Apex graphs with embeddings of face-width three

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Abstract

Aa apex graph is a graph which has a vertex whose removal makes the resulting graph planar. Embeddings of apex graphs having face-width three are characterized. Surprisingly, there are such embeddings of arbitrarily large genus. This solves a problem of Robertson and Vitray. We also give an elementary proof of a result of Robertson, Seymour, and Thomas [5] that any embedding of an apex graph in a nonorientable surface has face-width at most two.

1 Introduction

We follow standard graph theory terminology as used, for example, in [1].

Let Π be a (2-cell) embedding of a graph G into a *nonplanar* surface S, i.e. a closed surface distinct from the 2-sphere. Then we define the *face-width* $\mathrm{fw}(\Pi)$ (also called the *representativity*) of the embedding Π as the smallest number of (closed) faces of G in S whose union contains a noncontractible curve. It is not difficult to see (cf. [6, 7]) that a planar graph embedded in a nonplanar surface has face-width at most two. Robertson and Vitray asked [6] which apex graphs can be embedded in some surface with face-width three (or more). (A graph G is an *apex graph* if it contains a vertex v such that the vertex-deleted subgraph $G - v$ is planar.) Vitray proved

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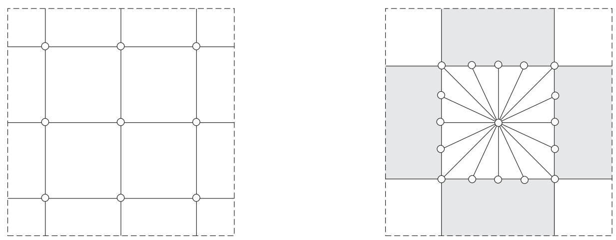
[8] that this cannot happen on the projective plane (cf. also [2]) and the results of Robertson, Seymour, and Thomas [5] on linkless embeddings of graphs in the 3-space imply that the same is true for every non-orientable surface. Vitray conjectured that the Cartesian product $C_3 \times C_3$ embedded in the torus (see Figure 1(a)) is a "generic" example. For instance, let G be a graph embedded in the torus as shown in Figure 1(b) where the shaded parts contain any plane graph. If v is the central vertex, then $G - v$ is planar, and if the shaded parts are "dense" enough, then the face-width is equal to 3. Clearly, this example generalizes $C_3 \times C_3$.



(a) (b)

Figure 1: $C_3 \times C_3$ and a generalization

In this paper we obtain a simple description of the general structure of apex graphs embedded with face-width 3. It turns out that the genus of such embeddings can be arbitrarily large. This disproves Vitray's conjecture. We also provide an elementary proof of the result of Robertson, Seymour, and Thomas [5] that every embedding of an apex graph in a nonorientable surface has face-width at most two.

2 Basic definitions

Let G be a connected graph. 2-cell embeddings of G in closed surfaces can be described in a purely combinatorial way by specifying:

(1) A *rotation system* $\pi = (\pi_v\,;\, v \in V(G))$, where for each vertex v of G, π_v is a cyclic permutation of edges incident with v. It represents the circular order of edges around v on the surface. The cyclic sequence $e, \pi_v(e), \pi_v^2(e), \pi_v^3(e), \ldots$ is called Π-*clockwise ordering* around v.

(2) A *signature* $\lambda : E(G) \to \{-1, 1\}$ whose meaning is as follows. By traversing an edge $e = uv$ on the surface, we see if the local rotations π_v and π_u are chosen consistently or not. If yes, then we have $\lambda(e) = 1$, otherwise we have $\lambda(e) = -1$.

The reader is referred to [3] for more details. We will use this description as a definition: An *embedding* of a connected graph G is a pair $\Pi = (\pi, \lambda)$ where π is a rotation system and λ is a signature. Having an embedding Π of G, we say that G is Π-*embedded*. The embedding Π is *nonorientable* if there is a cycle with an odd number of edges e having $\lambda(e) = -1$. We define Π-*facial walks* as closed walks in the graph that correspond to face boundaries of the corresponding topological embedding. If W is a walk, any subwalk of W is a *segment* of W.

If Π is an embedding of a graph G and H is a subgraph of G, then the *restriction* of Π to H is the embedding of H that is obtained from that of G by ignoring all edges in $E(G) \backslash E(H)$ and by restricting λ to $E(H)$. More precisely, if $e = uv \in E(H)$, then the successor of e in the clockwise ordering around v is the first edge of H in the sequence $\pi_v(e), \pi_v^2(e), \ldots$.

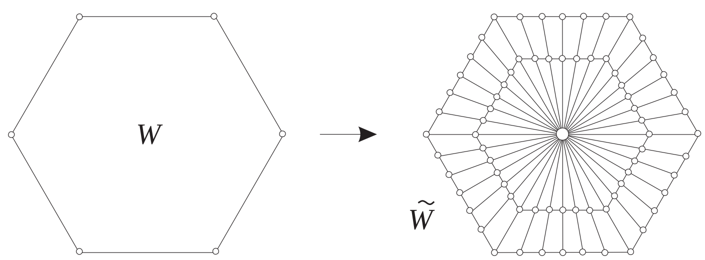


Figure 2: Filling up common faces

Let G be a 2-connected planar graph. Suppose that we have a fixed embedding Π of G into a nonplanar surface S such that $\mathrm{fw}(\Pi) \geq 2$. Let Π' be an embedding of G in the 2-sphere. If W is a Π-facial walk that is also Π'-facial, then we replace W by the graph $\tilde{W}$ as shown in Figure 2. Similarly, if W is a maximal common segment of a Π-facial walk and a Π'-facial walk, then we replace W by $\tilde{W}$ as shown in Figure 3. (As a special case, when W is just a path consisting of a single edge e of G, this operation is just a subdivision of e obtained by inserting five vertices of degree 2 on e.) When we do such replacements for all possible common facial walks and all maximal common facial segments W, we obtain a graph $\tilde{G}$ containing the (subdivided) graph G as a subgraph. Among all planar embeddings Π' of G we choose one such that Π and Π' are as close as possible. (This means that the number of common maximal segments of Π and Π' is as small as possible.) Then we say that $\tilde{G}$ is a *patch extension* of the embedding Π of G.

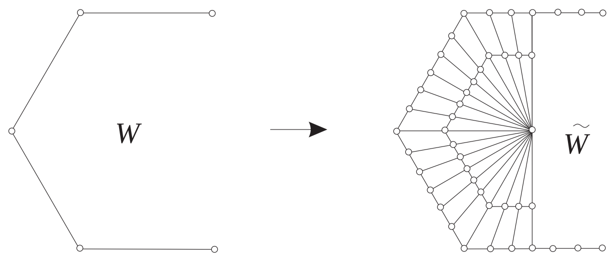


Figure 3: Filling up common facial segments

It is clear that embeddings Π' and Π can be extended to embeddings in the same surfaces, respectively, such that all triangles and quadrangles shown in Figures 2 and 3, respectively, are facial. In particular, $\tilde{G}$ is a planar graph. It is also easy to see that if G is 2-connected (3-connected, respectively), then so is $\tilde{G}$.

Let $\tilde{G}$ be the patch extension of a 2-connected Π-embedded planar graph G. Denote by $\tilde{\Pi} = (\tilde{\pi}, \tilde{\lambda})$ the corresponding embedding of $\tilde{G}$. The $\tilde{\Pi}$-facial walks that are not facial walks of the plane embedding of $\tilde{G}$ are the *patch facial walks* and the corresponding faces are called the *patch faces*. Vertices of $\tilde{G}$ of degree different from 2 that belong to two or more patch facial walks are *patch vertices*. Segments of patch facial walks joining patch vertices are also segments of facial walks of $\tilde{G}$ embedded in the plane. They are called *patch edges*. The *patch degree* of a patch vertex v is the number of patch facial walks that contain v.

Edges e and f of $\tilde{G}$ are *similar* if they both lie on the same patch edge or if they both lie on the same $\tilde{\Pi}$-facial walk that is not a patch facial walk. The smallest equivalence relation on $E(\tilde{G})$ containing the similarity relation is called the *patch equivalence*. Patch equivalence determines a partition of edges of $\tilde{G}$ into subgraphs of $\tilde{G}$. They are called *patches* of G (with respect to the embedding Π). It is convenient to view patches together with all

$\tilde{\Pi}$-facial and non-patch facial walks that they contain. Then patches can be viewed as subsets of the surface S with pairwise disjoint interiors. For more information on patches see [4].

Suppose that Π is an embedding of a graph G in a surface S such that $\mathrm{fw}(\Pi) = 2$. Then there are faces F_1, F_2 whose union contains a noncontractible closed curve γ in S that intersects the graph in exactly two points. Such a (noncontractible) curve is called a *2-curve*. It is clear that no 2-curve passes through a face of a patch. However, the following result was obtained by Mohar and Robertson [4].

Theorem 2.1 *Let G be a 2-connected planar graph that is Π-embedded in a nonplanar surface such that $\mathrm{fw}(\Pi) = 2$. Then for every patch face Φ and every patch vertex ν incident with Φ there is a 2-curve through Φ and ν.*

3 Apex graphs with face-width three

Suppose that G_0 is a 3-connected apex graph with an embedding Π_0 such that $\mathrm{fw}(\Pi_0) \geq 3$. Let v be a vertex of G_0 such that $G = G_0 - v$ is planar. Denote by Π the restriction of Π_0 to G. Clearly, G is 2-connected and Π is an embedding in the same surface as Π_0. Also, $\mathrm{fw}(\Pi) \geq \mathrm{fw}(\Pi_0) - 1$, and since G is planar, we have $\mathrm{fw}(\Pi_0) = 3$ and $\mathrm{fw}(\Pi) = 2$. Consequently, every Π-facial walk is a cycle of G. We shall denote by F_0 the Π-facial cycle that is not Π_0-facial. F_0 is uniquely determined, and it contains, in particular, all neighbors of v.

Consider the patches of Π. Denote by $\tilde{G}$ the patch extension of G, and let $\tilde{\Pi}$ be the corresponding extension of the embedding Π. Since G_0 is nonplanar, there is a patch face Φ_0 corresponding to F_0. Since $\mathrm{fw}(\Pi_0) = 3$, every 2-curve (with respect to Π) passes through F_0. Thus, every 2-curve with respect to $\tilde{\Pi}$ passes through Φ_0. Theorem 2.1 implies that every patch vertex lies on the boundary of the patch face Φ_0. Moreover, for every patch face $\Phi \neq \Phi_0$ and a patch vertex ν on the boundary of Φ, there is a 2-curve through Φ, ν, and Φ_0.

Lemma 3.1 *Every patch of Π is either just a path or a disk bounded by a cycle of $\tilde{G}$.*

Proof. Every patch edge is a segment of a facial walk in a plane embedding of $\tilde{G}$. Since $\tilde{G}$ is 2-connected, each patch edge is a path. Let P be a patch that is not just a path. It follows from the definition of patches that the interior (on the surface) of P is homeomorphic to an open connected subset of the plane that is bounded by patch edges. Let C be the cycle bounding Φ_0. Since every patch vertex is contained in C and since C does not enter the interior of P, the boundary walk of P is connected. If the boundary walk of P is not a cycle, it has a repeated patch vertex q. Since the patch vertices of P appear on C in the same order as in the planar embedding of $\tilde{G}$, there is a patch edge ε of P joining two appearances of q on the boundary of P. Since G is 2-connected, ε is a facial cycle in the plane embedding of G. Since the angle at q between the two appearances of ε is not filled up by a patch, this implies that the face-width of $\tilde{\Pi}$ is at most 1, a contradiction. $\square$

Patch faces are bounded by cycles since $\tilde{G}$ is 2-connected and has face-width 2. If two patch faces distinct from Φ_0 intersect in more than just a vertex, then their intersection is an edge since G_0 is 3-connected and $\mathrm{fw}(\Pi_0) = 3$.

In the proof of Theorem 3.3 below, we will use the following lemma.

Lemma 3.2 *Let C be a circle in the plane. Suppose that in the inside of C we have closed curves $\Phi_1, \ldots, \Phi_s$ ($s \geq 2$) each of which is composed of chords of C. (A* chord *is a straight line segment joining two points on C.) Suppose that only consecutive chords of the same curve can have an endpoint on the circle C in common. Let $C_1 \cup C_2$ be a partition of chords in $\Phi_1, \ldots, \Phi_s$ such that, for $i = 1, \ldots, s$, no two chords of Φ_i from the same set C_1 or C_2 cross inside C. Then the number of pairs of chords from C_1 that cross inside C has the same parity as the number of intersecting pairs of chords from C_2.*

Proof. If two consecutive chords γ, γ' of some Φ_i are in the same set, say in C_1, we split Φ_i at the common point x of γ and γ' and insert a new chord γ'' joining the two copies of x. We can do this change so that γ'' does not cross with any other chord and that other crossings remain the same as before. By putting γ'' in C_2 and repeating the procedure for all other cases, we get a system of curves satisfying the same conditions and having the same number of self-intersections of C_1 and of C_2. Therefore we may assume that, for $i = 1, \ldots, s$, no two consecutive chords of Φ_i are in the same set.

The proof proceeds by induction on the number of chords. The basic case is when $s = 2$ and each of Φ_1, Φ_2 is composed of four chords. This case is easily verified by considering all possibilities for the mutual placement of Φ_1 and Φ_2 (up to symmetries, there are 8 cases).

If $s > 2$, we just apply induction on each pair Φ_i, Φ_j, $1 \le i < j \le s$. Suppose now that $s = 2$ and that Φ_2 is composed of consecutive chords $\gamma_1, \ldots, \gamma_t$, where $t > 4$. Suppose that γ_2 is the shortest chord of Φ_2. Let α and β be the ends of γ_1 and γ_3, respectively, that are distinct from the ends of γ_2. Denote by γ the chord joining α and β. Let $\Phi_2' = \gamma \cup \gamma_1 \cup \gamma_2 \cup \gamma_3$ and let Φ_2'' be the closed curve composed of γ and $\gamma_4, \ldots, \gamma_t$. Suppose that $\gamma_1 \in C_1$. Since γ_2 is the shortest chord of Φ_2, Φ_2' has no self-intersections and no chord of Φ_2 that is in C_1 intersects the interior of γ (such a chord would either cross $\gamma_1 \in C_1$, $\gamma_3 \in C_1$, or γ_2; in the latter case we would get a shorter chord than γ_2). Therefore we can apply the induction to Φ_1, Φ_2' (with γ in C_2) and to Φ_1, Φ_2'' (with γ in C_1). Each intersection of γ with chords of Φ_1 is counted either in the first or in the second subproblem, but not both. The lemma now follows easily since γ intersects with an even number of chords from the closed curve Φ_1. $\square$

The patches of G are divided in two classes. Those that are embedded (in the plane embedding of $\tilde{G}$) in the interior of the facial cycle of Φ_0 are the *interior patches*. All others are the *exterior patches*. Now we present an elementary proof of a result of Robertson, Seymour, and Thomas [5].

Theorem 3.3 *Let G_0 be an apex graph embedded in a nonorientable surface. Then the face-width of the embedding is at most 2.*

Proof. Let G_0 be a Π_0-embedded graph where $\mathrm{fw}(\Pi_0) \ge 3$. By [6], every 3-connected component of G_0, except one, is planar and Π_0 restricted to that 3-connected component has the same face-width as Π_0. Π_0 restricted to any other 3-connected component H is an embedding of genus 0. Thus we may assume that G_0 is 3-connected. Hence, the discussion from the beginning of this section can be applied.

Suppose that Π_0 is a nonorientable embedding of G_0 having face-width 3. Then also the embedding $\tilde{\Pi}$ of $\tilde{G}$ is nonorientable. We may assume that the signature $\tilde{\lambda}$ is trivial (i.e., equal to 1) on all edges of Φ_0 and on all edges that are not incident with patch vertices. Since all patch vertices are on Φ_0 and the embedding is nonorientable, there is a patch facial walk Φ_1 that has a patch edge ε joining patch vertices τ and ν, say, with negative signature. Let P be the patch containing ε on its boundary. Since $\tilde{G}$ is planar, another patch edge δ of P incident with one of τ or ν, say ν, does not lie on Φ_0. Let Φ_2 be the patch face using δ. See Figure 4. Since $\mathrm{fw}(\Pi_0) = 3$, $\Phi_1 \ne \Phi_2$. Moreover, Φ_1 and Φ_2 intersect at ν (and at τ if $P = \varepsilon$), and they are disjoint elsewhere. We will show that this leads to a contradiction.

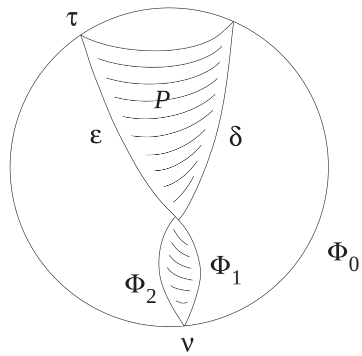


Figure 4: Patch faces Φ_1 and Φ_2 meet at ν

Φ_1 and Φ_2 use patch edges of interior and exterior patches. They can be interpreted as curves composed of chords of a circle in the plane (corresponding to Φ_0) as used in Lemma 3.2 with a partition $C_1 \cup C_2$ corresponding to the patch edges of the interior and exterior patches, respectively. The only thing that we need to change in order to apply the lemma is that we split the patch vertex ν (and τ if $P = \varepsilon$) so that ε and δ give rise to disjoint crossing chords. Since $\tilde{G}$ is planar, the only intersection of two chords from C_1 (or two chords from C_2) is between the two chords corresponding to ε and δ. This contradicts Lemma 3.2, and we are done. $\square$

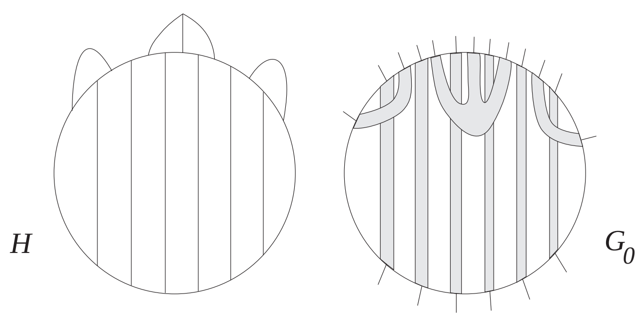


Figure 5: An example

On the other hand, there are orientable embeddings of apex graphs with face-width three that have arbitrarily large genera. Very general class of such embeddings is constructed as follows. Let H be a planar graph and F_0 a cycle of H such that vertices of C are of degree 3 and such that $H - V(C)$ consists of isolated vertices. Consider the orientable embedding of H such that C is a facial cycle, and such that all vertices of H lying inside C have the same rotation as in the plane and the vertices of H in the exterior of C have opposite rotation as in the plane. Let G be the patch extension of H, and let G_0 be obtained from G by adding a vertex v and joining it to some of the vertices on C. Let Π_0 be the corresponding embedding of G_0 extending the embedding of G such that the local rotation at v is determined be the sequence of its neighbors on C. If v is joined to at least one vertex in the interior of each of the segments of C shared with the patches, then this embedding has face-width three. More generally, we can replace every patch of G by a (dense) planar graph. Then we get many other examples. It is clear that these examples can be of arbitrarily large genus. A similar example is represented in Figure 5 where the pending outside edges all lead to the vertex v.

The above examples are not surprising. Less obvious is the fact that these examples are "generic" in the sense that any other embedding of an apex graph with face-width three can be described in the same way, except that the patches are not necessarily that "dense" as assumed above. In particular, some of the patches can be just edges or they may have just three patch vertices. Patches are not necessarily 2-connected. The arguments from the beginning of this section show that this is indeed the case: If we replace every patch P of the embedding of G by a vertex joined to all segments of $\Phi_0 \cap P$ (possibly just patch vertices of P), we get a planar graph H that is similar to the graph H used above, except that its vertices on the cycle C need not be cubic. This determines the patch structure of (orientable) embeddings of apex graphs having face-width 3. This description can be turned into a characterization of such embeddings by adding appropriate conditions on the structure of patches (in order to get face-width 3). Assuming that G_0 is 3-connected, these conditions can be formulated as follows:

(a) All Π-facial walks are cycles.

(b) If Φ_1, Φ_2 are patch facial walks distinct from Φ_0, the Π_0-facial walks of G_0 corresponding to Φ_1 and Φ_2 are either disjoint, they intersect in a single vertex, or they have a single edge in common.

(c) If F is a Π_0-facial walk corresponding to a patch facial walk distinct from Φ_0 and x, y are nonconsecutive vertices on F, then each of the two open segments from x to y on F_0 contains a neighbor of v.

Theorem 3.4 *There are orientable embeddings of apex graphs having face-width three and arbitrarily large genus. All such embeddings have the structure described above.*

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