$. Note that $\gamma'_2$ is not in $D_1$ and is not equivalent to $\gamma_2$. If $y_2 \in C \cap C'$, we repeat the same procedure at $y_2$ and obtain a contradiction. If $x_1 \in C \cap C'$, we see in the same way that there is a 1-curve $\gamma'_1$ through $x_1$, and we conclude as in the previous case by reaching a contradiction with the assumption that $\Pi'$ maximally coincides with $\Pi$. Hence $C \cap C' = \{x_2\}$. The cycle $C'$ consists of two patch edges that are $\Pi'$-facial segments. Since $G$ is 2-connected, the only $C'$-bridge embedded in the $\Pi'$-interior of $C'$ is the subgraph $H$ of $G$ bounded by $\gamma$ and $\gamma_2$. It is clear that $\Pi'$ can be chosen so that the induced embedding of $H \cup C'$ coincides with its embedding induced by $\Pi$. This yields a contradiction with the fact that $y_2$ is a patch angle.

If $\{x_1, y_1\} \cap \{x_2, y_2\} \neq \emptyset$, we may assume that $x_1 = x_2$. If $x_1 = y_1$ (or $x_2 = y_2$), then $F_1 = f_2$ ($F_1 = f_1$, respectively) is the required face. Hence $x_1 \neq y_1$ and $x_2 \neq y_2$. If $y_1 \in V(C')$, then we get a new 2-curve in $D_1$ through the face $f_2$ (say) containing $x_1$ and $y_1$ and the corresponding face $f_1$ or $g_1$ on the other side. Thus we may assume that $y_1 \notin V(C')$ and $y_2 \notin V(C)$. We shall distinguish two subcases.

(a) $D_1$ contains a patch face $\Phi$ distinct from $f_1, g_1, f_2, g_2$. By maximality of $\Gamma$, $\Phi$ contains just one vertex from $\{x_1, y_1\}$ and one vertex from $\{x_2, y_2\}$. In particular, $\Phi$ contains $y_1$ and $y_2$ and does not contain $x_1 = x_2$. We see as before that no 1/2-curve passes through the patch angles of $\Phi$. Hence, there is a 1-curve $\delta_i$ through $y_i$, $i = 1, 2$. Let $H$ be the subgraph of $G$ between $\delta_1$ and $\delta_2$. Then $H$ is a $\{y_1, y_2\}$-bridge in $G$. Since $\Pi'$ maximally coincides with $\Pi$, it is easy to see that $\Pi'$ coincides with $\Pi$ everywhere on $H$ except at $y_1$ and $y_2$. This contradicts the fact that $x_1$ is a patch vertex.

(b) There is no such face $\Phi$ as in (a). Let $H$ be the patch of $G$ in $D_1$. The restriction of $\Pi'$ to $H$ has two faces sharing $x_1$ but without any other vertices in common, and these two faces contain $y_1$ and $y_2$, respectively. Therefore, every $H$-bridge in $G$ is either attached to $x_1$ and $y_1$ or to $x_1$ and $y_2$ (and no other vertices). Since $C'$ is a $\Pi$-nonbounding cycle, one of these components has an edge on the left side of $C$ and an edge on the right side of $C'$ (where the right side of $C$ and the left side of $C'$ point in $D_1$). This implies that there is a $\Pi$-facial walk $\Psi$ that starts at $x_1$, say, on the left of $C$, leaves $D$, and returns to $C'$ at $x_1$ or $y_2$ on the right side. Then $\Psi$ and $f_2$ contain a 1/2-curve that leaves $D$ and crosses $C$, a contradiction.

This concludes our case analysis and establishes existence of faces $F_1$, ..., $F_{k-1}$. ∎

Suppose that $G$ contains disjoint homotopic $\Pi$-nonbounding cycles $C_1, C_2, C_3$ such that the cylinder $D$ bounded by $C_1$ and $C_2$ contains $C_3$. The assumptions of Theorem 6.2 are clearly satisfied in $D$ for any two non-crossing 1/2-curves $\gamma'$ and $\gamma''$ in the interior of $D$. Theorem 6.2 shows that a general patch structure of $G$ in $D$ follows a variety of patterns, whose most general examples are shown in Fig. 6.

Mohar, Robertson, and Vitray [4] described the general patch structure of planar graphs in the projective plane. They either look as a "double wheel" or as the octahedron structure shown in Figure 3. As a corollary of

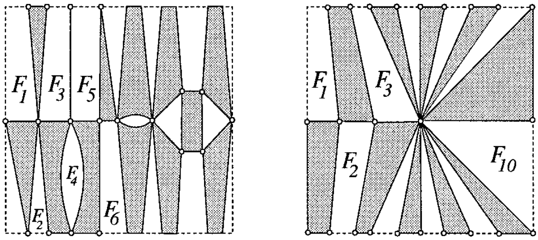


FIG. 6. A general patch structure.

Theorem 6.2 we will derive the corresponding result for the torus and the Klein bottle in case when there are three disjoint $\Pi$-nonbounding cycles and the face-width is two.

Suppose that $G$ is $\Pi$-embedded in the torus with face-width 2 and that $\gamma_1, ..., \gamma_k$ are noncrossing 2-curves. For $i=1, ..., k$ denote by $x_i, y_i$ and $f_i, g_i$ the vertices and patch faces (respectively) used by $\gamma_i$. Curves $\gamma_i$ and $\gamma_{i+1}$ (index $i+1$ taken modulo $k$) are homotopic and hence they bound a (degenerate) cylinder $D_i$. We may assume that the curves are naturally enumerated such that the union of $D_1, ..., D_k$ and $\gamma_1, ..., \gamma_k$ is the entire surface. If for each $i$, a segment of $\gamma_i$ can be joined to a segment of $\gamma_{i+1}$ by a curve in $D_i$ that is disjoint from $G$, then we say that the embedding of $G$ has the *chessboard structure*. This structure is *nondegenerate* if for each $i$, $\{x_i, y_i\} \cap \{x_{i+1}, y_{i+1}\} = \emptyset$. Such a structure with $k=6$ is represented in Fig. 7. Examples of degenerate chessboard structures are shown in Fig. 6. The same definition applies for the Klein bottle in which case we also require that the 2-curves $\gamma_i$ $(1 \leqslant i \leqslant k)$ are nonbounding.

The following is a corollary of Theorem 6.2.

COROLLARY 6.3. *Let $G$ be a 2-connected planar graph embedded in the torus or the Klein bottle with face-width 2. If $G$ contains three disjoint $\Pi$-nonbounding cycles, then the embedding has the chessboard structure.*

*Proof.* Disjoint $\Pi$-nonbounding cycles $C_0, C_1, C_2$ on the torus or the Klein bottle have the property that the removal of each of them leaves a

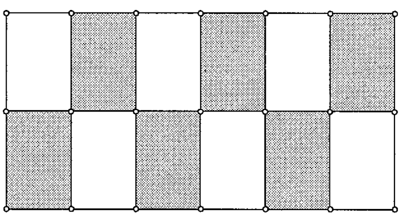

FIG. 7. A chessboard patch structure on the torus.

cylinder. By Theorem 4.9 there are 2-curves $\gamma_i$ crossing $C_i$ ($i=0, 1, 2$). Now we apply Theorem 6.2 to get the chessboard structure between $\gamma_i$ and $\gamma_{i+1}$ by using the cylinder $D$ obtained by cutting the surface along $C_{i+2}$ (indices modulo 3). ∎

Theorem 6.3 stimulated the following conjecture.

*Conjecture* 6.4. Suppose that $G$ is a 2-connected planar graph that is $\Pi$-embedded with face-width 2, and that $C_1$, $C_2$, $C_3$ are disjoint homotopic $\Pi$-nonbounding cycles. Let $k$ be the minimal number such that there exists a $k$-curve $\gamma$ that intersects each of $C_1$, $C_2$, $C_3$ exactly once. Then $k$ is equal to the maximal number $t$ of pairwise disjoint cycles $C'_1, ..., C'_t$ homotopic to $C_1$.

Clearly, $k \geqslant t$. By Corollary 6.3 we know that Conjecture 6.4 holds for the torus and the Klein bottle. Embeddings $\Pi$ in the torus that have face-width 2 and no two disjoint $\Pi$-noncontractible cycles are classified in [3]. Examples show that the requirement of Conjecture 6.4 about existence of three disjoint cycles $C_1$, $C_2$, $C_3$ cannot be entirely omitted.

## 7. GENERALIZED WHITNEY'S THEOREM

The patch degree of a patch vertex can be arbitrarily large. Examples on the projective plane or the torus are easy to construct. On the other hand, we will show in this section that the patches or the patch faces cannot be too complicated if we restrict our attention to a fixed surface $S$.

Suppose that $G$ is a 2-connected $\Pi$-embedded graph and $C$ is a $\Pi$-contractible cycle of $G$ such that only two vertices of $C$, say $v$ and $w$, have incident edges that are embedded in the $\Pi$-exterior of $C$. Denote by $D$ the $\Pi$-interior of $C$. Then we define a *Whitney 2-switching* of $\Pi$ (with respect to $C$) as a reembedding of $G$ such that the local rotation of each vertex in $D\backslash\{v, w\}$ is reversed, and local rotations at $v$ and $w$ are changed as follows. If $\pi_v = (e_1 e_2 \cdots e_d)$ where $e_1, ..., e_j$ are edges in $D$, then we change $\pi_v$ to $(e_j e_{j-1} \cdots e_1 e_{j+1} \cdots e_d)$, and similarly at $w$. This operation that preserves the underlying surface of the embedding generates an equivalence relation among embeddings of $G$. It was proved by Whitney [7] that any two embeddings of $G$ in the 2-sphere are equivalent.

Our main goal in this section is to prove a Whitney-type result for embeddings of planar graphs in an arbitrary fixed surface. For that purpose we also define a *k-switching operation* where $k > 1$ is an integer. Suppose that we have a patch $P$ whose interior in $S$ is homeomorphic to an open disk. Let $v_1, ..., v_k$ be the consecutive patch vertices that appear on the boundary of $P$, including possible multiple occurrences of the same vertex.

If there is a patch facial walk $W$ containing $v_1, ..., v_k$ (in this or in the reverse order), we can reembed $P$ in that face. Note that we may need to change the orientation in $P$ and that sometimes there is more than one possibility how to do the reembedding. However, we require that the faces of the patch extension $\tilde{G}$ in the patch $P$ are unchanged. More generally, if there is a closed curve $\gamma$ in $S$ without self-crossings (but possibly touching itself) that bounds an open disk $D$ in $S$ and intersects $G$ only in vertices $v_1, ..., v_k$, the same reembedding of $G \cap D$ into the face $W$ can be performed. Such a reembedding is called a *k-switching*. The Whitney 2-switching is a special case of this operation. If a $k$-switching does not change the underlying surface of the embedding, then it is invertible.

Two embeddings of $G$ in the same surface $S$ are *Whitney equivalent* if there is a sequence of 2-switchings, 3-switchings and 4-switchings transforming one embedding into the other. If a $k$-switching changes the underlying surface of the embedding, the Euler characteristic strictly decreases. Therefore all intermediate embeddings are also embeddings in $S$, and thus Whitney equivalence is an equivalence relation among embeddings of $G$ in $S$.

Embeddings $\Pi_1$ and $\Pi_2$ of planar graphs $G_1$ and $G_2$, respectively, in the same surface $S$ are *patch equivalent* if there is a homeomorphism of $S$ onto itself that induces a bijection on patch vertices, patch edges and patch faces of $\Pi_1$ and $\Pi_2$, respectively. We also say that $\Pi_1$ and $\Pi_2$ have the same *patch structure*. Our next result shows that up to Whitney equivalence, there are not too many patch structures of embeddings of planar graphs.

THEOREM 7.1. *For each surface S there is a finite number of patch structures such that any embedding of a planar 2-connected graph G in S is Whitney equivalent to an embedding having one of these structures as its patch structure.*

In the proof of Theorem 7.1 we shall use the following lemma.

LEMMA 7.2 [2]. *Let S be a closed surface of genus $g \geqslant 1$, and $p \geqslant 1$ an integer. Suppose that $\Gamma$ is a set of noncontractible simple closed curves in S that are either pairwise disjoint or they all pass through a point $x \in S$ and are disjoint elsewhere. If $|\Gamma| \geqslant 3pg$, then $\Gamma$ contains a subset of $p+1$ homotopic curves.*

*Proof.* By [2, Proposition 3.7] every set of $3g$ disjoint curves from $\Gamma$ contains a pair of homotopic curves. By [2, Proposition 3.6], every set of $3g$ curves passing through $x$ (and disjoint elsewhere) also contains a pair of homotopic curves. Now the lemma is immediate. ∎

*Proof of Theorem* 7.1. It suffices to show that every embedding $\Pi$ of a 2-connected planar graph $G$ in $S$ is Whitney equivalent to an embedding of $G$ in $S$ that has only a bounded number of patch edges.

Suppose that an embedding $\Pi$ of $G$ is Whitney equivalent to no embedding of $G$ in $S$ with fewer patch edges. We shall assume that the planar embedding $\Pi'$ of $G$ that determines the patches maximally coincides with $\Pi$. For every patch face $\Phi$ and a patch vertex $x$ that occurs $t \geqslant 2$ times on $\Phi$, we connect consecutive appearances of $x$ on $\Phi$ by $t-1$ 1-curves. Let $\Gamma_1$ be the set of the obtained 1-curves. For each simple and not bad patch angle take a 2-curve through it, and let $\Gamma_2$ be the set of the obtained 2-curves. We may assume that each 2-curve from $\Gamma_2$ passes through an angle that is not used by other curves from $\Gamma_1 \cup \Gamma_2$. By Corollary 4.6, it suffices to see that $|\Gamma_1| + |\Gamma_2|$ is bounded. We may assume that an arbitrary pair of curves from $\Gamma_1 \cup \Gamma_2$ intersects in the interior of a patch face $\Phi$ if and only if their angles in $\Phi$ interlace.

Suppose that $|\Gamma_1| \geqslant 216g^2$ where $g$ is the genus of $S$. By our selection of the 1-curves, any two curves from $\Gamma_1$ intersect at most once. If $\gamma_1, \gamma_2 \in \Gamma_1$ are homotopic and they intersect in a patch vertex $x$, then they bound an open disc and they are either equivalent, or $x$ is a cutvertex of $G$. None of these is possible. Hence, by Lemma 7.2, less than $3g$ curves from $\Gamma_1$ intersect in $x$. Similarly, if three homotopic curves from $\Gamma_1$ intersect in a point $z$ inside a patch face, two of them bound an open disc $D$ that contains the third one. It follows that the third 1-curve passes through a vertex of $G$ that is not a patch vertex since the embedding $\Pi'$ can be changed so that it matches the embedding of $G$ in $D$. Now Lemma 7.2 implies that $\Gamma_1$ contains a set of more than $|\Gamma_1|/(9g) \geqslant 24g$ pairwise disjoint 1-curves. The same lemma implies that this subset contains nine disjoint homotopic 1-curves $\gamma_1, ..., \gamma_9$. Assume that they are naturally enumerated. Let $D_1, ..., D_8$ be the corresponding cylinders between the consecutive curves. For $i = 1, ..., 7$, $D_i \cup D_{i+1}$ contains a $\Pi$-noncontractible cycle $C_i$. (Otherwise the embedding $\Pi'$ can be changed so that it matches the embedding of $G$ in $D_i \cup D_{i+1}$ which would contradict the fact that $\gamma_{i+1}$ crosses $G$ in a patch vertex.) Let $D$ be the cylinder between $C_1$ and $C_7$, and let $\gamma' = \gamma_4$, $\gamma'' = \gamma_6$. Theorem 6.2 implies that we can change the embedding of $G$ between $\gamma'$ and $\gamma''$ by using a sequence of 2/3/4-switchings so that the patch vertex of $\gamma_5$ disappears. This contradicts the minimality of the patch structure of our embedding. Hence, $|\Gamma_1| < 216g^2$.

Suppose now that there are patch vertices $x, y$ such that $p$ 2-curves from $\Gamma_2$ intersect $x$ and $y$. These curves give rise to $p$ simple arcs $\alpha_0, ..., \alpha_{p-1}$ from $x$ to $y$ such that every arc uses a patch angle that is not used by other arcs and is not used by any of the 1-curves from $\Gamma_1$. This implies that no two of these arcs are in the same patch face and hence they are internally disjoint. Let $\delta_i$ be the 2-curve composed of $\alpha_0$ and $\alpha_i$, $1 \leqslant i < p$. By contracting

$\alpha_0$ to a point, we get $p-1$ simple closed curves intersecting in a single point. If $p>6g$, then by Lemma 7.2, three of them, say $\delta_1, \delta_2, \delta_3$, are homotopic. This means that $\alpha_1, \alpha_2$ and $\alpha_2, \alpha_3$, respectively, bound two discs containing distinct $\{x, y\}$-bridges in $G$. Moreover, $\alpha_1$ and $\alpha_3$ (say) bound a disc that contains both of these components. Now, $\Pi'$ can be changed so that these components merge into a single patch, a contradiction. Therefore $p \leqslant 6g$. Let $\Gamma'_2$ be a maximal subset of $\Gamma_2$ such that no two curves from $\Gamma'_2$ use the same pair of patch vertices. By the above, it suffices to see that $|\Gamma'_2|$ is bounded.

Suppose that there are patch faces $\Phi$, $\Psi$ such that $p$ of the curves in $\Gamma'_2$ pass through $\Phi$ and $\Psi$. Let $\phi$ and $\psi$ be points in the interior of $\Phi$ and $\Psi$, respectively. Since no two of the curves use the same pair of patch vertices, these curves determine at least $\lceil \sqrt{2p} \rceil$ internally disjoint simple arcs from $\phi$ to $\psi$ consisting of segments of the curves. As above we see that no three of these arcs are homotopic, and hence $p \leqslant 18g^2$ by Lemma 7.2.

Suppose now that $\gamma_1, ..., \gamma_p$ are 2-curves in $\Gamma'_2$ that all pass through a patch vertex $x$ and a patch face $\Phi$ but any two of them use distinct second patch vertex and the patch face. As above we see that the vertex $x$ appears on $\Phi$ at most $3g$ times. Hence $\gamma_1, ..., \gamma_p$ contain a subset of at least $p/(3g)$ curves that intersect only in $x$. Let $\Gamma''_2$ be a maximal subset of $\Gamma'_2$ such that no two 2-curves from $\Gamma''_2$ intersect more than once. Let

$$r = \lfloor |\Gamma''_2|^{1/2}/(6g) \rfloor.$$

If a 2-curve $\gamma$ from $\Gamma''_2$ intersects $12rg$ curves of $\Gamma''_2$, then by Lemma 7.2 there is a subset $\Gamma = \{\gamma_1, ..., \gamma_r\}$ of $r$ pairwise homotopic nonequivalent 2-curves that all intersect $\gamma$ in the same point. If there is no such curve $\gamma$, then $\Gamma''_2$ contains a subset of $|\Gamma''_2|^{1/2}/2$ disjoint curves. By Lemma 7.2, $\Gamma''_2$ has a subset $\Gamma = \{\gamma_1, ..., \gamma_r\}$ of $r$ pairwise homotopic nonequivalent 2-curves that are pairwise disjoint. By the above it suffices to see that $r$ cannot be arbitrarily large.

Suppose that curves in $\Gamma$ are pairwise disjoint. We claim that there is a sequence of switchings yielding an embedding of $G$ in $S$ with fewer patch angles if $r$ is large enough. Suppose that $\gamma_1, ..., \gamma_r$ are naturally enumerated. For $i = 1, ..., r-1$, let $x_i, y_i$ be the patch vertices used by $\gamma_i$ and let $D_i$ be the cylinder bounded by $\gamma_i$ and $\gamma_{i+1}$. Denote by $D = D_1 \cup \cdots \cup D_{r-1}$. If 18 of the cylinders $D_i$ contain 1-curves, nine of these 1-curves are pairwise disjoint and homotopic. We get a contradiction as above. Otherwise, $q = \lfloor r/18 \rfloor$ consecutive cylinders, say $D_1, ..., D_q$, contain no 1-curves. Menger's theorem implies that each $D_i$, $1 \leqslant i \leqslant q$, contains disjoint paths $P_i$, $Q_i$ joining $x_i$ with $x_{i+1}$ (say) and $y_i$ with $y_{i+1}$, respectively. At least every second cylinder $D_i$ contains a path $R_i$ joining $P_i$ and $Q_i$ (otherwise $\Pi'$ could be changed and a patch vertex eliminated). Thus we can assume that the paths $R_i$ exist for all indices $i$ and that they are disjoint (by taking a subset of our curves if necessary).

Suppose that the 2-curves $\gamma_i$ are bounding. Then $\{x_i, y_i\}$ is a separating pair of $G$. Let $G_1$ be the subgraph of $G$ on the left side of $\gamma_2$ (where we assume that $D_2$ is on the right side of $\gamma_2$), and let $G_2$ be the subgraph of $G$ on the right side of $\gamma_4$. Also, let $G_3 = G \cap (D_2 \cap D_3)$. Then $G$ is edge-disjoint union of connected graphs $G_1$, $G_2$, and $G_3$. The embedding $\Pi$ restricted to $G_3$ is a planar embedding since $D_2 \cup D_3 \subset S$ is a cylinder. This embedding has a face $F_1$ containing $x_2, y_2$ and a face $F_2$ containing $x_4, y_4$. If $F_1 = F_2$, then $x_2, y_2$ and $x_4, y_4$ do not interlace in this face. The embedding $\Pi'$ restricted to $G_1$ contains $x_2$ and $y_2$ on the same face since there is a $G_1$-bridge in $G$ attached to $x_2$ and $y_2$. The same conclusions apply for $x_4, y_4$ in $\Pi'$ restricted to $G_2$. Therefore we can change $\Pi'$ into a planar embedding $\Pi''$ of $G$ that coincides with $\Pi'$ (or its inverse) on $G_1 \cup G_2$ and coincides with $\Pi$ on $G_3$. This is a contradiction with our choice of $\Pi'$.

We may now assume that $\gamma_1$ is nonbounding. We claim that $D'_i = D_i \cup \cdots \cup D_{i+3}$ contains a $\Pi$-noncontractible cycle, $i = 2, 3, ..., q-4$. Let $R$ be a path in the complement of $D'_i$ from $\{x_i, y_i\}$ to $\{x_{i+4}, y_{i+4}\}$. Consider the planar embedding $\Pi'$ restricted to the subgraph $H = \bigcup_{j=i+1}^{i+2} (P_j \cup Q_j \cup R_j)$. The $H$-bridge in $G$ containing $R_{i-1} \cup R_{i+4} \cup R$ is attached to $x_{i+1}$, $y_{i+1}$, $x_{i+3}$, and $y_{i+3}$. This implies that $H$ has all four vertices $y_{i+1}$, $x_{i+1}$, $x_{i+3}$ and $y_{i+3}$ in the same face and in that order. The same holds for the embedding $\Pi$ restricted to $H$ if $D'_i$ does not contain a $\Pi$-noncontractible cycle. Then $\Pi'$ can be replaced by a planar embedding $\Pi''$ of $G$ that coincides with $\Pi$ in $G \cap (D_{i+1} \cup D_{i+2})$ and coincides with $\Pi'$ elsewhere. This yields a contradiction with our choice of $\Pi'$.

Now, if $q$ is sufficiently large, then $D$ contains four disjoint $\Pi$-noncontractible cycles. We conclude as above in case of 1-curves by applying Theorem 6.2.

The above proof can also be used if $\gamma_1, ..., \gamma_r$ intersect in the interior of the same patch face. We may now assume that there is a patch vertex $x$ such that $\{\gamma_1, ..., \gamma_r\}$ intersect in $x$. Suppose that the curves are naturally enumerated. (In case when $\gamma_i$ are onesided, they cross at $x$, and then we extend the definition of naturally enumerated curves in the obvious way.) Let $y_i$ be other patch vertex used by $\gamma_i$. If two of the disks $D_i$ between $\gamma_i$ and $\gamma_{i+1}$ contain a 1-curve through $x$, then $G - x$ is disconnected. Therefore we may assume that for $i = 1, 2, ..., q = \lfloor r/3 \rfloor$, $G \cap D_i \backslash \{x\}$ contains a path $Q_i$ from $y_i$ to $y_{i+1}$. If the curves in $\Gamma$ are bounding, then we get a contradiction with the choice of $\Pi'$ as above (if $r$ is sufficiently large). The same arguments work if $\{x, y_1\}$ (and hence also $\{x, y_4\}$) is a separating set of the graph. Otherwise, there is a path $R$ from $y_1$ to $y_4$ that is internally disjoint from $x$ and from the discs $D_1$, $D_2$, $D_3$. We argue as above to show existence of paths $R_i \subset D_i$ $(i = 1, 2, 3)$ joining $Q_i$ with $x$. After possible reenumeration we may assume that $R_i \cap R_j = \{x\}$, $1 \leqslant i < j \leqslant 3$. The subgraph $H = \bigcup_{i=1}^{3} (Q_i \cup R_i) \cup R$ is homeomorphic to $K_4$ and has unique embedding in the sphere. If $G \cap D_i$ $(1 \leqslant i \leqslant 3)$ does not contain a $\Pi$-noncontractible cycle, then the embedding $\Pi$ coincides with $\Pi'$ on $G \cap D_i$ (by the choice of $\Pi'$). Then $D_i$ contains a single patch attached to $x$, $y_i$ and $y_{i+1}$. If $D_i$ and $D_{i+1}$ both have this property, then either $y_{i+1}$ is not a patch vertex, or the patch in $D_i$ can be reembedded by a 3-switching so that this patch and the patch in $D_{i+1}$ merge into a single match. In the latter case, either the 3-switching in $D_i$ or the 3-switching in $D_{i+1}$ does not change the surface $S$. This cannot happen by the minimality of $\Pi$. On the other hand, if $D_i$ contains a $\Pi$-noncontractible cycle of $G$, the path $Q_i$ can be subdivided by taking the patch vertices in its interior. The refined partition into disks $D_1, D_2, \ldots$ is easily seen to contain patches without $\Pi$-noncontractible cycles, and then the above proof works.

Above results show that $r$ is bounded by a constant depending only on $g$. This completes the proof. ∎

Given a surface $S$, let $\Pi_1, ..., \Pi_N$ be the basic patch structures of Theorem 7.1. To perform the change of an embedding with patch structure $\Pi_i$ into an embedding with patch structure $\Pi_j$ $(1 \leqslant i, j \leqslant N)$ we need to change only a bounded number of angles. By calling every such change a *generalized Whitney switching*, Theorem 7.1 can be formulated as follows.

COROLLARY 7.3. *Any two embeddings of a* 2-*connected planar graph in the same surface can be obtained from each other by performing a sequence of Whitney $k$-switchings for $k = 2, 3, 4$ and applying* (*at most one*) *generalized Whitney switching operation.*

The following specific problem also occurred to us. Let $G$ be a 5-connected planar triangulation. Change its planar rotation $\Pi'$ at each vertex (to a local rotation $\Pi$) so that no two $\Pi'$-consecutive edges are $\Pi$-consecutive. (For example, if the local rotation in the plane at a vertex $v$ of degree five is $\pi'_v = (e_1 e_2 \cdots e_5)$, we can change it to $\pi_v = (e_1 e_3 e_5 e_2 e_4)$ and similarly at vertices of larger degrees.) Then every patch is just an edge. *Question*: Can we get an embedding $\Pi$ of face-width two in this way? A negative answer to this question would support some further speculations that we have concerning Whitney equivalence of embeddings.

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