Projective plane and Möbius band obstructions*

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Abstract

Let S be a compact surface with possibly non-empty boundary ∂S and let G be a graph. Let K be a subgraph of G embedded in S such that $\partial S \subseteq K$. An *embedding extension* of K to G is an embedding of G in S which coincides on K with the given embedding of K. Minimal obstructions for the existence of embedding extensions are classified in cases when S is the projective plane or the Möbius band (for several "canonical" choices of K). Linear time algorithms are presented that either find an embedding extension, or return a "nice" obstruction for the existence of extensions.

1 Introduction

Let S be a fixed compact surface with possibly non-empty boundary ∂S. Let G be a graph and $K \subseteq G$. Suppose that we are given an embedding of K into a surface S such that $\partial S \subseteq K$. The *embedding extension problem* asks whether it is possible to extend the given embedding of K to an embedding of G, and any such embedding is said to be an *embedding extension* of K to G. An *obstruction* for embedding extensions is a subgraph Ω of $G - E(K)$ such that the embedding of K cannot be extended to $K \cup \Omega$. The obstruction is *bounded* if the branch size and the number of feet of Ω are bounded by certain given constant. If Ω is a bounded obstruction, then one can easily verify (in constant time) that no embedding extension to $K \cup \Omega$ exists, and hence Ω is a good verifier that there are no embedding extensions of K to G as well. In this paper, minimal obstructions for embedding extension problems in the Möbius band and the projective plane are classified for several "canonical" choices of K. It is interesting that minimal obstructions are not always bounded. They can be arbitrarily large but their structure is easily described. It is shown that one can always find "nice" obstructions. They have bounded branch size up to a subgraph, called millipede, that they may contain. Millipedes have rather simple structure (cf. Section 4) and they have two important properties: they admit just a bounded number of combinatorially distinct embeddings

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and they can be "compressed". Roughly speaking, this means that after changing a segment of a branch of K, the millipede turns into a bounded obstruction. We also present linear time algorithms that either find an embedding extension, or return a (minimal) "nice" obstruction for existence of extensions. The cases of k–Möbius band embedding extension problems with $k = 0$ or $k \geq 3$ and no local bridges always ends up with bounded obstructions (Theorems 3.2 and 7.1) while for $k = 1$ and 2 millipedes cannot be avoided (Theorems 5.1 and 8.2).

Algorithmic results of this paper are used as a basis in the design of a linear time algorithm for embedding graphs in the torus [12]. More generally, the author extended the results to a more complicated linear time algorithm for embedding graphs in an arbitrary fixed closed surface [17, 18]. The knowledge of the structure of the minimal obstructions in these and in some other cases [11, 15, 16] also leads to a reasonably short constructive proof of the Kuratowski theorem for general surfaces [18] (proved originally by Robertson and Seymour; cf. [19] and the graph minors papers preceding it).

The basic approach of this paper to the problem of embedding graphs in surfaces has been previously employed by other authors, most notably by Archdeacon and Huneke [2] (who proved the Kuratowski theorem for nonorientable surfaces), and by Filotti, Miller, and Reif [5] who designed a polynomial time algorithm for embedding graphs in a fixed orientable surface.

A list of minimal forbidden subgraphs for embedding graphs in the projective plane was determined by Glover, Huneke, and Wang [6], and it was proved by Archdeacon [1] that the list of 103 obstructions from [6] is complete. A part of our work is devoted also to this case since we also need a linear time procedure that discovers such an obstruction in any given non-projective graph. For this purpose, an extension of the linear time algorithm from [14] is presented (Theorem 6.1).

Embeddings in surfaces can be described combinatorially [7] by specifying a *rotation system* (for each vertex v of the graph G we have the cyclic permutation π_v of its neighbors, representing their circular order around v on the surface) together with a *signature* $\lambda : E(G) \to \{-1, 1\}$ having the property that a cycle of G has an odd number of edges e with $\lambda(e) = -1$ if and only if the cycle is one-sided on the surface (i.e., every open neighborhood of the cycle on the surface contains a Möbius band). In order to make a clear presentation of our algorithms, we have decided to use this description only implicitly. Whenever we say that we have an embedding (either given, obtained by some other algorithm, or produced inductively by our algorithm) we mean that we have such a combinatorial description. Whenever used, it is easy to see how one can combine the embeddings of some parts of the graph described this way into the embedding of larger species.

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for basic operations. This model was introduced by Cook and Reckhow [4]. More precisely, our model is the *unit-cost* RAM where operations on integers, whose value is $O(n)$, need only constant time (n is the order of the given graph).

2 Basic definitions

Let K be a subgraph of G. A *K–bridge* in G (or a *bridge* of K in G) is a subgraph of G which is either an edge $e \in E(G) \backslash E(K)$ (together with its endpoints) which has both endpoints in K, or it is a connected component of $G - V(K)$ together with all edges (and their endpoints) between this component and K. Each edge of a K–bridge R having an endpoint in K is a *foot* of R. The vertices of $R \cap K$ are the *vertices of attachment* of R. A vertex of K of degree different from 2 is a *main vertex* of K. For convenience, if a connected component of K is a cycle, then we choose an arbitrary vertex of it and declare it to be a main vertex of K as well. A *branch* of K is any path in K (possibly closed) whose endpoints are main vertices but no internal vertex on this path is a main vertex. If a K–bridge is attached to a single branch of K, it is said to be *local*. The number of branches of K is called the *branch size* of K.

Let G and H be graphs. Then we denote by $G - H$ the graph obtained from G by deleting all vertices of $G \cap H$ and all their incident edges. If $F \subseteq E(G)$, then $G - F$ denotes the graph obtained from G by deleting all edges in F. If K and L are subgraphs of G, then we say that a path P in G *joins* K and L if P is internally disjoint from $K \cup L$ and one of its ends is in K and the other end is in L.

One can define the concept of *3-connected components* of a graph [24, 8]. They can be viewed as subgraphs of G, where some edges (called virtual edges) correspond to paths in G. (We will also speak of 3-connected components when the graph is not 2-connected. In that case we define them to be the 3-connected components of the blocks of the graph.) A linear time algorithm for obtaining the 3-connected components of a graph was devised by Hopcroft and Tarjan [8].

Let K be a subgraph of G. Let G' be the graph obtained from G by adding three mutually adjacent vertices and joining each of them with all main vertices of K. If G' is 3-connected, then G is said to be *3-connected modulo K*. If this is the case, then every branch e of K can be replaced by a branch e' that is internally disjoint from $K - e$ and such that $K - e + e'$ has no local bridges on e'. Usually, this replacement can be performed in linear time [10].

There are very efficient (linear time) algorithms which for a given graph determine whether the graph is planar or not. The first such algorithm was obtained by Hopcroft and Tarjan [9] back in 1974. Extensions of this algorithm produce also an embedding (rotation system) whenever the given graph is found to be planar [3], or find a bounded obstruction — a subgraph homeomorphic to K_5 or $K_{3,3}$ — if the graph is non-planar [25, 26]. A subgraph of G homeomorphic to K_5 or $K_{3,3}$ is called a *Kuratowski subgraph* of G. So, there is a linear time algorithm that, given a graph G, either exhibits an embedding of G in the plane, or finds a Kuratowski subgraph of G. We will refer to this algorithm as *testing for planarity*.

For several objects (numbers, sets, or graphs) we will require to be *bounded*. This will mean that they are bounded above (for sets and graphs their cardinality and the branch size is bounded, respectively) by certain constant.

Suppose that $K \subseteq G$ is embedded in some surface. An *obstruction* in G is a subgraph Ω of $G - E(K)$ such that Ω has no embedding with certain properties; we say that Ω *obstructs* embedding extensions of K with these properties.

To measure the size of $\Omega \subseteq G - E(K)$ we will use the number $b(\Omega)$ which is equal to the number of branches of $K \cup \Omega$ that are contained in Ω. Then Ω is bounded if and only if $b(\Omega)$ is bounded. Note that $b(\Omega)$ can be different from the branch size of Ω.

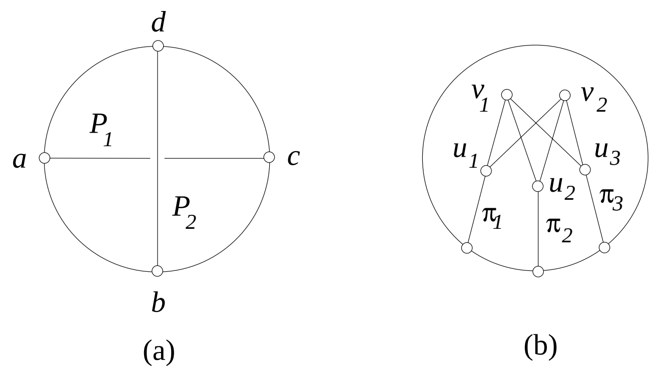

Figure 1: Disjoint crossing paths and a tripod

Let C be a cycle of a graph G. Two C–bridges B_1 and B_2 *overlap* if either B_1 and B_2 have three vertices of attachment in common, or there are four distinct vertices a, b, c, d which appear in this order on C and such that a and c are vertices of attachment of B_1, and b, d are vertices of attachment of B_2. In the latter case, B_1 and B_2 contain disjoint paths P_1 and P_2 whose ends a, c and b, d, respectively, interlace on C. Such paths (not necessarily in distinct bridges) will be referred to as *disjoint crossing paths*. See Figure 1(a). We will need another type of subgraphs of G that are attached to C. A *tripod* is a subgraph T of G that consists of two main vertices v_1, v_2 of degree 3, whose branches join them with the same triple of vertices u_1, u_2, u_3, together with three vertex disjoint paths π_1, π_2, π_3 joining u_1, u_2, and u_3 with C. Moreover, T intersects C only at the ends of π_1, π_2, and π_3. One or more of the paths π_i are allowed to be trivial, in which case $u_i \in C$. See Figure 1(b). If all three paths π_1, π_2, and π_3 are trivial (just vertices), then the tripod is said to be *degenerate*.

Let D be the closed unit disk in the euclidean plane. Given a graph G and a cycle C in G, we would like to find an embedding of G in D so that C is embedded on ∂D. Given G and C as above, we define the *auxiliary graph* $\tilde{G}$ for the disk embedding extension problem as the graph obtained from G by adding an additional vertex joined to all vertices on C. The following result has been proved by several authors [21, 22, 23, 20] with a corresponding linear time algorithm in [15].

Theorem 2.1 *Let G, C, D be as above, and let $\tilde{G}$ be the auxiliary graph of G with respect to C. There is a linear time algorithm that either finds an embedding of G in D with C on ∂D, or returns a bounded obstruction Ω. In the latter case, Ω is one of the following types of subgraphs of $G - E(C)$:*

(a) a pair of disjoint crossing paths,

(b) a tripod, or

(c) a Kuratowski subgraph contained in a 3-connected component of $\tilde{G}$ distinct from the 3-connected component of $\tilde{G}$ containing the auxiliary vertex and C.

3 k–Möbius band embedding extension problems

In this section we will consider certain embedding extension problems in the Möbius band. Let C be a cycle in a graph G, and for an integer $k \geq 0$, let $P_1, P_2, \ldots, P_k$ be vertex disjoint paths in G with their endpoints on C and with no interior points on C. Suppose, moreover, that the endpoints a_i, b_i of the paths P_i $(i = 1, \ldots, k)$ appear on C in order $a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k$. The embedding extension problem in the Möbius band with respect to the subgraph $K = C \cup P_1 \cup \cdots \cup P_k$, where K is embedded in such a way that C covers the boundary, will be referred to as the *k-Möbius band embedding extension problem.* We also assume when $k = 1$ that P_1 "crosses" the Möbius band so that the embedding is 2-cell. (This is automatically true for $k > 1$.) Cf. Figure 2.

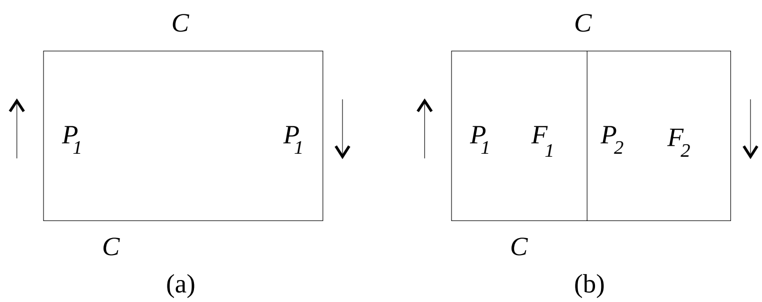

Figure 2: The 1- and 2-Möbius band embedding extension problems

In testing for the k-Möbius band embedding extensions of K to G we make use of the *auxiliary graph* $\tilde{G}$ which is obtained from G by adding an additional copy C' of the cycle C and joining each vertex of C' with its original in C.

Lemma 3.1 *Suppose that G is connected and that K is as above. If $k \neq 1$, then there is an embedding extension of K to G if and only if the Möbius band auxiliary graph $\tilde{G}$ is projective planar. Any projective embedding of $\tilde{G}$ determines a Möbius band extension. Similar situation is when $k = 1$ but in this case only embeddings of $\tilde{G}$ in the projective plane for which $C' \cup P_1 \cup a_1a_1' \cup b_1b_1'$ contains a non-contractible circuit are counted.*

Proof. It is clear that any embedding extension of K to G determines an embedding of $\tilde{G}$ in the projective plane (with the required property if $k = 1$).

Suppose now that $\tilde{G}$ is embedded in the projective plane. If C' is non-contractible, then the induced embedding of $\tilde{G} - C' = G$ is essentially a plane embedding. Moreover, it determines a plane embedding of G with C bounding a face. This is possible only when $k \leq 1$. Such an embedding is easily seen to give rise to a Möbius band extension: In case when $k = 0$ we get an embedding which is not 2-cell (replace any face by a Möbius band), while in case $k = 1$, we add signature –1 on any edge of P_1.

Suppose now that C' is contractible. Note that C' has only one bridge. If this bridge is embedded in the disk bounded by C', we have the same situation as above. Otherwise, removing the edges of C and contracting the edges between C and C' we get a graph isomorphic to G embedded in the projective plane with the cycle C' corresponding to

C and bounding a face. This embedding determines an embedding of G in the Möbius band having C on the boundary. If $k \neq 1$, this is necessarily a required extension. The same is true when $k = 1$ and $C' \cup P_1 \cup a_1 a_1' \cup b_1 b_1'$ contains a non-contractible circuit. $\square$

If there are local bridges attached to one or more of the paths P_i, we may get arbitrarily long chains of successively overlapping local bridges on P_i. There are examples of arbitrarily large minimal obstructions. On the other hand, local bridges can be eliminated efficiently (in linear time; see [14] and [10] for more details). Since we are usually allowed to change the paths P_i during the pre-processing time in applications that are using the obstructions, we will assume that there are no local bridges attached to any of the paths $P_1, \ldots, P_k$.

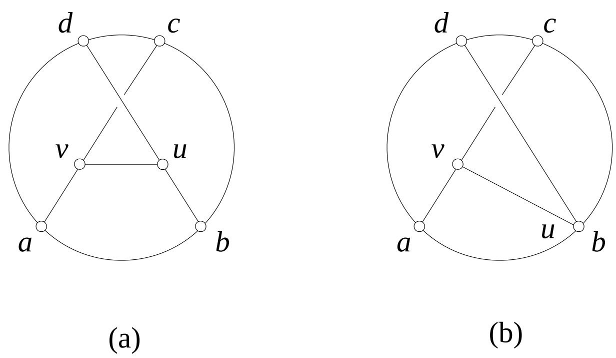

Figure 3: A dipod

Obstructions for k-Möbius band embedding extension problems with $k \geq 3$ are easy to find. They are not much more complicated than the closed disk obstructions classified in Theorem 2.1. Besides the disjoint crossing paths and the tripods, we get another type of obstructions. A *dipod* (with respect to the cycle C) is a subgraph H of G consisting of distinct vertices $a, b, c, d \in V(C)$ that appear on C in that order, distinct vertices v, u where $u \notin V(C)\backslash\{b\}$ and $v \notin V(C)$, and branches va, vc, vu, ub, and ud (Figure 3). The branches are internally disjoint from C. If $u = b$, the branch ub vanishes. Such a dipod is said to be *degenerate*. See Figure 3(b). Note that a dipod contains a pair of disjoint crossing paths. We also define a *triad* (with respect to a subgraph K of G) as a subgraph of G consisting of a vertex $x \notin V(K)$ and three paths joining x with K that are pairwise disjoint except at their common end x.

For $K = C \cup P_1 \cup \cdots \cup P_k$ embedded in the Möbius band with C on its boundary, let $F_1, \ldots, F_k$ be the faces of K. We suppose that for $i = 1, 2, \ldots, k$, ∂F_i contains P_i and P_{i+1} (index modulo k).

Theorem 3.2 *Let K be a subgraph of G for the k-Möbius band embedding extension problem, where $k \geq 3$. Suppose that no K–bridge in G is attached just to a path P_i of K, $1 \leq i \leq k$. Then there is no embedding extension of K to G if and only if $G - E(K)$ contains a subgraph Ω of one of the following types:*

(a) A path joining two vertices of K that do not lie on the boundary of a common face of K, or (when $k = 3$) a triad attached to P_1, P_2, and to P_3.

(b) A tripod attached to the boundary of one of the faces F_i. Not all three attachments of the tripod lie on just one of the paths P_i, P_{i+1} on ∂F_i.

(c) A pair of disjoint crossing paths with respect to the boundary of one of the faces F_i. None of the two paths is attached to just one of the paths P_i, P_{i+1} on ∂F_i.

(d) A dipod with respect to the boundary cycle of some F_i. In this case, the vertices a, c, and d from the definition of dipod all lie on one of the paths P_i, or P_{i+1}, while $b \in \partial F_i$ does not lie on that path.

(e) A Kuratowski subgraph contained in a 3-connected component L of the auxiliary graph $\tilde{G}$ of G, where L is distinct from the 3-connected component containing the auxiliary cycle C'.

There is a linear time algorithm that either finds an embedding extension of K to G, or returns an obstruction Ω which fits one of the above cases.

Proof. We can find embedding extensions, if they exist at all, by testing projective planarity of the auxiliary graph $\tilde{G}$ (Lemma 3.1) using the algorithm of [14]. Suppose now that embedding extensions do not exist. Our goal is to show how to find an obstruction Ω.

Since $k \geq 3$ and there are no local bridges at the paths P_j, every K–bridge is embeddable in at most one of the faces F_i. If one of the bridges contains a path whose ends do not belong to the boundary of the same face, then this path is clearly an obstruction for embedding extendibility. If a bridge B of K does not have all of its vertices of attachment on the boundary of a single face F_i, then B either contains such a path, or it contains a triad attached to P_1, P_2, and P_3. The latter case is needed only when $k = 3$. So, we have (a). Otherwise, every K–bridge is attached to ∂F_i for exactly one i, $1 \leq i \leq k$. Therefore, there is no embedding extension if and only if for some i, $1 \leq i \leq k$, we have a closed disk obstruction (cf. Theorem 2.1) in the subgraph G_i consisting of $C_i = \partial F_i$ and all the K–bridges attached to C_i. By Theorem 2.1, obstruction to the (G_i, C_i) disk embeddability is either a pair of disjoint crossing paths, a tripod, or a Kuratowski subgraph in a 3-connected component of $\tilde{G}_i$ not containing the auxiliary vertex. In the latter case, $\tilde{G}_i$ is the auxiliary graph of G_i with respect to C_i for the disk embedding extension problem. Since there are no local bridges attached to the paths P_i and P_{i+1}, the 3-connected components of $\tilde{G}_i$ not containing the auxiliary vertex are also 3-connected components of $\tilde{G}$. Consequently, a Kuratowski subgraph obstruction in G_i gives (e).

Suppose now that in G_i we have a tripod T. If T is not local on P_i and not local on P_{i+1}, we have (b). Otherwise, assume all three attachments of T are on P_i. Denote by $v_1, v_2, u_1, u_2, u_3, \pi_1, \pi_2, \pi_3$ the elements of T as they are shown on Figure 1, and suppose that π_2 is attached to P_i between π_1 and π_3. Construct a path P, internally disjoint from C, that connects $C - P_i$ with an interior vertex x of T. The existence of P is guaranteed since the bridge(s) containing T is (are) not local on P_i. If x is on π_s for some $s \in \{1, 2, 3\}$, then we can replace the segment of π_s from x to P_i by P and get a tripod satisfying (b). If x is an interior vertex of the branch u_2v_1, then $T \cup P$ contains a dipod satisfying (d). By the symmetries of T, the only essentially different remaining case is when x is on the branch u_1v_1, where $x \neq u_1$ but possibly $x = v_1$. Let Q_1 be the path $Pxv_1u_2\pi_2$ and let Q_2 be the path $\pi_1u_1v_2u_3\pi_3$ in $T \cup P$. If Q_1 and Q_2 are in the same K–bridge of G, then we can find a path P' from Q_1 to Q_2 that is disjoint from C, and $Q_1 \cup Q_2 \cup P'$ is a dipod satisfying (d). On the other hand, if Q_1 and Q_2 are in different K–bridges B_1, B_2, respectively, let P' be a path from the interior of Q_2 to C that is disjoint from P_i. Such a path exists, again, because B_2 is not local on P_i. Now, $Q_1 \cup Q_2 \cup P'$ contains disjoint crossing paths satisfying (c), unless the endpoints of P and P' on C coincide. But in this case, $Q_1 \cup Q_2 \cup P'$ is a degenerate dipod with the attachment b (see Figure 3(b)) corresponding to the common point of P and P'.

It remains to consider the case of disjoint crossing paths, say Q_1 and Q_2, obtained as an obstruction in G_i. If both Q_1 and Q_2 are attached locally to P_i, we change one of them so that it has an attachment on $C - P_i$. For this purpose, the same method as above can be applied. If just one of the paths (possibly after the previous change) is local on P_i, the same procedure can be applied as it was undertaken above with the paths Q_1 and Q_2 in case of the tripods. We either get a dipod or disjoint crossing paths satisfying (d) or (c), respectively.

It is easy to perform the above construction in linear time. To find disk obstructions we use Theorem 2.1, and to find paths P, P', etc., we can use standard graph search algorithms. Most of other tasks are just constant time operations. $\square$

4 Millipedes

Suppose that K is 2-cell embedded in some surface and that F_1 and F_2 are non-singular faces of K whose boundaries have a branch e and a vertex x ($x \notin V(e)$) in common. We choose and fix an orientation of e so that we will be able to speak about the "left" and the "right" side of e. We say that K–bridges B and B' *overlap* in a face of K if they cannot be simultaneously embedded in that face.

For the purpose of the following definition we assume that all bridges of K in G are bounded. If this were not the case, the bridges B_i° appearing in the definition should be replaced by their H–subgraphs (cf. [15, 16]). A *thin millipede* in G *based* on e and with *apex* x is a subgraph M of $G - E(K)$ which can be expressed as $M = B_1^\circ \cup B_2^\circ \cup \cdots \cup B_m^\circ$ ($m \geq 7$) where:

(M1) Each of B_1° and B_m° is a K-bridge in G. Moreover, $B_1^\circ \cup B_2^\circ \cup B_3^\circ$ is uniquely embeddable in $F_1 \cup F_2$. Let F_α be the face containing B_1° under this embedding. Similarly, $B_{m-2}^\circ \cup B_{m-1}^\circ \cup B_m^\circ$ is uniquely embeddable, and let F_β be the face containing B_m°. If m is even, then $\alpha = \beta$. If m is odd, then $\alpha \neq \beta$.

(M2) $B_2^\circ, \ldots, B_{m-1}^\circ$ are distinct K-bridges that are attached to e and to x and are not attached to K elsewhere.

(M3) For each $i = 1, 2, \ldots, m-1$, B_i° and B_{i+1}° overlap in F_1 and in F_2.

(M4) For $i > 1$ and $i + 2 \leq j < m$, B_i° and B_j° overlap neither in F_1 nor in F_2. Similarly, B_1° and B_j°, $3 \leq j < m$, do not overlap in F_α, and B_i°, B_m° ($1 < i \leq m-2$) do not overlap in F_β. Additionally, $B_1^\circ \cup B_m^\circ$ can be embedded in $F_\alpha \cup F_\beta$.

Property (m3) implies that in any simultaneous embedding of $B_i^\circ \cup B_{i+1}^\circ$, the two bridges lie in distinct faces. The parity condition on m in (M1) therefore implies that a thin millipede obstructs embedding extensions of K to G.

Our notion of millipedes is slightly different from the concept of millipedes introduced in [15]. The millipedes in [15] can be shorter (i.e., $m < 7$ is allowed) and their subgraphs B_i° are allowed to be proper subgraphs of bridges in order that the millipedes become minimal obstructions. On the other hand, after eliminating superfluous branches in bridges B_i°, we can get from our thin millipedes a millipede in the sense of [15].

We will also need *skew millipedes*. They are defined similarly as thin millipedes. Their faces F_1 and F_2 share a branch e and an edge $f = xy$ such that $F_1 \cup F_2 \cup e \cup f$ contains a Möbius band. (We will also allow that f is a segment of a branch of K such that no K–bridge is attached to the interior of f.) The bridges $B_1^\circ, \dots, B_m^\circ$ satisfy (M1) and (M3), while (M2) and (M4) are replaced by:

(M2′) $B_2^\circ, \dots, B_{m-1}^\circ$ are distinct K-bridges. If i is even ($1 < i < m$), then B_i° is attached to e and to x (and not elsewhere). If i is odd ($1 < i < m$), then B_i° is attached to e and to y (and not elsewhere).

(M4′) For $i > 1$ and $i + 2 \le j < m$, B_i° and B_j° do not overlap in F_α if either $i \not\equiv \alpha \pmod 2$, or $j \equiv \alpha \pmod 2$ (or both). They do not overlap in $F_{3-\alpha}$ if either $i \equiv \alpha \pmod 2$, or $j \not\equiv \alpha \pmod 2$ (or both). For $3 \le j < m$, $B_1^\circ \cup B_j^\circ$ can be embedded in F_α. For $1 < i \le m - 2$, $B_i^\circ \cup B_m^\circ$ can be embedded in F_β. Additionally, $B_1^\circ \cup B_2^\circ \cup B_3^\circ \cup B_{m-2}^\circ \cup B_{m-1}^\circ \cup B_m^\circ$ can be embedded in $F_1 \cup F_2$.

Equivalent definition of a skew millipede is that (M2′) and the last condition in (M4′) hold and after contracting the edge $f = xy$, we get a thin millipede. If M is a skew millipede, then the *apex* of M consists of corresponding vertices x and y. In referring to a *millipede*, we mean either a thin or a skew millipede. If M is a millipede, it is an obstruction for embedding extensions of K to G. By (M2) and (M4) ((M4′), respectively), every millipede is a "minimal" obstruction in the sense that no bridge in M is redundant.

If M is a millipede, we define $\partial M = B_1^\circ \cup B_m^\circ$, and the rest, $M^\circ = B_2^\circ \cup \cdots \cup B_{m-1}^\circ$, is called the *central part* of M. The following lemma is obvious by (M3), (M4), and (M4′).

Lemma 4.1 *Let M be a millipede. Then M° has exactly two combinatorially different embeddings in $F_1 \cup F_2$. Under any such embedding, the bridges $M^{even} = B_2^\circ \cup B_4^\circ \cup B_6^\circ \cup \cdots$ are all embedded in the same face (F_1, or F_2), and $M^{odd} = B_3^\circ \cup B_5^\circ \cup B_7^\circ \cup \cdots$ are all embedded in the other face.*

In Figure 4, the two embeddings of the central part of a skew millipede are shown.

Lemma 4.2 *$B_1^\circ \cup B_2^\circ \cup B_3^\circ \cup B_{m-2}^\circ \cup B_{m-1}^\circ \cup B_m^\circ$ has unique embedding in $F_1 \cup F_2$ and under this embedding, B_1° and B_3° are in F_α, B_2° is in $F_{3-\alpha}$, B_{m-2}° and B_m° are in F_β, and B_{m-1}° is in $F_{3-\beta}$.*

Proof. Existence and uniqueness follow easily from (M1) and (M4) (or trivially by (M4′)). The location of bridges in particular faces is determined by (M3). □

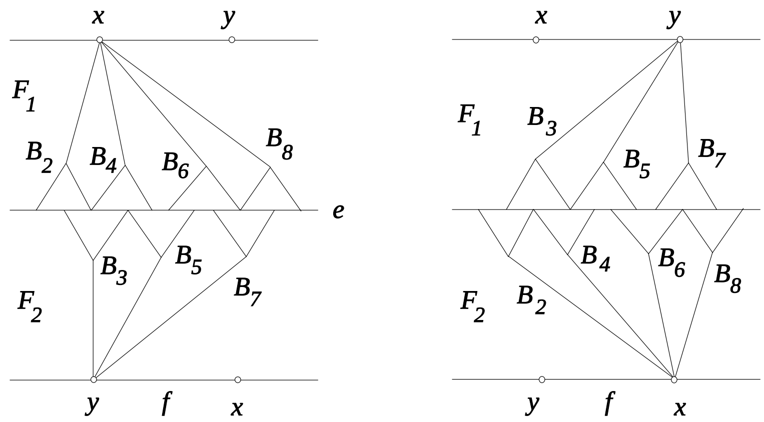

Figure 4: Two embeddings of the central part of a skew millipede

Let M be a millipede. For $i = 2, 3, \ldots, m-1$, denote by l_i and r_i the leftmost and the rightmost attachment of B_i° on e, respectively.

Lemma 4.3 *Suppose that B_2° is attached on e to the left of B_4°. Then for $i = 2, 3, \ldots, m-2$, B_i° is attached on e to the left of $B_{i+2}^\circ, B_{i+3}^\circ, \ldots, B_{m-1}^\circ$. Moreover, if $3 \le i \le m-2$, then*

$$l_{i-1} \prec l_i \prec r_{i-1} \preceq l_{i+1} \prec r_i \prec r_{i+1} \tag{1}$$

where the relation $\prec$ means being more to the left on e.

Proof. Let $2 \le i \le m-1$. Since $m \ge 6$, it follows by (M4) (or (M4$'$)) that B_i° can be embedded in F_1 and in F_2. Denote by J_i the open segment on e from l_i to r_i. By (M3) we see that $J_{i-1} \cap J_i \neq \emptyset$ ($i \neq 2$) and that $J_i \cap J_{i+1} \neq \emptyset$ ($i \neq m-1$). By (M4) (or (M4$'$)) we see that the intervals J_i, J_j are disjoint for $j \neq i-1, i, i+1$. We claim that J_i is to the left of J_j ($j \ge i+2$). This can be established by induction. As the basis of induction we have the assumption that J_2 is to the left of J_4. If J_5 is to the left of J_2, then J_4 and J_5 would not fulfil (M3). So, J_5 is to the right of J_2. Now, J_3 intersects J_2 and not J_5. Thus J_3 is to the left of J_5, etc. This gives that $l_i \prec r_{i-1} \preceq l_{i+1} \prec r_i$.

It remains to see that for $3 \le i \le m-2$, we have $l_{i-1} \prec l_i$ and $r_i \prec r_{i+1}$. Suppose that $l_{i-1} \succeq l_i$. Since J_{i+1} is to the right of J_{i-1} and since $J_{i+1} \cap J_i \neq \emptyset$, it follows that $J_{i-1} \subseteq J_i$. If $i \ge 4$, this is contradictory since $J_{i-2} \cap J_{i-1} \neq \emptyset$ and $J_{i-2} \cap J_i = \emptyset$. The proof of the remaining case, $l_2 \prec l_3$, is contained in the proof of Lemma 4.4. (Note that application of Lemma 4.3 in the proof of Lemma 4.4 does not involve the unsettled cases $l_2 \prec l_3$ and $r_{m-2} \prec r_{m-1}$.) The proof that $r_i \prec r_{i+1}$ is similar. □

From now on we will always assume that the left and the right side of e are defined such that (1) holds.

Lemma 4.4 *Each of B_1° and B_m° has an attachment that is neither on e nor in the apex of M. Also, B_1° and B_m° have attachments r_1 and l_m, respectively, on e such that $l_2 \prec r_1 \preceq l_3$ and $r_{m-2} \preceq l_m \prec r_{m-1}$.*

Proof. By symmetry, we give the proof only for B_1°. If M is thin, the claims are easy to see by uniqueness of embedding of $B_1^\circ \cup B_2^\circ \cup B_3^\circ$ and by (M4). Suppose now that M is a skew millipede. Consider the embedding of $L = B_1^\circ \cup B_2^\circ \cup B_3^\circ \cup B_{m-2}^\circ \cup B_{m-1}^\circ \cup B_m^\circ$ in $F_1 \cup F_2$. By Lemma 4.2 we get the following. If $\alpha = \beta$, then B_3° and B_{m-2}° are both in F_α. Also, m is even, so B_3° is attached to y, and B_{m-2}° is attached to x. Consequently, x is to the right of y in F_α. The same conclusion follows when $\alpha \neq \beta$. By Lemma 4.3, every vertex between l_3 and r_{m-2} on e is contained in some open segment (l_i, r_i) on e, $3 \leq i \leq m-2$. Consequently, (M4$'$) implies that B_1° is embedded to the left of B_3° in F_α. In particular, B_1° cannot be attached to x. If B_1° would be attached only to e and to y, the embedding of $B_1^\circ \cup B_2^\circ \cup B_3^\circ$ would not be unique. Thus, we have an attachment of B_1° out of e and the apex of M. B_1° overlaps with B_2° in F_α. By the above, this is possible only if it has an attachment r_1 on e such that $l_2 \prec r_1$. Since B_3° is to the right of B_1° in F_α, we have $l_2 \prec r_1 \preceq l_3$. $\square$

Let M be a millipede based on e, defined with respect to certain 2-cell embedding of K. Suppose that we have another 2-cell embedding of K such that e appears on the boundary of two distinct faces F_1' and F_2' and such that each vertex of attachment of bridges in M° appears at most once on $\partial F_1'$ and at most once on $\partial F_2'$. Such an embedding of K is said to be *M–nonsingular*. By Lemma 4.3 it is easy to see that Lemma 4.1 remains valid with respect to arbitrary M–nonsingular embeddings of K with a slight difference that M° may not have an embedding in $F_1' \cup F_2'$ if the apex of M is not contained in $\partial F_1'$ and in $\partial F_2'$.

Let M be a millipede. Let f' be the rightmost foot of B_2° on e. Subdivide f' by inserting a new vertex v_2 of degree 2. Introduce similarly vertices v_3 in B_3°, and v_{m-2}, v_{m-1} in $B_{m-2}^\circ, B_{m-1}^\circ$, respectively (in the latter two cases with respect to their leftmost feet). If m is even, then add to M the edges $f_1 = v_2 v_{m-2}$ and $f_2 = v_3 v_{m-1}$. If m is odd, then add edges $f_1 = v_2 v_{m-1}$ and $f_2 = v_3 v_{m-2}$. Finally, delete bridges $B_4^\circ, B_5^\circ, \ldots, B_{m-3}^\circ$ from M. Denote the obtained graph by $\tilde{M}$ and call it the *squashed millipede*. This way we reduce the size of M, while essentially preserving its embedding extension properties (Lemma 4.1) with respect to M–nonsingular embeddings of K. Note that all bridges in M^{even} (and similarly in M^{odd}) are replaced by a single bridge. To preserve the interference of M with other bridges of K, we apply another change described in the sequel. (As pointed out by one of the referees, squashing a millipede is really just a standard "cut and paste" technique from topology).

Lemma 4.3 implies that $l_2 \prec l_3 \prec r_2$ and $l_{m-1} \prec r_{m-2} \prec r_{m-1}$. Let $D \subseteq G - E(K)$ be the union of all K–bridges that have an attachment on e strictly between l_3 and r_{m-2}. By Lemma 4.1, every K–bridge in D is "blocked" by M. Even more, if we have an M–nonsingular embedding of K with corresponding faces F_1', F_2', then under any embedding of $R = B_2^\circ \cup B_3^\circ \cup B_{m-2}^\circ \cup B_{m-1}^\circ$ in $F_1' \cup F_2'$, the bridges of D are blocked between the left "barrier", $B_2^\circ \cup B_3^\circ$, and the right one, $B_{m-2}^\circ \cup B_{m-1}^\circ$. We distinguish two cases:

(SQ1) *$M^\circ \cup D$ can be embedded in $F_1 \cup F_2$.* By Lemma 4.1 and its extension to M–nonsingular embeddings as mentioned above, we see that for any M–nonsingular embedding of K, the central part of the squashed millipede $\tilde{M}$ has same embedding extensions in $F_1' \cup F_2'$ as $M^\circ \cup D$. Therefore we can replace the bridges from $M^\circ \cup D$

in G by the squashed millipede $\tilde{M}$ and an arbitrary M–nonsingular embedding of K can be extended to the obtained graph if and only if it can be extended to G.

(SQ2) *$M^\circ \cup D$ cannot be embedded in $F_1 \cup F_2$.* In this case, $M^\circ \cup D$ contains a bounded obstruction Ω for extending the embedding of K to an embedding of $K \cup M^\circ \cup D$. Let us note that such an obstruction can be obtained in linear time as follows. Suppose first that D contains a bridge B' distinct from B_2° and B_{m-1}° that is not attached only to the segment between l_3 and r_{m-2} of e and to the apex of the millipede. Then $\Omega = B' \cup B_2^\circ \cup B_3^\circ \cup B_{m-2}^\circ \cup B_{m-1}^\circ$ is a bounded obstruction. (Note that Ω does not necessarily obstruct extensions of arbitrary M–nonsingular embeddings of K.) The next possibility is when there is a K–bridge B in D that is embeddable in at most one of the faces F_1, F_2. Excluding the previous case, M is a skew millipede and B is attached to e and to both vertices in the apex of M. Let p be a vertex of attachment of B on e that is between l_3 and r_{m-2}. Then $l_i \prec p \prec r_i$ for some i, $3 \le i \le m-2$ (Lemma 4.3). It is easy to see that $B \cup B_{i-1}^\circ \cup B_i^\circ \cup B_{i+1}^\circ$ is an obstruction. Otherwise, every bridge in $M^\circ \cup D$ can be embedded in F_1 and in F_2. Theorem 5.1 can be applied to get an obstruction Ω contained in $M^\circ \cup D$. Suppose that $\Omega = B'_1 \cup \cdots \cup B'_t$ is a millipede. By Lemma 4.4, B'_1 has an attachment out of e and the apex. Therefore, M is a skew millipede and Ω is a thin millipede with apex x, say. By Lemma 4.3 applied on M, the vertex r'_3 of B'_3 is between l_i and r_i for some i, $3 \le i \le m-2$. If $B_i \ne B'_4$, this implies that B_i, B'_3, B'_4 mutually overlap and thus form a bounded obstruction. Otherwise, consider the vertex $l'_4 = l_i$ covered by B_{i-1}. Since B_{i-1} is attached to y, it follows that B_{i-1}, B_i, B'_3 form a bounded obstruction. Thus, we may assume that Ω is not a millipede, and Theorem 5.1 implies that Ω is bounded, $b(\Omega) \le 13\beta_0$.

Let M be a millipede in G and suppose that case (SQ1) applies. Let us define graphs G' and $\tilde{G}$ as follows. Let $G' = (G \backslash D) \cup (D \cap M)$. In other words, G' is a subgraph of G obtained by deleting the "superfluous" bridges in D. To get $\tilde{G}$, add to G' the edges f_1, f_2 as introduced above, and remove bridges $B_4^\circ, \ldots, B_{m-3}^\circ$. Note that $\tilde{M}$ is contained in $\tilde{G}$. The operation of replacing G by $\tilde{G}$ and M by $\tilde{M}$ is called *squashing* of the millipede M. For convenience, the replacement of M by the bounded obstruction Ω obtained in (SQ2) is also called *squashing* of M. In this case, G remains unchanged ($\tilde{G} = G' = G$) but M is replaced by a bounded obstruction Ω that obstructs extensions of the same embedding of K as M does. It is important that any obstruction for embedding extensions of M–nonsingular embeddings of K to $\tilde{G}$ or to G' is also an obstruction for G.

Suppose again that (SQ1) applies for M. For $i = 2, \ldots, m-1$, let P_i be a path in B_i° joining l_i and r_i (internally disjoint from K). By Lemma 4.3 we can define a path e' from l_4 to r_{m-3} as follows. The path e' starts with P_4, continues on e from r_4 to l_6, uses P_6, the segment on e from r_6 to l_8, etc. It stops either with P_{m-3} (if m is odd), or with P_{m-4} and the segment from r_{m-4} to r_{m-3} (if m is even). One can define similarly a path e'' from l_4 to r_{m-3} which uses the paths $P_5, P_7, \ldots$ and the corresponding segments on e. By Lemma 4.3, e' and e'' are internally disjoint. Let us now change G, K, and M as follows. First, replace in K the segment on e from l_4 to r_{m-3} by e' and denote the obtained subgraph of G by $\overline{K}$. Denote by $\overline{M}$ the subgraph of $G - E(\overline{K})$ composed of $B_1^\circ, B_2^\circ, B_3^\circ, B_{m-2}^\circ, B_{m-1}^\circ, B_m^\circ$ together with the path e''. Finally, let $\overline{G}$ be the subgraph of G obtained by replacing $M \cup D$ and the segment of e from l_4 to r_{m-3} with $\overline{M} \cup e'$. Note that numerous K–bridges in $M \cup D$ have been replaced by only 5 or 6 $\overline{K}$–bridges in $\overline{G}$ (depending on the parity of m). The performed operation which replaces G by $\overline{G}$, K by $\overline{K}$, and M by $\overline{M}$ is called the *compression* of M. It gives rise to a bounded subgraph of $M \cup e$ which has the same extension properties as $M \cup e$. Given an embedding of K, we will consider the embedding of $\overline{K}$ which is unchanged on $K \cap \overline{K}$ and has e' embedded in the same way as the corresponding segment of e in K. For convenience, the replacement of M by a bounded obstruction Ω in (SQ2) (and leaving K and G unchanged) will also be referred to as a *compression* of M.

Proposition 4.5 *Let M be a millipede with respect to certain embedding of K such that (SQ1) applies. Let $\tilde{G}$ and $\overline{G}$ obtained after squashing and compression of M, respectively. Suppose that we have another M–nonsingular embedding ϕ of K. Then every embedding extension of ϕ to G gives rise to embeddings of $\tilde{G}$ and $\overline{G}$ that coincide on the intersection of G with $\tilde{G}$, or $\overline{G}$, respectively. Conversely, having an embedding extension of ϕ to $\tilde{G}$ (or $\overline{G}$), the embedding of $G \cap \tilde{G}$ (or of $\overline{G}$, respectively) can be extended to an embedding of G.*

Proof. Easy to see since embedding extensions of M° and of central parts of $\tilde{M}$ and $\overline{M}$ are the same as under the original embedding of K with respect to which the millipede M is defined. $\square$

Proposition 4.5 will be used in Sections 6 and 8 where different embeddings of K will be considered. A millipede for one of the embeddings will not obstruct extensions of other embeddings of K. But after squashing or compressing we will be able to pretend that the obstruction M is bounded (since M turns into $\tilde{M}$, or $\overline{M}$). On the other hand, in case of squashing (case (SQ1)), the edges f_1, f_2 in the obstructions defined this way have to be replaced by $B_4^\circ \cup \cdots \cup B_{m-3}^\circ$ at the very end.

5 2–Möbius band embedding extension problem

The 2–Möbius band embedding extension problem has been solved by Juvan and Mohar in [13]. To state the next result in a more compact form, we introduce the following notation. If $\Omega \subseteq G - E(K)$ contains a millipede $M = B_1^\circ \cup B_2^\circ \cup \cdots \cup B_m^\circ$, then let $b^\circ(\Omega) = b(\Omega \backslash (B_4^\circ \cup B_5^\circ \cup \cdots \cup B_{m-3}^\circ))$. If Ω does not contain millipedes, then $b^\circ(\Omega) = b(\Omega)$. Let β_0 be the maximal size $b(B)$ over all bridges B with respect to this embedding extension problem. By the results of [16] we can in linear time replace all bridges B by their subgraphs $\tilde{B} \subseteq B$ such that $b(\tilde{B}) \leq 13$ and the embedding extension problem with respect to new bridges is equivalent to the original one. Therefore we can assume that $\beta_0 \leq 13$.

Theorem 5.1 *([13]) Let $K = C \cup P_1 \cup P_2$ be a subgraph of a graph G for the 2–Möbius band embedding extension problem. Suppose that no K–bridge in G is local on one of the paths P_1, P_2. There is a linear time algorithm that either finds an embedding extension of K to G, or returns an obstruction Ω for embedding extendibility. In the latter case, Ω is either bounded and contains at most 13 bridges, or it is a millipede based on one of the paths P_1, P_2 and with apex on the other path. Consequently,*

$$b^\circ(\Omega) \leq 13\beta_0 \,.$$

Theorem 5.1 shows that there is a smallest number $\beta_2 \leq 13$ such that for any 2–Möbius band embedding extension problem, there is an obstruction Ω which is either bounded and contains at most β_2 bridges or it is a millipede. By repeating the algorithm of Theorem 5.1 on bridge-deleted subgraphs of Ω we can achieve that our algorithm actually finds such an obstruction with $b^\circ(\Omega) \leq \beta_0\beta_2$.

6 Projective plane obstructions

It is known [1, 6] that there are exactly 103 forbidden subgraphs for graphs being embeddable in the projective plane. We have the following extension of the algorithm in [14].

Theorem 6.1 *There is a linear time algorithm that for a given graph G either finds an embedding of G in the projective plane, or exhibits a subgraph Ω of G homeomorphic to one of the 103 forbidden subgraphs for embeddability in the projective plane. In particular, the branch size of Ω is at most 22.*

Proof. To be somewhat shorter in the following explanation, we use the algorithm of [14] to find an embedding of G in the projective plane if such an embedding exists (although the following algorithm does the same job). Thus we assume from now on that G cannot be embedded in the projective plane. We will show how to find a bounded obstruction. It is clear that once we have a bounded obstruction, we can get a minimal one (i.e., one among the 103 forbidden subgraphs) in constant time by successively removing superfluous branches (using the algorithm of [14]). It follows by [1, 6] that the branch size of the obtained obstruction will be at most 22.

Suppose now that G cannot be embedded in the projective plane. The procedure to find a bounded obstruction $\Omega \subseteq G$ will follow the algorithm of [14]. As the first step we find a Kuratowski subgraph K in G. Then we reduce the problem to 3-connected case by replacing G with the 3-connected component containing K. If this component is projective planar, then another 3-connected component contains a Kuratowski subgraph K' and the obvious combination of K and K' gives rise to a bounded projective plane obstruction. Similarly, we may assume that K has no local bridges. They can be either removed, or a bounded obstruction is found; cf. [14].

As the next step we replace every bridge B of K in G by a bounded bridge $\tilde{B} \subseteq B$ such that the following holds. If $B_1, \ldots, B_k$ is an arbitrary set of K–bridges, then any embedding of $K \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_k$ in the projective plane can be extended to an embedding of $K \cup B_1 \cup \cdots \cup B_k$. Such subgraphs $\tilde{B}$ can be obtained in linear time as shown in [16, Corollary 3.6].

After replacing every K–bridge B in G by the corresponding subgraph $\tilde{B}$, we get a subgraph of G that also cannot be embedded in the projective plane (by the property explained above). Replace G by the obtained graph and note that after doing that, all K–bridges in G are bounded.

Let $\mathcal{B}_0$ be the set of bridges of K whose attachments to K are not limited to two of the branches of K. The number of bridges in $\mathcal{B}_0$ is bounded if $K \cup \mathcal{B}_0$ admits an embedding in the projective plane. It can be shown that every bridge in $\mathcal{B}_0$ is embeddable in at most one ϕ–face for any embedding ϕ of K in the projective plane. Thus, a very rough estimate for the number of such bridges is one bridge for each triple of branches or main vertices of K. All together this is less than $\binom{15}{3}$. If $\mathcal{B}_0$ contains more bridges, they give rise to a bounded obstruction. Otherwise, we may assume that $\mathcal{B}_0$ is bounded.

There is only a bounded number of essentially different embeddings of K in the projective plane. Now, for every such embedding ϕ of K, we will find an obstruction $\Omega(\phi)$ for embedding extensions of K to G. We will show that $\Omega(\phi)$ will be equal to one of the following:

(a) A bridge B that is not embeddable in any of the ϕ–faces of K, or a pair of bridges embeddable in exactly one ϕ–face of K and overlapping in that face.

(b) A 2-Möbius band obstruction with respect to two faces F, F' of K.

(c) Union of four bridges $B_1(F'), B_2(F'), B_1(F''), B_2(F'')$ (see the remaining part of the proof for their definition) together with at most four additional bridges.

(d) A combination of (b) and (c): Union of bridges $B_1(F'), B_2(F'), B_1(F''), B_2(F'')$ together with a 2–Möbius band obstruction with respect to faces F and F' where all bridges that are overlapping in F with $B_1(F'')$ and $B_2(F'')$ are considered as being uniquely embeddable in F'.

A linear time algorithm that discovers such an obstruction is as follows.

Fix an embedding ϕ of K in the projective plane and try to extend it to G as follows.

Step 1. For each face of K, determine which K–bridges can be embedded in this face. This can be done by testing for planarity in linear time since K has only a bounded number of faces. If some of the bridges cannot be embedded in any face of K, we have (a). Next, we check for each face F of K if all bridges that are embeddable only in F can be simultaneously embedded in F. This test can be done in linear time by applying Theorem 2.1, and we either get all positive answers, or an obstruction of type (a) in which case we stop.

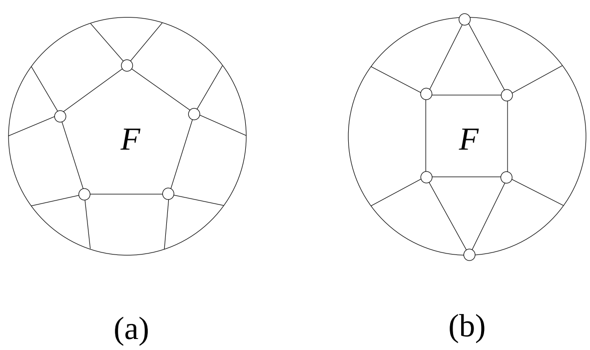

Figure 5: Embeddings of K_5

Step 2. We will assign to each of K–bridges in which of the faces of K it should be embedded. After making the decision for all of the bridges, we will perform a planarity test (Theorem 2.1) for the bridges in each of the faces to check whether an embedding

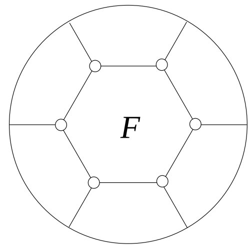

Figure 6: The embedding of $K_{3,3}$ in the projective plane

of G of the determined kind exists, or not. From now on we only discuss the way how to split the bridges among the faces of K. First of all, our job is trivial for K–bridges in $\mathcal{B}_0$ and for all other bridges which were found in Step 1 to have embeddings in a single face. Since K has no local bridges, it is easy to see that the remaining bridges all have exactly two distinct embeddings with respect to K. Let $\mathcal{B}_1$ be the set of these bridges. Denote by F the ϕ–face of K as designated in Figures 5 and 6 where all types of embeddings of K in the projective plane are exhibited. It is easy to see that one of the two ϕ–faces allowing an embedding of a bridge from $\mathcal{B}_1$ is equal to F. We say that bridges in $\mathcal{B}_1$ that admit embeddings into faces F and F' are of *type* F'.

Step 3. For each face $F' \neq F$ of K, choose all bridges that have been determined to be in F', and add all bridges of type F'. By testing for disk obstructions (Theorem 2.1), we can check if all these bridges can be simultaneously embedded in F'. If this happens, then we can embed all of them in F' since they will not block any of the remaining bridges. (In other words, we can eliminate these bridges from G.) If not, then in every embedding of these bridges at least one of the bridges of type F' will be embedded in F. Let us select an overlapping pair $B_1 = B_1(F')$ and $B_2 = B_2(F')$. Under any embedding of $B_1 \cup B_2$ extending the embedding of K, either B_1 or B_2 will lie in F.

Step 4. For each face $F' \neq F$ for which the above test was not successful we check if all bridges, that have been assigned to F or to F' together with the bridges of type F', can be simultaneously embedded in $F \cup F'$. This can be done by solving a 2-Möbius band embedding extension problem (Theorem 5.1). We call this procedure (F, F')*–test*. Getting an obstruction in an (F, F')–test, we have (b). Otherwise, we assume that the (F, F')–tests are all positive.

If K is homeomorphic to K_5, its embeddings in the projective plane are as shown in Figure 5. If K is embedded as in Figure 5(a), then the (F, F')–test is required for at most one face $F' \neq F$. (Otherwise we have (c).) An embedding is obtained if this test is positive. Since we know that G has no embeddings in the projective plane, this is not possible. Thus we have an obstruction of type (b). If K is embedded as in Figure 5(b), then there is only one face F' which might need to be tested for. The result is the same as above.

The remaining case is when K is homeomorphic to $K_{3,3}$. Up to symmetries, K is embedded as in Figure 6. Now, the situation is slightly more complicated. If there is just one (F, F')–test that is necessary, we get an obstruction as in above cases. Suppose that all three tests (for faces F', F'', F''') are necessary. Suppose that $\Omega' = B_1(F') \cup B_2(F') \cup$

$B_1(F'') \cup B_2(F'') \cup B_1(F''') \cup B_2(F''')$ is not an obstruction (otherwise we have (c)). Then $B_1(F'), B_1(F''), B_1(F''')$ are just edges joining the triple of main vertices of K in one of the bipartition classes of $K_{3,3}$. Moreover, Ω' has a unique embedding extending the embedding of K, and under this embedding the three edges are all embedded in F. Now it is easy to see that every remaining bridge has at most one face of $K \cup \Omega'$ that it can be embedded in. An obstruction is obtained as a non-embeddable bridge, or a pair of overlapping bridges added to Ω'. This yields case (c).

Suppose now that there are two necessary tests, say for the faces F' and F''. Then we have an obstruction of type (c) unless the only bridges of type F' or F'' that will be embedded in F will be attached at one side only to a main vertex of K, say x, and to a branch e' (respectively e'') on the other side of the face F. Any embedding of $B_1(F'') \cup B_2(F'')$ forces all the bridges of type F', that do not attach as required, to be in F'. Similarly for F''. If x is the same for all embeddings of $B_1(F'), B_2(F'), B_1(F''), B_2(F'')$, then we solve the embedding extension problem by recalling to two independent 2-Möbius band embedding extension problems. First we solve the problem for F, F', and then for F, F''. In each of these tests, the bridges that are not attached only to x and e' (x and e'', respectively) are considered to be embeddable only in F' (respectively, in F''). In at least one of these tests we will get an obstruction, and its union with $B_1(F'), B_2(F'), B_1(F'')$, and $B_2(F'')$ is an obstruction for extending ϕ to an embedding of G. This type of an obstruction is of form (d). It is also possible that $\Omega' = B_1(F') \cup B_2(F') \cup B_1(F'') \cup B_2(F'')$ has embeddings with different vertices playing the role of x. It turns out that for each of the two possible embeddings of Ω', the projective plane is dissected in such a way that every remaining bridge is embeddable in at most one of the faces. We get an obstruction by means of one or two bridges added to Ω' (for each embedding). All together we have at most 4 bridges in addition to Ω', so we have (c).

To get obstructions of types (b), or (d), we need to apply Theorem 5.1. If we get a millipede, say M, in $\Omega(\phi)$, we perform the compression of M. In case (SQ1) when the compression changes the subgraph K, we should not forget about the following. If some previously obtained obstruction $\Omega(\phi')$ contains some bridge that is in D (see (SQ1) for the definition of D), replace the K–bridges in $\Omega(\phi')$ that are contained in D by the compressed millipede $\overline{M}$. By Proposition 4.5, the new $\Omega(\phi')$ still obstructs embedding extensions of ϕ'.

Finally, let Ω be the union of all obstructions $\Omega(\phi)$. Because of compression and since the K–bridges in G are bounded, Ω is bounded. It is clear that $K \cup \Omega$ is an obstruction for embeddability of G in the projective plane. Since this obstruction is bounded, we are done. It is easy to check that the running time of the algorithm is linear. □

7 0–Möbius band obstructions

Theorem 7.1 *Let C be a cycle of a graph G that is embedded on the boundary of the Möbius band. If there is no embedding extension to G, then $G - E(C)$ contains an obstruction Ω with $b(\Omega) \leq 48$. Such an obstruction can be found in linear time.*

Proof. Let $\tilde{G}$ be the auxiliary graph for the Möbius band embedding extension problem. By using Theorem 6.1, we find a bounded obstruction $\Lambda \subseteq \tilde{G}$ for embeddability of $\tilde{G}$ in the projective plane. By Lemma 3.1, $G_1 = (\Lambda \cup C) \cap G \subseteq G$ determines an obstruction $\Omega_1 = G_1 - E(C)$ for our 0–Möbius band embedding extension problem. Unfortunately, $b(\Omega_1)$ can be large. To make it bounded we employ the following procedure.

If G_1 contains a component or a block Q that does not contain C, then Q is also a component or a block of Λ. By minimality of Λ, we either have $Q = \Lambda$ (in which case we take $\Omega = \Lambda$ and stop), or Q is a Kuratowski subgraph. It is easy to see that $G_1 \backslash Q$ contains a subgraph H such that $H \cup C$ cannot be embedded in the disk with C on its boundary. Then $H \cup Q$ is an obstruction for our 0–Möbius band problem. We apply Theorem 2.1 on $(G_1 \backslash Q, C)$ to find a disk obstruction $H \subseteq G_1 \backslash Q$. Then $\Omega = H \cup Q$ is a required obstruction with $b(\Omega) \leq b(H) + b(Q) + 2 \leq 24$.

Excluding the above case, G_1 is 2-connected. Apply Theorem 2.1 to find a disk obstruction $D_0 \subseteq G_1 - E(C)$. If D_0 is a Kuratowski subgraph in a 3-connected component of the disk auxiliary graph of G_1, we also find two disjoint paths connecting D_0 with C. (Such paths, possibly of length 0, exist by Menger's Theorem and they can be found in linear time by standard flow techniques.) We add these paths to D_0 and denote the obtained graph by D_0 as well.

Let B be a C–bridge in G_1 that is disjoint from D_0 (including vertices of attachment). If B is attached to distinct components of $C - D_0$, let P be a path in B joining such distinct parts. If D_0 is a tripod or a Kuratowski subgraph obstruction, then $\Omega = D_0 \cup P$ is a bounded obstruction. Otherwise, D_0 is a pair of disjoint crossing paths. Note that $\Omega' = D_0 \cup P$ consists of three disjoint paths based on C. By the standard method of [14] we can change the branches of Ω' into paths joining the same pairs of vertices so that we have no local $(C \cup \Omega')$–bridges in G_1 (or we get a bounded obstruction). If Ω' is not an obstruction, we apply Theorem 3.2 with $k = 3$ and $K = C \cup \Omega'$ to get a required bounded obstruction.

From now on we may assume that every C–bridge in G_1 disjoint from D_0 is attached to just one of components of $C - D_0$. Consider one of components of $C - D_0$ and the set $\mathcal{B}$ of all C–bridges disjoint from D_0 that are attached to this component. Any embedding of D_0 in the Möbius band forces $\mathcal{B}$ to be embedded "locally". By using Theorem 2.1, we either find an obstruction D_1 (in which case $\Omega = D_0 \cup D_1$ is a required obstruction with $b(\Omega) \leq 23$), or we get a "local" embedding of $\mathcal{B}$. If this embedding obstructs any of the remaining C–bridges (this can be checked by using Theorem 2.1), we easily get a bounded 0–Möbius band obstruction. Otherwise we can remove $\mathcal{B}$ from G_1 and the resulting graph still has no embedding extension.

If the above tests have not ended up with a bounded obstruction, we are left with the graph $D \subseteq G_1 - E(C)$ consisting of D_0 together with all C–bridges in G_1 that are not disjoint from D_0. Every branch of Λ, except possibly two, gives rise to at most two branches in D, and possible exceptions can contribute together at most four additional branches. By Theorem 6.1 we thus have $b(D) \leq 48$. Since D is an obstruction, we are done. $\square$

8 1–Möbius band embedding extension problem

Let F be the face of the Möbius band Σ shown in Figure 7, where S and S' determine the boundary of the Möbius band, and the horizontal sides denoted by π ($\pi \subset \partial F$) are assumed to be identified (after a twist of one of them). Let G be a graph and let K be a subgraph of G that is embedded in Σ so that F is its face. In particular, $\alpha, \beta \in V(K)$ and S, S', π are branches of K joining α and β. It will be convenient to treat S and S' as open segments (i.e., $\alpha, \beta \notin S, S'$) and to have π as a closed branch, i.e., $\alpha, \beta \in \pi$. Denote by $\mathcal{B}$ the set of K–bridges in G. First, we will present a solution for the 1–Möbius band embedding extension problem under two additional assumptions:

(i) No bridge $B \in \mathcal{B}$ has an attachment in S'.

(ii) Every bridge $B \in \mathcal{B}$ has an attachment in S (no bridge is local on π).

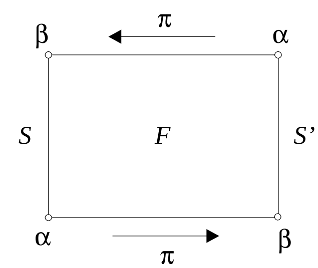

Figure 7: The dissected Möbius band

Millipedes for 1–Möbius band embedding extension problems are defined similarly as in case of 2–Möbius band embedding extension problems. They are subgraphs of $G - E(K)$ consisting of a path P joining two vertices of K and a thin or a skew millipede M with respect to the graph $P \cup K \subseteq G$ as the 2–Möbius band subgraph where M is based on K and has its apex on P.

Suppose that we have a 1–Möbius band embedding extension problem. Let $\beta_0 \leq 13$ and $\beta_2 \leq 13$ be as defined in Section 5.

Lemma 8.1 *There is a linear time algorithm that, given G and K satisfying the above assumptions (i) and (ii), either*

(a) exhibits an embedding extension of K to G in Σ, or

(b) returns an obstruction Ω for embedding extensions of K to G in Σ. The obstruction Ω contains at most one millipede (based on π) and

$$b^\circ(\Omega) \leq 8 + 8\beta_0(\beta_2 + 2)\,.$$

Proof. First of all, we test for an embedding extension by using the algorithm of [14]. If we get (a), then we stop. Otherwise we know that no embedding extension exists. Then we change G so that it is 3-connected modulo K (cf. [14]). If this operation is not successful, we get a bounded obstruction in the form of a Kuratowski subgraph in one of the K–bridges and we stop. (Here we need (ii) in order to show that this is an obstruction.) Next, we embed the K–bridges that have no attachment on π. This is achieved as follows. If B has all its attachments in S, it has essentially unique embedding in F. Such an embedding can be found (simultaneously for all such bridges) in linear time by using a planarity test. If such a bridge B cannot be embedded in F, we get a required obstruction. The same holds if B can be embedded in F but it overlaps on S with another K–bridge. If we stopped up till now, we have an obstruction Ω with $b(\Omega) \leq 12$ (Theorem 2.1). Therefore we may assume that all the bridges that are local on S are embedded in F and that their embedding does not comply with the rest. Hence, we can "forget" about these bridges and continue with the remaining graph.

Next we try to extend the embedding of K to G in such a way that no foot will be attached to π at the upper occurrence of π on ∂F. Such an embedding will be called a *lower embedding*. (Similarly we define *upper embeddings*.) To get an obstruction for lower embeddings we can use Theorem 2.1. Since G is 3-connected modulo K and since we know that no embedding extension exists, we get either a pair of disjoint crossing paths, or a tripod. Note that the obtained obstruction T obstructs also upper embeddings. In case of disjoint crossing paths, it is easy to see that they can be changed either into a required obstruction (if both ends of one of the paths are on S), into a pair of paths attached to K as shown in Figure 8, or into a dipod which is attached to ∂F as shown in Figure 9 (possibly $s = t$). In case when we have a tripod, we do the following. If T is attached at least twice to S, then it is an obstruction for embedding extensions of K to G and we stop. If it is attached three times to π, we take a path P disjoint from π that joins S with T. Then $T \cup P$ either contains a pair of disjoint crossing paths (this case was considered above) or a tripod that is attached to S. Thus, we may assume that T is attached once to S and twice to π. If T cannot be embedded in F, we stop. Otherwise, at least one of its attachment paths on π is degenerate. Up to symmetries, we have the case of Figure 10 (possibly $t = s$ and/or $b = a$). In case (a) of Figure 10 we automatically have $b = a$. All vertices shown in Figures 8–10 are distinct except that t and s can coincide and that we may have $b = a$ in Figure 10(b).

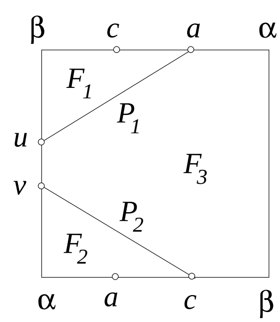

Figure 8: Disjoint crossing paths in F

In each of the above three possibilities for the lower embedding obstruction T (Figures 8, 9, 10), we will assume that a is as close to α as possible, and that c is as close to β as possible. More precisely, in the first case we will assume that no $(K \cup T)$–bridge in G with an attachment on the open branch au is attached to π closer to α than a. In case of the dipod, no $(K \cup T)$–bridge in G with an attachment at u or on one of the open branches au, cu, or tu is attached to π closer to α than a. Similarly in case of tripods.

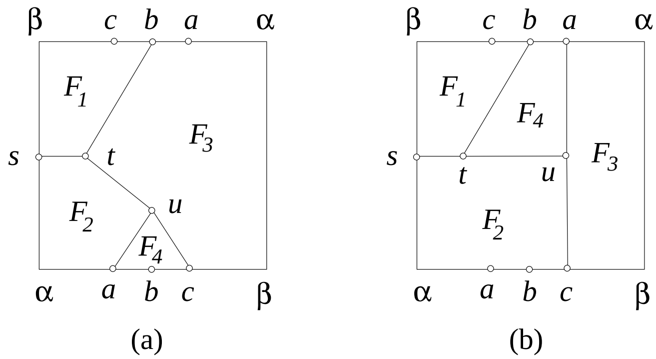

Figure 9: Embeddings of a dipod

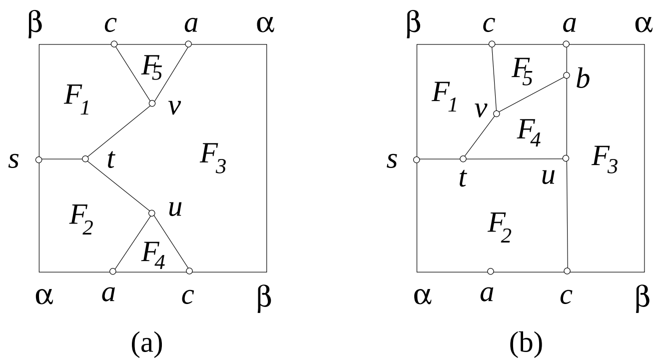

Figure 10: Embeddings of a tripod

Similar assumptions can then be made for c as well. This can be achieved in linear time by standard graph search procedures. (In case of a tripod we are allowed to change T into a dipod.) We will refer to these properties as *extremality* of T.

Our next goal is to change T so that $K \cup T$ has no local bridges in G. Since G is 3-connected modulo K, it is also 3-connected modulo $K \cup T$, and thus the algorithm of [14] can be used to eliminate local bridges.

Let $K' = K \cup T$. We will fix an embedding of T in F and denote the faces of K' by F_j ($j = 1, 2, \ldots, p$) as shown in Figure 8 ($p = 3$), Figure 9 ($p = 4$), or Figure 10 ($p = 5$), depending on T and the chosen embedding of T. We split K'–bridges in G in the following classes. A bridge is a 0–*bridge* if it cannot be embedded in any of the faces F_j. It is a j–*bridge* ($1 \le j \le p$) if it can be embedded in F_j but cannot be embedded in other faces. It is an ij–*bridge* ($1 \le i < j \le p$) if it can be embedded in F_i and in F_j. Since there are no local K'–bridges, none of the bridges is embeddable in three faces. Thus the above classification determines a partition of K'–bridges. It also follows that every ij–bridge is attached to π.

We can replace every K'–bridge B by a bounded subgraph of B such that any embedding extension to the reduced graph G can be (trivially) extended to an embedding extension of the original graph. Such a bounded subgraph B' of B can be obtained, for example, by using the results of [16, Corollary 3.6]. By the same result, $b(B') \le \beta_0$. (Here $b(B')$ is considered with respect to K'.) We will assume from now on that K'–bridges in G are bounded and that their sizes fulfil the above bound.

Several times we will perform the following operation called the *F_j–test*. We will try to embed in F_j all the j–bridges together with all ij– and ji–bridges ($i \in \{1,\dots,p\}\backslash\{j\}$). If an embedding is found, the test is *positive*. Otherwise, an obstruction for such an embedding will be returned in the form of two bridges that are overlapping in F_j. The two bridges will be denoted by $B_1(F_j)$ and $B_2(F_j)$. An F_j–test can be performed in linear time by using Theorem 2.1. Similarly, applying Theorem 5.1 to solve the 2–Möbius band embedding extension problem with respect to faces F_i and F_j and all i–bridges, j–bridges, ij–bridges, will be called an *(F_i, F_j)–test*. It will either return an embedding in which case the test will be declared as *positive*, or we will get a 2–Möbius band obstruction $\Omega(F_i, F_j)$. We make sure that $b^\circ(\Omega(F_i, F_j)) \le \beta_0\beta_2$ (cf. the remark after Theorem 5.1). Again, $b^\circ(\cdot)$ is considered with respect to K'. If $\Omega(F_i, F_j)$ contains a millipede M that is not based on π, we are allowed to compress M. Note that in case (SQ1) when compression changes a segment Q of a branch of T, no K'–bridge (except those in D that are removed anyway) is attached to Q. We will consider the compression as a part of the (F_i, F_j)–test. Similarly, if M is based on π, we perform squashing of M.

Now, we will show for each of the three cases for T how to get a required obstruction.

Case 1: T is a pair of disjoint crossing paths. In this case, K' has unique embedding extending the embedding of K (Figure 8). If there is a 0-bridge B, we are done by taking $\Omega = T \cup B$. By the above, $b(\Omega) \le 2\beta_0 + 4$. (The factor 2 at β_0 is added since each foot of B can produce a new branch of Ω contained in T.) Otherwise, we perform the F_1–test and the F_2–test. If both are positive, the obstruction is among the 3-bridges — two of them overlap. Denote the union of these two bridges by Ω_1. They can be found by means of Theorem 2.1. If both test are negative, then let $\Omega_1 = B_1(F_1) \cup B_2(F_1) \cup B_1(F_2) \cup B_2(F_2)$. In both cases, $\Omega = T \cup \Omega_1$ is an obstruction with $b(\Omega) \le 8\beta_0 + 4$.

As for the remaining possibility we may assume that the F_2–test was positive and that F_1–test was negative. The 2– and 23–bridges can be embedded in F_2 where they do not interfere with the remaining bridges. They can thus be omitted. An obstruction $\Omega_1 = \Omega(F_1, F_3)$ can now be found by performing (F_1, F_3)–test.

Case 2. T is a dipod. In this case, there are four ways for embedding T in F. Two of them are shown in Figure 9. The other two are obtained by up/down symmetry. We will first show how to get obstructions for extending the embedding of K' in each of the four possible cases. They will be denoted by $\Omega_1, \dots, \Omega_4$, respectively. Clearly, $\Omega = T \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$ is an obstruction for embedding extensions of K to G. We will take care in choosing $\Omega_1, \dots, \Omega_4$ so that Ω will be as required. In particular, Ω will contain at most one millipede. As far as their size is concerned, it suffices to show that $b^\circ(\Omega_i) \le 2\beta_0 + \beta_0\beta_2$ (the size measured with respect to K'). Moreover, Ω_i will be a union of K'–bridges. Thus, $b^\circ(\Omega) \le b(T) + 2\,b^\circ(\Omega_1) + \cdots + 2\,b^\circ(\Omega_4)$, and the required bound on the size of the obstruction Ω follows.

Fix one of the four possible embeddings of K'. Suppose first that the embedding is the one represented in Figure 9(a). Having a 0-bridge, we get a bounded obstruction for extending this embedding of K'. Suppose that this is not the case. Then there are no 12–, 14–, or 24–bridges. We perform the F_1–test, the F_2–test, and the F_4–test. If all three tests are positive, then the obstruction is among 3-bridges. An overlapping pair of 3-bridges gives rise to an obstruction Ω_1 with $b(\Omega_1) \le 2\beta_0$. If just one of the tests fails, say F_j–test, then we get an obstruction Ω_1 by (F_j, F_3)–test. Then $b^\circ(\Omega_1) \le \beta_0\beta_2$. If Ω_1 contains a millipede, it is based either on $c\beta$ (if $j = 1$), αa (if $j = 2$), or on ab (if $j = 4$). If F_1–test and F_4–test are negative, then $\Omega_1 = B_1(F_1) \cup B_2(F_1) \cup B_1(F_4) \cup B_2(F_4)$ is an obstruction with $b(\Omega_1) \le 4\beta_0$.

Suppose now that F_4–test is positive while F_1–test and F_2–test are negative. In this case, any embedding of $B_1(F_2) \cup B_2(F_2)$ blocks F_3-embeddability of any 13–bridge that is attached to an interior point of the branch tb. Thus, all such 13–bridges should be embedded in F_1. Similarly, all 23–bridges attached to the interior of tu or to u must be in F_2. Now we solve the (F_1, F_3)–test under these new restrictions. (We will refer to such embedding extension problems as *restricted* 2–Möbius band problems. It is easy to see how one should define auxiliary graphs in order to restrict embeddability of bridges in question so that Lemma 5.1 can be used for its solution.) Obtaining an obstruction Ω_1' for this 2–Möbius band problem, we let $\Omega_1 = \Omega_1' \cup B_1(F_2) \cup B_2(F_2)$. Clearly, $b^\circ(\Omega_1) \le 2\beta_0 + \beta_0\beta_2$ as required. Possible millipede is based on $c\beta$ and has apex t. If the above (F_1, F_3)–test is positive, we solve the restricted 2–Möbius band problem in $F_2 \cup F_3$. Here we definitely get an obstruction and conclude as explained above.

The only remaining possibility is when the F_1–test is positive while F_2–test and F_4–test are negative. If $\Omega_1' = B_1(F_2) \cup B_2(F_2) \cup B_1(F_4) \cup B_2(F_4)$ is an obstruction, we are done. If not, consider an embedding of Ω_1'. It is easy to see (by extremality of T and since there are no local K'–bridges) that bridges of Ω_1' that are in F_3 are attached to π only at vertex a. We are left to solve two restricted 2–Möbius band problems (only bridges attached to π only at a are allowed in both faces). One of the two restricted problems must fail. Suppose that this is the test with F_3 and F_4. The corresponding obstruction, say Ω_{34}, contains bridges that overlap in F_4. These bridges can be taken as $B_1(F_4)$ and $B_2(F_4)$. Consequently, $\Omega_1 = B_1(F_2) \cup B_2(F_1) \cup \Omega_{34}$ is an obstruction whose size is bounded as required.

Suppose now that the embedding of K' is as represented in Figure 9(b). Again, we may assume that there are no 0-bridges. There are no 13–, 14– or 34–bridges, but we can have 12–, 23–, and 24–bridges. We first perform F_3–test. If it is negative, $B_1(F_3) \cup B_2(F_3)$ obstruct embeddings of 12–bridges in F_2. They also block embeddings of 24–bridges in F_2 except those whose only attachment on π is the vertex a. If F_1–test is negative, we can take $\Omega_2 = B_1(F_1) \cup B_2(F_1) \cup B_1(F_3) \cup B_2(F_3)$ and we have $b(\Omega_2) \le 4\beta_0$. Otherwise, we perform F_4–test. If negative, we do also the (F_2, F_4)–test where only non-blocked 24–bridges are allowed in F_2. If an obstruction Ω_2' is found, then $\Omega_2 = \Omega_2' \cup B_1(F_3) \cup B_2(F_3)$ is a required obstruction. If the F_4–test or the restricted (F_2, F_4)–test is positive, an obstruction Ω_2 will be found by solving the corresponding (F_2, F_3)–test. By extremality of T, every 23–bridge is attached to π only at the vertex a. Therefore the (F_2, F_3)–test is independent of the embedding obtained with (F_2, F_4)–test.

The other possibility is when the F_3–test was positive. Then we perform F_1–test and F_4–test. If both are positive, the obstruction Ω_2 can be found in the form of two overlapping 2–bridges. If just one is positive, we reduce the problem to the (F_2, F_4)–test or (F_2, F_1)–test, and we get an obstruction of the 2–Möbius band problem. Finally, if F_1–test and F_4–test are both negative, we take $L = B_1(F_1) \cup B_2(F_1) \cup B_1(F_4) \cup B_2(F_4)$ as a “blockage” and then we are left to solve two restricted 2-Möbius band problems in

$F_1 \cup F_2$ and $F_4 \cup F_2$, respectively. One of them fails and gives us an obstruction Ω_2 with $b^\circ(\Omega_2) \leq 2\beta_0 + \beta_0\beta_2$. The above restricted 2-Möbius band problems depend on L. If L itself is not an obstruction, then there are three possibilities:

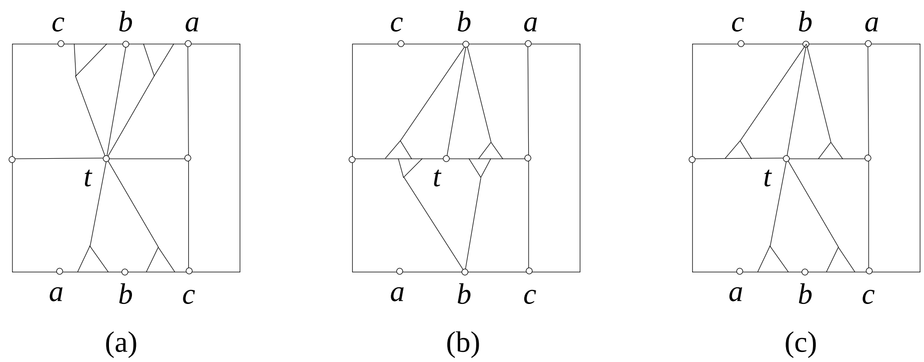

Figure 11: Cases for L

(a) In every embedding of L, the bridge $B_1(F_4)$ or $B_2(F_4)$ that is in F_2 is attached to tu only at t (cf. Figure 11(a) where bridges in F_1 and in F_4 can have further attachments than just those shown).

(b) In every embedding of L, the bridge $B_1(F_4)$ or $B_2(F_4)$ that is in F_2 is attached to ab only at b (cf. Figure 11(b)).

(c) Cases from (a) and (b) appear under different embeddings of L (cf. Figure 11(c)).

Cases (a) and (b) are handled easily (restricted 2-Möbius band problems). In case (c), L has to be exactly as shown in Figure 11(c) except that they do not need to have three vertices of attachment. L has two embeddings and under any of them, all bridges have at most one face where they can go. An obstruction will be obtained by means of two (or one) overlapping bridges for each embedding of L. In this case, $b(\Omega_2) \leq 8\beta_0$.

We get obstructions $\Omega_3 \subseteq G - E(K')$ obstructing extensions of the embedding of K' symmetric to the one in Figure 9(a), and Ω_4 (embedding of K' symmetric to the one in Figure 9(b)) in the same way as above.

The final outcome are obstructions $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ for extending each of the four embeddings of K'. It is possible that Ω_i contains a compressed millipede $\overline{M}$ (where K' has been changed). If this is the case, any K'–bridge in Ω_j ($j < i$) that intersects the central part of M or is attached to the part of the branch of K' that is changed during the compression is to be replaced by the compressed millipede $\overline{M}$. If there are no such bridges in Ω_j, then the only change that we make in Ω_j is the appropriate change of K'. This way we make sure that K' is the same in all four cases and that the changed obstructions still obstruct corresponding embedding extensions. This change has to be made before proceeding to the $(i+1)$st embedding of T.

We will now explain what steps should be undertaken if we have a millipede in Ω_j ($1 \leq j \leq 4$) in order that the final obstruction Ω will not contain more than one millipede.

In all the following, M will be a millipede and $B_1^\circ, B_2^\circ, \ldots, B_m^\circ$ its bridges. Let us remark that M is a millipede with respect to some embedding of K'. Other embeddings of K' are M–nonsingular but M does not need to obstruct them.

Suppose first that we have a millipede M in Ω_1. If M is based on αa, its apex is t (by extremality of T). We assume that B_2° is attached on π closer to α than B_{m-1}°. In particular, B_2° is attached on π closer to α than a. Thus $B_2^\circ \cup B_3^\circ \cup st$ contains a dipod T' which has some "nicer" properties than T. Suppose that T' is obtained from the H–graph Y of B_2° (a minimal connected subgraph of B_2° containing a foot at t and two extreme feet on π) together with st and a path in B_3° from t to a vertex on π between the attachments of Y (Lemma 4.3). Then we repeat the whole process with T' instead of T. By Lemma 4.4, B_1° has an attachment out of αa and t. By extremality of T and by (i), this attachment is on S or in the interior of the branch st. Thus, B_1° is embeddable only in F_2 and B_1° can serve as a (very small) obstruction Ω_1 for the embedding of $K \cup T'$ as in Figure 9(a). In particular, Ω_1 is not a millipede. If Ω_3 (with respect to the new dipod T') contains a millipede, then this millipede obstructs both embeddings corresponding to Figure 9(b) as well. (It seems that there is an exception: if there is a millipede with apex u in T'. However, this cannot happen by our choice of T' since such a millipede would be contained in the original bridge B_2° which is too small.) Thus, we can take $\Omega_2 = \Omega_4 = \Omega_3$ so that the union of all four obstructions Ω_j will contain just one millipede. To summarize: If Ω_1 contained a millipede based on αa, we have changed T in such a way that either all four new obstructions contain at most one millipede, or such that none of Ω_1 and Ω_3 contains a millipede.

Having a millipede in Ω_1 based on $c\beta$, we perform similar change. Afterwards we can take $\Omega_3 := B_1^\circ$. We have the same conclusion as above.

A millipede in Ω_1 based on ab has its apex x (or x, y) on uc. By Lemma 4.3, $B_2^\circ \cup B_3^\circ \cup st \cup tu \cup ux$ (plus xy if the millipede is skew) contains a dipod with vertices a and c corresponding to the extreme attachments of B_2° on π. We repeat the search of obstructions with the new dipod. Since B_2° is bounded, a possible millipede in Ω_1 cannot be based on ab. Similarly for Ω_3.

If we succeeded for Ω_1 to contain no millipedes, we perform the same procedure as above on Ω_3 (if necessary). The final conclusion is: either $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ contain just one millipede, or else Ω_1 and Ω_3 do not contain millipedes. In the latter case, we can have millipedes in Ω_2 and Ω_4. Consider a millipede M in Ω_2. M may be based on ab with apex on tu, or based on bc with apex on st. Millipedes based on αa are excluded by extremality of T. We may assume that Ω_2 does not obstruct the fourth embedding (otherwise we take $\Omega_4 = \Omega_2$). Then the apex of M contains t. In this case we can use M to change the path tb so that it uses the millipede and attaches to π as close as possible to a (if M is based on ab), or c (if based on bc). Repeat the whole procedure for obtaining $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ with the new dipod. If we come to this point again, we claim that a and c have not been changed. Since the new dipod satisfies the extremality property, the only place where a and c might have been changed is in the process of eliminating millipedes from Ω_1 or Ω_3. However, in that case not many feet are attached between a and c and hence a millipede cannot be based between a and c. Thus, we know that b is as close to a (respectively, to c) as possible. So, no millipede based on ab (respectively, on bc) can arise. If we have millipedes in Ω_2 and in Ω_4, they are based on the same branch (ab, or bc). If Ω_2 is not an obstruction for the fourth embedding, then either B_1° or B_m° (boundary bridges in the millipede M of Ω_2) is attached to the branch tb. Otherwise, the re-embedding of tb would not change obstruction properties of the millipede. Similarly, the bridge B_1' (say) of the millipede M' in Ω_4 is attached to tb, or else Ω_4 can be used in place of Ω_2. Assume that B_1° and B_1' are attached to tb. Let $L = B_1^\circ \cup B_2^\circ \cup B_3^\circ$ and $L' = B_1' \cup B_2' \cup B_3'$. If $L \cup L'$ obstructs the second embedding, we can take it as Ω_2 and stop. Otherwise, both millipedes have apex t and are based on the same segment of π, say on bc. Consider an embedding of $L \cup L'$ extending the second embedding of K'. Then $B_1^\circ, B_3^\circ, B_1', B_3'$ are embedded in F_1 and B_2°, B_2' are in F_2. After obtaining the millipede M, we performed its squashing. Thus, if $B_2' \neq B_2^\circ$, we have $r_2' \preceq l_2$ (with the obvious meaning of notation). By Lemma 4.3 applied on M' and Lemma 4.4 applied on M, we get $l_3' \prec r_2' \preceq l_2 \prec r_1$. This is a contradiction since then B_3' overlaps in F_1 with B_1°. Consequently, we have $B_2' = B_2^\circ$. Because of squashing of M, it follows that $M^\circ \subseteq M'^\circ$. Hence, we may replace Ω_2 by $M' \cup B_1^\circ \cup B_m^\circ$, and we have reached our goal.

Finally, let $\Omega = T \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$. By the above, Ω contains at most one millipede and $b^\circ(\Omega) \leq 5 + 8(2\beta_0 + \beta_0\beta_2)$.

Case 3. T is a tripod. We will follow the same lines of conclusions as in Case 2. For each of the four possible embeddings of $K' = K \cup T$ (Figure 10 with up/down symmetries), we will find an obstruction Ω_i ($i = 1, 2, 3, 4$). Suppose that we get a millipede in Ω_i. Then $B_2^\circ \cup B_3^\circ$ contains a dipod. Identifying this dipod, we can use Case 2 to get a solution. Thus we will assume (without explicitly mentioning this trick) that no obstructions found in the sequel contain millipedes based on π.

Consider the first embedding of K' (Figure 10(a)). Perform F_i–tests, $i = 1, 2, 4, 5$. If three of the tests are negative, say ith, kth, and jth, then $\Omega_1 = B_1(F_i) \cup B_2(F_i) \cup B_1(F_j) \cup B_2(F_j) \cup B_1(F_k) \cup B_2(F_k)$ is a bounded obstruction. If two of the tests are negative, say the ith and the jth, then $\Omega_1' = B_1(F_i) \cup B_2(F_i) \cup B_1(F_j) \cup B_2(F_j)$ is usually a bounded obstruction. Possible exceptions are only when $\{i, j\} = \{1, 5\}$, $\{2, 4\}$, or $\{1, 2\}$. Suppose that we have such an exception. Consider first the case $\{1, 5\}$. If $B_1(F_5)$ or $B_2(F_5)$ is attached to π at a vertex distinct from a and c, we get a dipod in $T \cup B_1(F_5)$ and we can use Case 2 to get a solution. Similarly, if $B_1(F_1)$ or $B_2(F_1)$ is attached to π at a vertex distinct from c. (Here we also need to use the fact that Ω_1' is not an obstruction to exclude the case for $B_j(F_1)$ being attached only to t and π.) Then it is clear that we are left with two "restricted" 2–Möbius band problems, similarly as we had in Case 2. The same method works for other exceptions when $\{i, j\} = \{2, 4\}$ or $\{1, 2\}$.

If all F_i–tests performed above are positive, an obstruction can be found in the form of two overlapping 3-bridges (using Theorem 2.1). Suppose now that exactly one F_i–test is negative. If this is the F_4–test, then the solution is obtained by performing the (F_3, F_4)–test. If F_1–test is negative, we reduce the problem to the (F_1, F_3)–test. Similarly for the other two possibilities.

Similarly we get a bounded obstruction Ω_3 for extending the third embedding of K' (symmetric to the first one).

We now proceed with the second embedding (Figure 10(b)). Perform F_i–tests, $i = 1, 2, 3, 4, 5$. We may assume that F_5–test is positive (or we have a bounded obstruction). Same with the F_4–test if $b \neq a$. Thus we may assume from now on that $b = a$. (If not, being the same or not will not be important at all.) Suppose now that F_3–test is negative. Then $B_1(F_3) \cup B_2(F_3)$ blocks any 12–bridges being embedded in F_2. Thus the F_1–test must be positive (or we have a bounded obstruction $\Omega_2 = B_1(F_1) \cup B_2(F_1) \cup B_1(F_3) \cup B_2(F_3)$). Then we get an obstruction either in the (F_2, F_3)–test, or in the (F_2, F_4)–test. (These two tests are independent since by extremality of T we may assume that 23–bridges are attached to π only at the vertex a.) It remains to see what happens when the F_3–test is positive. If F_1–test and F_4–test are both positive then we find an obstruction in the form of overlapping 2-bridges by using Theorem 2.1. If just one of these two tests is negative, then Ω_2 is obtained by solving either the (F_1, F_2)–test or the (F_2, F_4)–test. In the remaining case, $B_1(F_1) \cup B_2(F_1) \cup B_1(F_4) \cup B_2(F_4)$ may be an obstruction. If not, then $B_1(F_1)$, say, is attached only to t and c, and $B_1(F_4)$ is attached only to t and a. After embedding these bridges (uniquely), every remaining bridge has at most one face to go. An obstruction is easily found.

The fourth embedding of K' is symmetric to the above.

Finally, let $\Omega = T \cup \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$. By the above, Ω contains no millipedes and $b(\Omega) \leq 8 + 8\beta_0(\beta_2 + 2)$.

It is easy to see that the overall time complexity of the above algorithm can be made to be linear in the number of edges of G. Least obvious spots have been commented during the description of the algorithm. □

By an application of Lemma 8.1 we can solve the general 1–Möbius band embedding extension problem.

Theorem 8.2 *Let G, K be an instance of a 1–Möbius band embedding extension problem, and let $\beta_1 = 11 + 8\beta_0(\beta_2 + 2)$. There is a linear time algorithm that either*

(a) exhibits an embedding extension of K to G in the Möbius band, or

(b) returns an obstruction Ω for embedding extensions of K to G. The obstruction Ω contains at most one millipede (based on $\pi \subset K$) and $b^\circ(\Omega) \leq \beta_1$.

Proof. At the beginning we perform usual reductions to get the case when G is 3-connected modulo K (or we get a very small obstruction). We will distinguish two cases.

Case 1: There exists a path P from S to S' disjoint from π. Such a path can be found in linear time by standard graph search procedures. Change P if necessary so that it contains no local bridges [14]. Then apply Theorem 5.1 to get an embedding (in which case we have (a) and stop), or an obstruction Ω'. If Ω' contains a millipede based on P, we use the compression in order to make Ω' bounded. Let $\Omega = \Omega' \cup P$. Then Ω is an obstruction for embedding extensions. Since $b^\circ(\Omega') \leq \beta_0\beta_2$, we have $b^\circ(\Omega) \leq 2\beta_0\beta_2 + 2$.

Case 2: No path across. K–bridges can be classified as (S)–bridges (those attached to S), or (S')–bridges (attached to S'). Try to get a lower embedding of all (S)–bridges and an upper embedding of all (S')–bridges. If successful, we have (a). Otherwise, suppose that a lower embedding of (S)–bridges does not exist. Let T be an obstruction of one of the types shown in Figures 8, 9, 10. As usual, we may assume that $K \cup T$ does not have local bridges and that all bridges are bounded. If any (S')–bridge is attached to π between a and c, a path in such a bridge together with T forms an obstruction if T is not a dipod. If T is a dipod, let a' be the attachment of (S')–bridges on the open segment ac of π as close to a as possible. Let c' be defined similarly (as close to

c as possible). Let P' be a path in (S')–bridges joining a' with S', and let P'' be a path joining c' and S'. Make sure that $b(P' \cup P'') \le 3$. If $a \prec a' \prec b \prec c' \prec c$, then $\Omega = T \cup P' \cup P''$ is a bounded obstruction. Otherwise, let us first suppose that either $a' \neq b$, or $c' \neq b$. Then $T \cup P' \cup P''$ has unique embedding (or it is an obstruction), and we can get a solution in the same way as explained below for the case when no (S')–bridge is attached between a and c. If $a' = c' = b$, then we take $P' = P''$ and $T \cup P'$ has two embeddings. For each of the two cases we either get a bounded obstruction for embedding extension to (S')–bridges (using Theorem 2.1), or an obstruction Ω' for extending the embedding of T (using Lemma 8.1 with S modified as explained below.) Finally, Ω is the union of these two obstructions together with T and some paths in (S')–bridges. If one of the two obstructions contains a millipede, then this obstruction obstructs also the other embedding. Thus, Ω contains at most one millipede and it is easy to see that $b^\circ(\Omega)$ is bounded as claimed.

Suppose now that (S')–bridges are not attached in the interior of the segment ac on π. For any embedding of T in F, the (S')–bridges have unique embedding extension (if any at all). We can apply Theorem 2.1 to either obtain a bounded obstruction T' for embeddings of (S')–bridges, or getting an actual embedding extension. In the former case we can stop by taking $\Omega = T \cup T'$. In the latter case, let β' be an attachment of (S')–bridges on the segment $c\beta \subset \pi$ closest to c (if any), and let α' be an attachment of (S')–bridges on the segment $\alpha a \subset \pi$ closest to a (if any). Let T' be paths within (S')–bridges joining α' and β' with S'. Note that T' can be chosen so that $b(T') \le 3$. Then any embedding of T' extending any embedding of T in F obstructs (S)–bridges being attached below on $\beta'\beta$ and above on the segment $\alpha'\alpha$. Pretend now that S is extended by adding the segments $\alpha\alpha'$ and $\beta\beta'$ on the left and use Lemma 8.1 to solve the corresponding embedding extension problem for (S)–bridges satisfying conditions (i) and (ii). To solve this task we can use the same T that was discovered above. Note that in cases from the previous paragraph where $T' = P' \cup P''$, the subgraph T changes into disjoint crossing paths (Figure 8) since vertex a or c becomes part of S.) An embedding extension combined with the embedding of (S')–bridges gives (a), while an obstruction Ω' for this problem together with T' forms a required obstruction Ω. Note that $b^\circ(\Omega)$ is larger by at most 3 than the corresponding branch size of Ω'. $\square$

References

[1] D. Archdeacon, A Kuratowski Theorem for the projective plane, J. Graph Theory 5 (1981) 243–246.

[2] D. Archdeacon, P. Huneke, A Kuratowski theorem for nonorientable surfaces, J. Combin. Theory Ser. B 46 (1989) 173–231.

[3] N. Chiba, T. Nishizeki, S. Abe, T. Ozawa, A linear algorithm for embedding planar graphs using PQ-trees, J. Comput. System Sci. 30 (1985) 54–76.

[4] S. A. Cook, R. A. Reckhow, Time bounded random access machines, J. Comput. Syst. Sci. 7 (1976) 354–375.

[5] I. S. Filotti, G. L. Miller, J. Reif, On determining the genus of a graph in $O(v^{O(g)})$ steps, in "Proc. 11th Ann. ACM STOC," Atlanta, Georgia (1979) pp. 27–37.

[6] H. Glover, J. P. Huneke, C. S. Wang, 103 graphs that are irreducible for the projective plane, J. Combin. Theory, Ser. B 27 (1979) 332–370.

[7] J. L. Gross, T. W. Tucker, Topological graph theory, Wiley-Interscience, New York, 1987.

[8] J. E. Hopcroft, R. E. Tarjan, Dividing a graph into triconnected components, SIAM J. Comput. 2 (1973) 135–158.

[9] J. E. Hopcroft, R. E. Tarjan, Efficient planarity testing, J. ACM 21 (1974) 549–568.

[10] M. Juvan, J. Marinček, B. Mohar, Elimination of local bridges, submitted.

[11] M. Juvan, J. Marinček, B. Mohar, Obstructions for simple embeddings, submitted.

[12] M. Juvan, J. Marinček, B. Mohar, Embedding graphs in the torus, submitted.

[13] M. Juvan, B. Mohar, Obstructions for the 2–Möbius band embedding extension problem, SIAM J. Discrete Math, in press.

[14] B. Mohar, Projective planarity in linear time, J. Algorithms 15 (1993) 482–502.

[15] B. Mohar, Obstructions for the disk and the cylinder embedding extension problems, Combin. Probab. Comput. 3 (1994) 375–406.

[16] B. Mohar, Universal obstructions for embedding extension problems, submitted.

[17] B. Mohar, Embedding graphs in an arbitrary surface in linear time, Proc. 28th Ann. ACM STOC, Philadelphia, ACM Press, 1996, pp. 392–397.

[18] B. Mohar, A linear time algorithm for embedding graphs in an arbitrary surface, submitted.

[19] N. Robertson, P. D. Seymour, Graph minors. VIII. A Kuratowski theorem for general surfaces, J. Combin. Theory, Ser. B 48 (1990) 255–288.

[20] N. Robertson, P. D. Seymour, Graph minors. IX. Disjoint crossed paths, J. Combin Theory, Ser. B 49 (1990) 40–77.

[21] P. D. Seymour, Disjoint paths in graphs, Discrete Math. 29 (1980) 293–309.

[22] Y. Shiloach, A polynomial solution to the undirected two paths problem, J. Assoc. Comput. Mach. 27 (1980) 445–456.

[23] C. Thomassen, 2–linked graphs, Europ. J. Combin. 1 (1980) 371–378.

[24] W. T. Tutte, Connectivity of graphs, Univ. Toronto Press, Toronto, Ontario; Oxford Univ. Press, London, 1966.

[25] S. G. Williamson, Embedding graphs in the plane — algorithmic aspects, Ann. Discrete Math. 6 (1980) 349–384.

[26] S. G. Williamson, Depth-first search and Kuratowski subgraphs, J. ACM 31 (1984) 681–693.