Circle packings of maps in polynomial time*

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Abstract

The Andreev-Koebe-Thurston circle packing theorem is generalized and improved in two ways. First, we get simultaneous circle packings of the map and its dual map so that, in the corresponding straight-line representations of the map and the dual, any two edges dual to each other are perpendicular. Necessary and sufficient condition for a map to have such a primal-dual circle packing representation in a surface of constant curvature is that its universal cover is 3-connected (the map has no "planar" 2-separations). Secondly, an algorithm is obtained that given a map M and a rational number $\varepsilon > 0$ finds an ε-approximation for the radii and the coordinates of the centres for the primal-dual circle packing representation of M. The algorithm is polynomial in $|E(M)|$ and $\log(1/\varepsilon)$. In particular, for a map without planar 2-separations on an arbitrary surface we have a polynomial time algorithm for simultaneous geodesic convex representations of the map and its dual so that only edges dual to each other cross, and the angles at the crossings are arbitrarily close to $\frac{\pi}{2}$.

Proposed running head: Circle packings in polynomial time

Math. Subj. Class. (1991) *Primary:* 05 C 10, 05 C 85, 51 M 10, 52 C 15
Secondary: 05 B 40, 51 M 20

*This article appeared in: *Europ. J. Combin.* 18 (1997) 785–805.
†Supported in part by the Ministry of Science and Technology of Slovenia, Research Project J1–7036.
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1 Introduction

Let Σ be a surface. A *map* on Σ is a pair (G, Σ) where G is a connected graph that is 2-cell embedded in Σ. Given a map $M = (G, \Sigma)$, a *circle packing* of M is a set of (geodesic) circles (disks) in a Riemannian surface Σ' of constant curvature $+1$, 0, or -1 that is homeomorphic to Σ, one circle for each vertex of G, such that the following conditions are fulfilled:

(i) the interiors of circles are pairwise disjoint open disks,

(ii) for each edge $uv \in E(G)$, the circles corresponding to u and v touch, and

(iii) by putting a vertex v_D in the centre of each circle D and joining v_D by geodesics with all points on the boundary of D where the other circles touch D and where D touches itself, we get a map on Σ' which is isomorphic to M.

Because of (iii) we also say to have a *circle packing representation* of M. The obtained map on Σ' is said to be a *straight-line* representation of M. Simultaneous circle packing representations of a map M and its dual map M^* are called a *primal-dual circle packing representation* of M if for any two edges $e = uv \in E(M)$ and $e^* = u^*v^* \in E(M^*)$ which are dual to each other, the circles C_u, C_v corresponding to e touch at the same point as the circles C_{u^*}, C_{v^*} of e^*, and C_u, C_{u^*} cross each other at that point perpendicularly. Having a primal-dual circle packing representation, each pair of dual edges intersects at the right angle. The obtained representations of the maps M and M^* on Σ' are easily seen to be *convex*, i.e., if x, y are points in the same face F of M (or M^*), then in F there is a geodesic (not necessarily a shortest one) joining x and y.

It was proved by Koebe [7], Andreev [1, 2], and Thurston [12] that if M is a triangulation, then it admits a circle packing representation. The proofs of Andreev and Thurston are existential (using a fixed point theorem) but Colin de Verdière [4, 5] found a constructive proof by means of a convergent process. In this paper we present an algorithm that for a given reduced map M (see Section 3 for the definition) and a given rational number $\varepsilon > 0$ finds an ε-approximation for a circle packing of M into a surface of constant curvature (either $+1$, 0, or -1). The time used by our algorithm is polynomial in the size of the input (the number of edges of M plus the size of ε, i.e., $\max\{1, \lceil \log(1/\varepsilon) \rceil\}$). Cf. Theorem 5.5.

We generalize the result of Andreev–Koebe–Thurston to the most general maps that admit primal-dual circle packing representation (reduced maps, i.e., maps with 3-connected universal cover). In particular, every map with a 3-connected graph has a primal-dual circle packing representation. This extends the results of Pulleyblank and Rote (private communication) and Brightwell and Scheinerman [3] about circle packings of 3-connected planar graphs. With these results we not only characterize maps which admit convex representations but also prove a far reaching generalization, to arbitrary surfaces, of a conjecture of Tutte (settled in [3]) that a 3-connected planar graph and its dual admit simultaneous straight-line drawing in the plane (with the vertex corresponding to the unbounded face at the infinity) such that each pair of dual edges is perpendicular. We also obtain results about uniqueness of primal-dual circle packings. The reader is referred to the last section.

It is worth mentioning that our proofs establishing existence and uniqueness of primal-dual circle packings are elementary. The basic idea relies on the interpretation (due to Lovász) of Thurston's proof. Unfortunately, the proof that our algorithms run in polynomial time requires more work. However, in view of the diversity of possible applications of circle packings in computational geometry, graph drawing and in computer graphics (cf., e.g., [6, 8, 9]), we think that this additional work is worth its effort.

For example, a by-product of our results is a polynomial time algorithm for the following combinatorial problem. Given a reduced map M_0, find simultaneous convex representations of M_0 and its dual map M_0^* on a surface with constant curvature, such that each edge of M_0 crosses only with its dual edge in M_0^* and the angle at which they cross is between $\frac{\pi}{2} - 10^{-1994}$ and $\frac{\pi}{2} + 10^{-1994}$. Not only that our results show that there is such a representation, but using the circle packing algorithm up to a certain precision one really gets such a representation in time bounded by a polynomial in $|E(M_0)|$.

Reduced maps are more general than submaps of triangulations in the sense that they may contain loops or parallel edges. Therefore, our results in particular prove the existence of circle packings of more general maps than implied by the Andreev–Koebe–Thurston's Theorem. More important, we get a characterization of such maps (Corollary 5.6).

2 Primal-dual circle packings

Let $M_0 = (G_0, \Sigma)$ be a map on Σ. Define a new map $M = (G, \Sigma)$ whose vertices are the vertices of G_0 together with the faces of M_0, and whose edges correspond to the vertex-face incidence in M_0. The embedding of G is obtained simply by putting a vertex in each face F of M_0 and joining it to all the vertices on the boundary of F. If a vertex of G_0 appears more than once on the boundary of the face, then we get multiple edges at F but their order around F is determined by the order of the vertices on the boundary of F. The map M and the graph G are called the *vertex-face map* and the *vertex-face graph*, respectively. (Sometimes also the name *angle map* and *angle graph* is used.) Note that G is bipartite and that every face of M is bounded by precisely four edges of G.

From now on we assume that M_0 is a given map on a closed surface Σ and that M and G are its vertex-face map and vertex-face graph, respectively. We will use the notation $V = V(G)$ throughout the paper. We will denote by n and m the number of vertices and edges of G, respectively. It follows by Euler's formula that

$$m = 2(n - \chi(\Sigma)) \tag{1}$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. If $S, T \subseteq V(G)$, then $E(S)$ denotes the set of edges with both endpoints in S, and $E(S,T)$ is the set of edges with one endpoint in S and the other in T. Although $E(S,T) = E(T,S)$, we emphasize that, in order to simplify the notation, $uv \in E(S,T)$ will not only mean the membership but will also implicitly assume that $u \in S$, $v \in T$.

Having a primal-dual circle packing representation of M_0 in a surface Σ', we have a circle for each vertex of G. Let r_v be the radius of the circle corresponding to the vertex $v \in V(G)$. Clearly, the primal-dual circle packing representation in Σ' gives rise to a straight-line representation of M. Consider a vertex v of M. It is surrounded by quadrilaterals. If $vuv'u'$ is one of them (Figure 1), then its diagonals are perpendicular

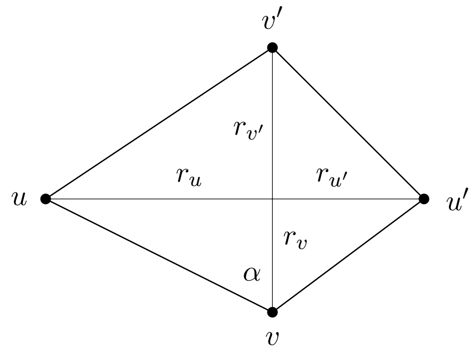

Figure 1: A basic quadrangle

and have length $r_v + r_{v'}$ and $r_u + r_{u'}$, respectively. Assume now that Σ' has constant curvature +1 (*spherical* case), 0 (*Euclidean* case), or −1 (*hyperbolic* case). By elementary geometry (spherical, Euclidean, or hyperbolic, respectively) we get the following formula for the angle $\alpha = \alpha(r_u, r_v)$ as shown on Figure 1:

$$\alpha = \alpha(r_u, r_v) = \begin{cases} \operatorname{arctg}(\operatorname{tg} r_u / \sin r_v)\,, & \text{spherical case} \\ \operatorname{arctg}(r_u/r_v)\,, & \text{Euclidean case} \\ \operatorname{arctg}(\operatorname{th} r_u / \operatorname{sh} r_v)\,, & \text{hyperbolic case} \end{cases} . \tag{2}$$

Since the total sum of the angles around a vertex is 2π, we have a necessary condition for a set of radii $r = (r_v \mid v \in V(G))$ to be the radii of a primal-dual circle packing:

$$\varphi_v = \sum_{vu \in E(G)} \alpha(r_u, r_v) = \pi\,, \quad v \in V(G) \tag{3}$$

where the sum is taken over all edges vu that are incident to v in G. It is important that (3) is also sufficient.

Proposition 2.1 *Let M be the vertex-face map of a map M_0 on Σ. Let $\alpha(r_u, r_v)$ be defined by (2) with the spherical, Euclidean, or hyperbolic case, depending on whether the Euler characteristic of Σ is positive, zero, or negative, respectively. Then $r = (r_v \mid v \in V(M))$ are the radii of a primal-dual circle packing representation of M into a surface with constant curvature +1, 0, or –1, respectively, if and only if $r_v > 0$, $v \in V(G)$, and the angle condition (3) is satisfied.*

Proof. Necessity of (3) is obvious. To prove the converse, it suffices to realize the universal cover $\tilde{M}$ of M as the corresponding primal-dual circle packing.

Suppose that D is a subcomplex od $\tilde{M}$ which is homeomorphic to a closed disk in the plane. Since the graph of $\tilde{M}$ is 2-connected, all closed faces of $\tilde{M}$ are disks. If e is an edge on ∂D, let F_e be the face containing e which is not in D. If $\partial D \cap \partial F_e$ is connected, then $D \cup F_e$ is a closed disk in the plane. If not, let C be the outer cycle of $D \cup F_e$, and let $D(e) \supseteq D \cup F_e$ be the disk bounded by C. If there is an edge f of $\partial D \backslash \partial F_e$ which is in the interior of $D(e)$, then $D(f) \subset D(e)$. This shows that there exists such an edge f for which $D(f) = D \cup F_f$. Continuing with $D(f)$ instead of D, we see that there is a sequence $D_1 = D, D_2 = D \cup F_f, D_3, \ldots, D_k = D(e)$ such that each D_{i+i} is a disk obtained from D_i by adding a face ($i = 1, \ldots, k-1$). This shows that there is a sequence $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$ of subcomplexes of $\tilde{M}$ such that $\tilde{M} = \cup_{i=1}^{\infty} D_i$, D_1 is a closed face of $\tilde{M}$, and D_{i+1} is homeomorphic to a disk (except for the spherical case when the last member D_k is the 2-sphere) and is obtained from D_i by adding a closed face of $\tilde{M}$ ($i = 1, 2, 3, \ldots$).

Suppose first that $\chi(\Sigma) < 0$ (the hyperbolic case). If we realize D_1, D_2, D_3, etc., respectively, by pasting together hyperbolic quadrangles (constant curvature -1) with appropriate angles, the angle condition implies that D_i is locally isomorphic to the hyperbolic plane at every interior point. In the limit we get a simply connected surface with constant negative curvature, and it is well known that this must be the hyperbolic plane. Finally, the universal covering projection determines a primal-dual circle packing representation of M on a surface with constant curvature –1.

The same proof works in the Euclidean case, and also in the spherical case (when $\tilde{M}$ is finite, and so the sequence $D_1, D_2, \ldots, D_k$ is finite and then D_i, $i < k$ are disks, but D_k is the 2-sphere). □

3 Reduced maps

From now on we will assume that $M_0 = (G_0, \Sigma)$ is a given map on a surface Σ with $\chi(\Sigma) \leq 0$, and that M and G are its vertex-face map and vertex-face graph, respectively.

Vertices $x, y \in V(M_0)$ (with the possibility $x = y$) are said to be a *planar 2-separation* if there are internally disjoint simple paths π_1, π_2 from x to y on Σ such that:

(i) π_1, π_2 meet $G_0 \subset \Sigma$ only at their endpoints x, y.

(ii) The closed walk $\pi_1 \pi_2^{-1}$ bounds an open disk $D \subset \Sigma$.

(iii) D contains a vertex or a face of M_0.

The map M_0 is *reduced* if it contains no planar 2-separations. Maps with 3-connected graphs are reduced but we can have a reduced map whose graph is not 3-connected, or even not simple. For example, the toroidal map on Figure 2 has loops at each of its two vertices and 4 parallel edges with the same endpoints but it is still reduced.

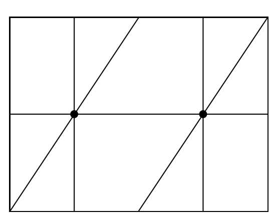

Figure 2: A reduced toroidal map

Proposition 3.1 *Let $M_0 = (G_0, \Sigma)$ be a map on the surface Σ with non-positive Euler characteristic, $\chi(\Sigma) \leq 0$. Then the following conditions are equivalent:*

(a) The map M_0 is reduced.

(b) The graph of the universal cover of M_0 is 3-connected.

(c) The graph G_0 has no vertices of degree less than 3, no faces of size less than 3 and does not contain vertices x, y and two internally disjoint paths P_1, P_2 from x to y such that the closed walk $P_1 P_2^{-1}$ bounds a disk D on Σ and the only vertices on $P_1 \cup P_2$ that have a neighbour out of D are x and y.

(d) If there is a closed walk of length at most 4 in the vertex-face graph G that bounds a disk D in Σ, then D is a face of M.

(e) For every proper non-empty subset $S \subset V(G)$ of vertices of G we have:

$$2|S| - |E(S)| \geq 2\chi(\Sigma) + 1\,. \tag{4}$$

Proof. (a) $\Leftrightarrow$ (b): Let $\tilde{M} = (\tilde{G}, \tilde{\Sigma})$ be the universal cover of M_0. If $\tilde{G}$ is not 3-connected, then either it has at most three vertices, contains a loop, a pair of parallel edges, or there are vertices $\tilde{x}, \tilde{y}$ whose removal disconnects the graph. Since $\tilde{G}$ is infinite, the first possibility cannot occur. In case of a loop, let $\tilde{x} = \tilde{y}$ be the vertex of the loop, and in case of a parallel pair, let $\tilde{x}, \tilde{y}$ be the endpoints of the edges. In each of these cases, as well as in the case when $\tilde{x}, \tilde{y}$ form a cutset, it is easy to see that there are internally disjoint paths $\tilde{\pi}_1, \tilde{\pi}_2$ from $\tilde{x}$ to $\tilde{y}$ such that the properties (i)–(iii) hold. Consider their projections π_1, π_2 in Σ. Since each of $\tilde{\pi}_1, \tilde{\pi}_2$ lies within a face of $\tilde{M}$, π_1 and π_2 are simple paths. In order to show that they determine a planar 2-separation in M_0, we must show that π_1, π_2 are internally disjoint, they bound a disk D, and D contains either a vertex or a face. It is clear that π_1, π_2 are internally disjoint if they lie in distinct faces of M_0. But being in the same face and intersecting in its interior, also their lifts $\tilde{\pi}_1$ and $\tilde{\pi}_2$ in $\tilde{M}$ would internally intersect. By the homotopy lifting property of covering spaces, π_1, π_2 are homotopic paths (relative to their endpoints). Therefore they bound a disk, call it D. This disk is the projection of the disk $\tilde{D}$ in $\tilde{M}$ bounded by $\tilde{\pi}_1$, and $\tilde{\pi}_2$. The projection of the vertex or the face contained in $\tilde{D}$ is a vertex or a face of M_0 that is contained in D. Therefore, π_1 and π_2 determine a planar 2-separation in M_0.

Conversely, if M_0 contains a planar 2-separation, then also $\tilde{M}$ does. If the disk D of the 2-separation of M_0 and its complement D' each contain a vertex, then it is clear that G_0 and $\tilde{G}$ are not 3-connected. On the other hand, if D (or its complement) contains a face but no vertex, then this face is bounded by a loop or a parallel edge pair which also implies that G_0 and $\tilde{G}$ are not 3-connected. The last possibility is that D contains a vertex $v \in V(M_0)$ but D' contains neither a vertex nor a face of M_0. Since $\chi(\Sigma) \leq 0$, D' is not a disk. This means that the complement of the lift of D in the universal cover contains another copy of the lift of v. Then $\tilde{G}$ is not 3-connected.

(b) $\Rightarrow$ (c): A face of size 1 or 2 lifts to the face of the same size in the universal cover. Its existence thus contradicts the 3-connectivity of $\tilde{G}$. The same holds in case of vertices of degree 1 or 2. Suppose now that we have paths P_1, P_2 in the graph G_0 joining vertices x, y and having properties stated in (c). $P_1 P_2^{-1}$ bounds a disk in Σ, and so its lift to the universal cover also bounds a disk $\tilde{D}$ in $\tilde{\Sigma}$. Denote by $\bar{D}$ the closure of $\tilde{D}$. We may assume that $\bar{D}$ is topologically a closed disk, since otherwise we could replace either P_1 or P_2 by a trivial path. The only vertices of $\tilde{G}$ in $\bar{D}$ that have an adjacent edge out of $\bar{D}$, are the pre-images $\tilde{x}, \tilde{y}$ of x and y, respectively. Since $\tilde{G}$ is 3-connected, the only possibility for this to happen is that out of $\bar{D}$ there is at most one edge. But this is not possible since $\chi(\Sigma) \leq 0$ implies that $\tilde{G}$ is infinite.

(c) $\Rightarrow$ (d): Excluding walks of length 2 is easy. Since G is bipartite, the other possibility to consider is a closed walk $vuv'u'$ of size 4 in G, bounding a disk D in Σ, where D is not a face of the vertex-face map M. Suppose that D is minimal in the sense that D does not properly contain a disk with the same properties (corresponding to another walk). If v has no neighbour in D, let $vuwu'$ be the boundary of the quadrangular face in D containing v. Then either w corresponds to a vertex of degree 2, or a face of size 2 in M_0, or D is not minimal. By symmetry, we may thus assume that all vertices v, u, v', u' have neighbours in D. In the same way we also see that v and v' (same for u and u') have no common neighbours in D. Suppose that v, v' correspond to vertices of M_0, and let P_1, P_2 be paths in M_0 from v to v' going through the neighbours of u, and through the neighbours of u', respectively. By the minimality of D, P_1 and P_2 are internally disjoint simple paths in G_0 joining v and v' and bounding a disk contained in D. It is easy to see that P_1 and P_2 have the property forbidden by (c). A contradiction.

(d) $\Rightarrow$ (e): Let $S \subset V(G)$, $S \neq \emptyset$, and let $G|S$ be the subgraph of G induced by S. If $G|S$ consists of isolated vertices and edges, (4) is clearly satisfied (since $\chi(\Sigma) \leq 0$). Since isolated vertices and edges only increase the left hand side of (4), we may thus assume that $G|S$ does not have isolated vertices. Consider $G|S$ as a graph embedded in Σ, and denote by $\mathcal{F}(S)$ the set of the faces of $G|S$. Euler's formula then reads:

$$|S| - |E(S)| + \sum_{F \in \mathcal{F}(S)} \chi(F) = \chi(\Sigma)\,. \tag{5}$$

For $F \in \mathcal{F}(S)$, let size(F) denote the length of the facial walk(s) corresponding to F. Then it suffices to show

$$2|E(S)| = \sum_{F \in \mathcal{F}(S)} \text{size}(F) \geq 4 \sum_{F \in \mathcal{F}(S)} \chi(F) + 2 \tag{6}$$

since this inequality and (5) imply (4). We note that $\chi(F) > 0$ only when F is a disk. In this case, we have by (d) that size$(F) \geq 4 = 4\chi(F)$. Also by (d), if F is a face containing a vertex from $V(G)\backslash S$, then either F is a disk and size$(F) \geq 6 = 2 + 4\chi(F)$, or F is not a disk, in which case size$(F) \geq 2 \geq 2 + 4\chi(F)$. All this clearly implies (6).

(e) $\Rightarrow$ (b): Suppose that the universal cover graph $\tilde{G}$ of M_0 is not 3-connected. Then it either contains a loop, or a pair of parallel edges, or there are vertices $\tilde{x}, \tilde{y}$ (possibly $\tilde{x} = \tilde{y}$) such that $\tilde{G} - \tilde{x} - \tilde{y}$ is disconnected. Having a loop or parallel edges in $\tilde{G}$, we have a contractible loop or homotopic edges with the same endpoints in M_0. Then the vertex-face graph G contains a non-facial digon or a non-facial 4-gon bounding an open disk D. We get the same conclusion when $\tilde{G} - \tilde{x} - \tilde{y}$ is disconnected. Let S be the set of vertices of G that do not lie in D. Then it follows easily by Euler's formula that $2|S| - |E(S)| = 2\chi(\Sigma)$ (or $= 2\chi(\Sigma) - 1$ in case of a digon), which contradicts (e). □

Corollary 3.2 *The dual map M_0^* of M_0 is reduced if and only if M_0 is reduced.*

Proof. This is clear by equivalence of (a) and (d) in Proposition 3.1 since the property (d) is the same for M_0 as for M_0^*. □

Corollary 3.3 *If M_0 is a reduced map, then its vertex-face graph G has no vertices of degree 2 or less.*

4 Computation of radii

In this section we describe a procedure which finds appropriate radii satisfying (3). We will give details only for the hyperbolic case. The spherical and the Euclidean case are not very different.

Given a set of "radii" $r = (r_v \mid v \in V(G))$, i.e., for each vertex $v \in V(G)$ we have a positive number $r_v > 0$, one can define corresponding "angles" in analogy with (3):

$$\varphi_v = \sum_{uv \in E(G)} \alpha(r_u, r_v)\,, \qquad v \in V(G) \tag{7}$$

where the sum is over all edges uv that are incident to v in G. (In case of multiple edges between u and v, each such edge gives its contribution.) We write

$$\vartheta_v = \varphi_v - \pi\,, \tag{8}$$

and use the function

$$\mu(r) = \sum_{v \in V} \vartheta_v^2 \tag{9}$$

to measure how far from the required radii satisfying (3) is our choice of r.

Angles and the corresponding radii will be computed by means of an iteration process. Call the radii $r = (r_v \mid v \in V)$ *normalized* if $\sum_{v \in V} \vartheta_v = 0$. (In case when $\chi(\Sigma) = 0$ we also require that $\min_{v \in V} r_v = 1$, and if $\chi(\Sigma) > 0$, then $\max_{v \in V} r_v \le \frac{\pi}{2}$.) The following two lemmas, which can easily be proved, will be used routinely in the sequel.

Lemma 4.1 *If $\chi(\Sigma) \le 0$, then for any $r = (r_v \mid v \in V)$, $r_v > 0$, there is a unique constant $\tau > 0$ such that τr is normalized. If r° is normalized and $r_v \ge r_v^\circ$ for every $v \in V$, then $\tau \le 1$. If $r_v \le r_v^\circ$ for every $v \in V$, then $\tau \ge 1$. If we have strict inequality for some v, then also the inequality for τ is strict.*

Lemma 4.2 *For any $v, u \in V(G)$, $v \ne u$, we have $\partial\vartheta_v/\partial r_v < 0$, $\partial\vartheta_v/\partial r_u > 0$ if u is adjacent to v, and $\partial\vartheta_v/\partial r_u = 0$ otherwise.*

Given a normalized $r = (r_v \mid v \in V)$, order the vertices $u_1, u_2, \ldots, u_n$ of G such that $\vartheta_{u_1} \ge \vartheta_{u_2} \ge \cdots \ge \vartheta_{u_q} \ge 0 > \vartheta_{u_{q+1}} \ge \cdots \ge \vartheta_{u_n}$. Let

$$\sigma(r) = \max_{q \le i < n} (\vartheta_{u_i}^- - \vartheta_{u_{i+1}})\,, \tag{10}$$

where $\vartheta_{u_q}^- = 0$ and $\vartheta_{u_i}^- = \vartheta_{u_i}$ if $i > q$. Let t be the smallest index i where the maximum in (10) is attained. (We define $\sigma(r) = 0$ and $t = n$ if $q = n$.) Set $S = S(r) = \{u_1, \ldots, u_t\}$, and let r' be defined by

$$r'_v = \begin{cases} \beta r_v\,, & \text{if } v \in S \\ \gamma r_v\,, & \text{otherwise} \end{cases} \tag{11}$$

where $\beta \geq 1$ and $\gamma > 0$ are constants such that r' is normalized. It follows by Lemma 4.1 that $\gamma \leq 1$. Let $(\vartheta'_v;\, v \in V)$ be the values ϑ corresponding to r', and let

$$f(\beta, \gamma) = \sum_{v \in S} (\vartheta_v - \vartheta'_v).$$

(It follows by Lemma 4.1 that γ is uniquely determined by β, and hence $f(\beta,\gamma)$ really depends only on β as far as for the given β there exists a γ such that r' is normalized.) Call the pair (β,γ) *suitable* if $\frac{\sigma(r)}{6} \leq f(\beta,\gamma) \leq \frac{\sigma(r)}{2}$. It will be proved by Lemma 4.6 that $\vartheta'_v \geq \vartheta'_u$ for all $v \in S$ and $u \notin S$ whenever (β,γ) is suitable.

Starting with an arbitrary normalized set of radii, we perform the following process until we get an r with $\mu(r) \leq \varepsilon/2$. At each step we first determine $\sigma = \sigma(r)$ and the set $S = S(r) \subset V$. Then we find a suitable pair (β,γ). Such a pair always exists, and it can be found by bisection as described by Lemma A.1. Finally, the radii r' for the next iteration are determined by (11). It should be remarked that at each repetition of this step, the value of ϑ_v decreases for every $v \in S$, and that ϑ_v increases for $v \notin S$. Moreover, if $\vartheta_v < 0$ at a certain step of the process, then ϑ_v remains negative ever since. This process will be referred to as *Process A*. Its formal description as a polynomial time algorithm is given in Appendix A. The rest of this section is devoted to the proof that Process A works as expected, and that it finds the solution in time that is bounded by a polynomial in $|E(M_0)|$ and the size of ε.

Having defined $\alpha(r_u, r_v)$ by (2), we write

$$\overline{\alpha}(r_u, r_v) = \alpha(r_u, r_v) + \alpha(r_v, r_u)\,. \tag{12}$$

If $S \subset V$ is a proper non-empty subset of the vertex set V, then the following formula holds

$$\begin{aligned} \sum_{v \in V} \vartheta_v \;&=\; -\tfrac{\pi}{2}(|E(S)| + 2\chi(\Sigma)) + \sum_{vu \in E(S)} \overline{\alpha}(r_u, r_v) + \sum_{vu \in E(S,T)} \alpha(r_u, r_v) \\ &\quad - \sum_{vu \in E(T)} (\tfrac{\pi}{2} - \overline{\alpha}(r_u, r_v)) - \sum_{vu \in E(T,S)} (\tfrac{\pi}{2} - \alpha(r_u, r_v)) \end{aligned} \tag{13}$$

where $T = V \backslash S$. This can be easily proved using the following equality: $2\chi(\Sigma) + |E(S)| = 2n - m + |E(S)| = 2n - |E(T)| - |E(T,S)|$. Also,

$$\sum_{v \in S} \vartheta_v = -\pi|S| + \sum_{vu \in E(S)} \overline{\alpha}(r_u, r_v) + \sum_{vu \in E(S,T)} \alpha(r_u, r_v)\,. \tag{14}$$

On the other hand, if r is normalized and $s = |S|$, $e_s = |E(S)|$, $e_t = |E(T)|$, $e_{st} = |E(S,T)| = |E(T,S)|$, then

$$\sum_{v \in S} \vartheta_v = -\sum_{u \notin S} \vartheta_u$$

$$= \pi(n-s) - \tfrac{\pi}{2}(e_t + e_{st}) + \sum_{vu\in E(T)} (\tfrac{\pi}{2} - \overline{\alpha}(r_u, r_v)) + \sum_{vu\in E(T,S)} (\tfrac{\pi}{2} - \alpha(r_u, r_v))$$
$$= \tfrac{\pi}{2}(2\chi(\Sigma) - (2s - e_s)) + \sum_{vu\in E(T)} (\tfrac{\pi}{2} - \overline{\alpha}(r_u, r_v)) + \sum_{vu\in E(T,S)} (\tfrac{\pi}{2} - \alpha(r_u, r_v))\,.$$

Since M_0 is reduced, we get by Proposition 3.1(e):

$$\sum_{v\in S} \vartheta_v \le -\tfrac{\pi}{2} + \sum_{vu\in E(T)} (\tfrac{\pi}{2} - \overline{\alpha}(r_u, r_v)) + \sum_{vu\in E(T,S)} (\tfrac{\pi}{2} - \alpha(r_u, r_v))\,. \tag{15}$$

Given $r = (r_v \mid v \in V)$ we define

$$\Omega(r) = \max_{v\in V} r_v \qquad \text{and} \qquad \omega(r) = \min_{v\in V} r_v\,.$$

In the following two lemmas we will show that the values $\Omega(r)$ and $\omega(r)$ cannot be too large, or too small, respectively.

Lemma 4.3 *Let $\omega^\circ = \operatorname{Ar\,ch}(\operatorname{ctg}(\frac{\pi n}{2m}))$. Then $\frac{1}{m} < \omega^\circ < \log(6|\chi(\Sigma)|)$, and for any normalized r, $\omega(r) \le \omega^\circ \le \Omega(r)$.*

Proof. By (1), we have $m - 2n = -2\chi(\Sigma) \ge 2$. Therefore $\frac{\pi}{4} - \frac{\pi n}{2m} = \frac{\pi}{4m}(m-2n) \ge \frac{\pi}{2m} \ge \frac{1}{m}$. It follows that $\operatorname{ctg}\frac{\pi n}{2m} \ge \operatorname{ctg}(\frac{\pi}{4} - \frac{1}{m}) \ge 1 + \frac{1}{m} > \operatorname{ch}\frac{1}{m}$. Thus $\omega^\circ > \frac{1}{m}$.

For the upper bound we will also use the fact that $n \ge 2$:

$$\operatorname{ctg}\frac{\pi n}{2m} \le \frac{2m}{\pi n} = \frac{4}{\pi}(1 + \frac{|\chi|}{n}) \le 2 + |\chi| \le 3|\chi|$$

where $\chi = \chi(\Sigma)$. Then $\omega^\circ \le \operatorname{Ar\,ch}(3|\chi|) < \log(6|\chi|)$. The rest is clear by Lemma 4.1. □

Lemma 4.4 *The set of radii given by $r_v = \omega^\circ$, $v \in V(G)$, is normalized. Starting Process A with these radii, all radii r in the process satisfy*

$$\Omega(r) \le 2\log m \qquad \textit{and} \qquad \omega(r) \ge m^{-n}. \tag{16}$$

Proof. The radii ω° are normalized since $\vartheta_v = \pi - \deg(v)\frac{n\pi}{2m}$. At the beginning of Process A we have

$$\vartheta_v = -\pi + \deg(v)\frac{\pi n}{2m} \ge -\pi + \frac{4n}{m}$$

where we used Corollary 3.3. Let us consider ϑ_v ($v \in V(G)$) in a general step. Let w be a vertex with $r_w = \Omega(r)$. Since the minimal value of ϑ_v, $v \in V$, never decreases during the process, we have:

$$-\pi + \frac{4n}{m} \le \min_{v\in V} \vartheta_v \le \vartheta_w \le -\pi + \sum_{uw\in E(G)} \operatorname{arctg}\frac{1}{\operatorname{sh}\Omega(r)} \le -\pi + \frac{m}{\operatorname{sh}\Omega(r)}\,.$$

By (1) we see that $m \ge 2n \ge 4$. Therefore $2\log m > 1$ and to prove the bound on $\Omega(r)$ we may assume that $\Omega(r) \ge 1$. Now, the first inequality in (16) follows from the above inequality by using (37).

To prove the second bound, we note that during Process A, if $\vartheta_v \ge 0$ at a certain time, then it was non-negative all the time from the very beginning. This implies that such a vertex v was in the set S at every step, hence $r_v \ge \omega^\circ \ge \frac{1}{m}$ (cf. Lemma 4.3). Suppose now that $\omega = \omega(r) < m^{-n}$. There is an integer k, $1 \le k < n$, such that for all vertices $u \in V$, either $r_u \le m^{k-1}\omega$, or $r_u \ge m^k\omega$. Let $S' = \{v \in V \mid r_v \le m^{k-1}\omega\}$. Then $S' \ne \emptyset$ since it contains the vertex v with $r_v = \omega$. On the other hand, if $\vartheta_u \ge 0$, then $r_u \ge m^{-1}$ (as shown above). This proves that $V\backslash S' \ne \emptyset$. Therefore (15) (applied for the set $V\backslash S'$) and Lemmas B.1 and B.2 imply:

$$\begin{aligned} 0 &\le -\sum_{v\in S}\vartheta_v = \sum_{u\notin S'}\vartheta_u \\ &\le -\tfrac{\pi}{2} + \sum_{vu\in E(S')}(\tfrac{\pi}{2} - \overline{\alpha}(r_u, r_v)) + \sum_{vu\in E(S',V\backslash S')}(\tfrac{\pi}{2} - \alpha(r_u, r_v)) \\ &\le -\tfrac{\pi}{2} + |E(S')|2\omega m^{k-1} + |E(S', V\backslash S')|\frac{1.55}{m} \le -\tfrac{\pi}{2} + 1.55 < 0\,. \end{aligned}$$

This is a contradiction. □

Lemma 4.5 *Given $r = (r_v \mid v \in V)$, let $\omega = \omega(r), \Omega = \Omega(r)$. Suppose that in Process A either $\beta \ge \frac{1}{\omega}\log m$, or $\gamma \le \omega/(2m\Omega)$. Then $f(\beta,\gamma) \ge \sigma(r)/2$.*

Proof. Since r is normalized, we have $\sum_{v\in S}\vartheta_v \ge \sigma(r)$, where $S = S(r)$. Therefore it suffices to see that $\sum_{v\in S}\vartheta'_v \le 0$. Suppose that $\beta \ge \frac{1}{\omega}\log m$. By (14) and Lemma B.3 we have:

$$\begin{aligned} \sum_{v\in S}\vartheta'_v &= -\pi|S| + \sum_{vu\in E(S)}\overline{\alpha}(\beta r_u, \beta r_v) + \sum_{vu\in E(S,T)}\alpha(\gamma r_u, \beta r_v) \\ &\le -\pi|S| + |E(S)|6\exp(-\beta\omega) + |E(S,T)|3\exp(-\beta\omega) \\ &\le -\pi|S| + 3(2|E(S)| + |E(S,T)|)\frac{1}{m} \end{aligned}$$

where $T = V\backslash S$. If $|E(S)| = 0$, then the last row above is clearly negative. If $|E(S)| > 0$, then $|S| \ge 2$, and the same conclusion holds.

To see the same in case when γ is small, we will apply (15) and Lemmas B.1 and B.2. By the choice of Process A we have $\beta \ge 1$. Then

$$\begin{aligned} \sum_{v\in S}\vartheta'_v &\le -\tfrac{\pi}{2} + \sum_{vu\in E(T)}(\tfrac{\pi}{2} - \overline{\alpha}(\gamma r_u, \gamma r_v)) - \sum_{vu\in E(T,S)}(\tfrac{\pi}{2} - \alpha(\beta r_u, \gamma r_v)) \\ &\le -\tfrac{\pi}{2} + |E(T)|2\gamma\Omega + |E(T,S)|\frac{2\gamma\Omega}{\min\{\beta\omega, 1\}} \\ &\le -\tfrac{\pi}{2} + \frac{2m\gamma\Omega}{\omega} \le -\tfrac{\pi}{2} + 1 < 0\,. \end{aligned}$$

□

Lemma 4.6 *If $f(\beta,\gamma) \le \sigma(r)/2$, then for arbitrary vertices $v \in S, u \notin S$ we have $\vartheta'_v \ge \vartheta'_u$.*

Proof. Recall that $\vartheta_v \geq \vartheta_u + \sigma(r)$ for all $v \in S$, $u \notin S$. Also, $\vartheta'_v \leq \vartheta_v$ for $v \in S$ and $\vartheta'_u \geq \vartheta_u$ for $u \notin S$. If for $v \in S, u \notin S$ we have $\vartheta'_v < \vartheta'_u$, then either $\vartheta_v - \vartheta'_v > \sigma(r)/2$, or $\vartheta'_u - \vartheta_u > \sigma(r)/2$. The first case clearly implies that $f(\beta,\gamma) > \sigma(r)/2$. The same with the alternative since $f(\beta,\gamma) = \sum_{v\in S}(\vartheta_v - \vartheta'_v) = \sum_{u\notin S}(\vartheta'_u - \vartheta_u)$. □

Lemma 4.7 *If* $r_v = \mathrm{Ar\,ch}(\mathrm{ctg}(\frac{n\pi}{2m}))$ *for each* $v \in V$*, then* $\mu(r) < 15n^2$.

Proof. In this case we have $\varphi_v = \deg(v)\frac{n\pi}{2m}$ and $|\vartheta_v|^2 \leq \pi^2 + (\frac{n\pi}{2m}\deg(v))^2$. Hence

$$\begin{aligned}
\mu(r) &\leq n\pi^2 + \Big(\frac{n\pi}{2m}\Big)^2 \sum_{v\in V} \deg(v)^2 \\
&\leq n\pi^2 + \Big(\frac{n\pi}{2m}\Big)^2 \Big(\sum_{v\in V} \deg(v)\Big)^2 = \pi^2 n(n+1) < 15n^2 .
\end{aligned}$$

□

Lemma 4.8 *If* r' *is the new value for the function* r *obtained in one step of Process A, then*

$$\mu(r') \leq (1 - \frac{1}{3n^4})\mu(r) . \tag{17}$$

Proof. Using the notation of the Process, let $t_1 = \min_{v\in S}\vartheta_v$, $t_2 = \max_{u\notin S}\vartheta_u$. Then $t_1 - t_2 \geq \sigma$. Since (β,γ) is suitable, there is a number t_3 between t_2 and t_1, such that for every $v \in S, u \notin S$, $\vartheta'_v \geq t_3 \geq \vartheta'_u$. Since $\beta \geq 1$ and $\gamma \leq 1$ we have $\vartheta_v \geq \vartheta'_v$ for $v \in S$, and $\vartheta_u \leq \vartheta'_u$ for $u \notin S$. Then

$$\begin{aligned}
\mu(r) - \mu(r') &= \sum_{v\in V}(\vartheta_v^2 - \vartheta_v'^2) \\
&= \sum_{v\in S}(\vartheta_v + \vartheta'_v)(\vartheta_v - \vartheta'_v) + \sum_{u\notin S}(\vartheta_u + \vartheta'_u)(\vartheta_u - \vartheta'_u) \\
&\geq \sum_{v\in S}(t_1 + t_3)(\vartheta_v - \vartheta'_v) + \sum_{u\notin S}(t_2 + t_3)(\vartheta_u - \vartheta'_u) \\
&= \sum_{v\in S}(t_1 - t_2)(\vartheta_v - \vartheta'_v) \geq \sigma \sum_{v\in S}(\vartheta_v - \vartheta'_v) = \sigma f(\beta,\gamma) \geq \frac{\sigma^2}{6} .
\end{aligned}$$

To get (17) we combine above bound with (20) below. In deriving (20) we assume that $V = \{1,2,\ldots,n\}$ and that $\vartheta_1 \geq \cdots \geq \vartheta_q \geq 0 > \vartheta_{q+1} \geq \cdots \geq \vartheta_n$. Then $|\vartheta_{q+i}| \leq i\sigma$, $1 \leq i \leq n-q$. This implies that

$$\sum_{j=1}^{q} \vartheta_j = \sum_{j=q+1}^{n} |\vartheta_j| \leq \sigma\frac{n^2}{2} \tag{18}$$

and

$$\sum_{j=q+1}^{n} \vartheta_j^2 \leq \sigma^2 \sum_{i=1}^{n-q} i^2 \leq \sigma^2\frac{n^3}{3} . \tag{19}$$

From (18), (19), and $n \geq 2$ it follows that

$$\mu(r) = \sum_{j=1}^{n} \vartheta_j^2 \leq \Big(\sum_{j=1}^{q} \vartheta_j\Big)^2 + \sum_{j=q+1}^{n} \vartheta_j^2 \leq \sigma^2 \frac{n^4}{4} + \sigma^2 \frac{n^3}{3} \leq \sigma^2 \frac{n^4}{2} . \tag{20}$$

The proof is complete. □

5 Uniqueness of primal-dual circle packings

In this section we will show that for an arbitrary $\varepsilon > 0$ we can find in polynomial time an ε-approximation for the centres and the radii of a primal-dual circle packing if we can solve in polynomial time δ-approximation computation of the radii. We will provide the details only for the hyperbolic case.

Lemma 5.1 *Let M_0 be a reduced map on a surface Σ with the Euler characteristic $\chi = \chi(\Sigma) < 0$. Suppose that $r = (r_v \mid v \in V(G))$ are positive numbers associated with vertices of the vertex-face graph G of M_0 such that $\sum_{v \in V} \vartheta_v = 0$. If $\mu(r) \leq 1$ then*

$$\max_{v \in V} r_v \leq \log m . \tag{21}$$

If $\mu(r) \leq 1/(4nm)$, then

$$\min_{v \in V} r_v \geq (2m)^{-n} . \tag{22}$$

Proof. The first part is easy. Let w be the vertex for which $r_w = \max_{v \in V} r_v$. Since $\mu(r) \leq 1$ we have $\vartheta_w \geq -1$. Thus

$$2 < \varphi_w = \sum_{uw \in E} \operatorname{arctg} \frac{\operatorname{th} r_u}{\operatorname{sh} r_w} \leq \sum_{uw \in E} \frac{1}{\operatorname{ch} r_w} \leq \frac{m}{\operatorname{ch} r_w} .$$

Consequently, $\operatorname{ch} r_w < m/2$. This implies (21).

By Lemma 4.3, there is a vertex with $r_v \geq 1/m$. Consequently, to prove (22), it suffices to show that for an arbitrary non-empty proper subset $T \subset V$ of vertices of G we have

$$a \leq 2mb \tag{23}$$

where $a = \min_{v \in T} r_v$ and $b = \max_{u \notin T} r_u$. Moreover, it suffices to prove (23) only in case when $b \leq 1/(4m^2)$ which we assume henceforth. We may replace a by any smaller number, thus we may assume that $a \leq 1$. By the Cauchy-Schwartz inequality we get

$$\mu(r) \geq \sum_{v \in S} |\varphi_v - \pi|^2 \geq \frac{1}{s} \sum_{v \in S} |\varphi_v - \pi| \geq \frac{1}{s} \sum_{v \in S} \varphi_v - \pi$$

where $s = |T|$. It follows that

$$\sum_{v \in S} \varphi_v \leq \pi s + n\mu(r) \leq \pi s + \frac{1}{4m} . \tag{24}$$

Since $b \leq 1$ and for $u, v \in T$, $r_u \leq b, r_v \leq b$, it follows by Lemma B.1 that $\overline{\alpha}(r_u, r_v) \geq \frac{\pi}{2} - 2b$. If $v \in T$, $u \notin T$, then by Lemma B.2 $\alpha(r_u, r_v) \geq \alpha(a, b) \leq \frac{\pi}{2} - 2b/a$. Let $e_s = |E(T)|, e_t = |E(V \backslash T)|$, and let e_{st} be the number of edges from T to $V \backslash T$. Then we have by (24):

$$\begin{aligned} \pi s \quad + \quad \frac{1}{4m} &\geq \sum_{v \in S} \varphi_v = \sum_{vu \in E(T)} \overline{\alpha}(r_u, r_v) + \sum_{vu \in E(T, V \backslash T)} \alpha(r_u, r_v) \\ &\geq \quad e_s(\tfrac{\pi}{2} - 2b) + e_{st}(\tfrac{\pi}{2} - \frac{2b}{a}) \,. \end{aligned}$$

After re-arranging and using the assumption, $b \leq 1/(4m^2)$ we get:

$$\frac{\pi}{2} \cdot \frac{2s - e_s}{e_{st}} + \frac{3}{4me_{st}} \geq \frac{\pi}{2} - \frac{2b}{a} \,. \tag{25}$$

By Proposition 3.1(e) for the vertex set $V \backslash T$, we have $2(n-s) - e_t \geq 2\chi + 1$. Since $2n - m = 2\chi$, we get $2s - e_s \leq e_{st} - 1$ which implies that

$$\frac{2s - e_s}{e_{st}} \leq 1 - \frac{1}{e_{st}} \,. \tag{26}$$

From (25) and (26) we conclude:

$$\frac{2b}{a} \geq \frac{1}{2e_{st}}(\pi - \frac{3}{2m}) \,. \tag{27}$$

Since $m \geq 4$ and $e_{st} \leq m$, this implies (23). □

Although the above lemma resembles very much on Lemma 4.4, its main advantage is that it does not assume the radii to arise from the computation of Process A.

It remains to show that our polynomial time convergence process always converges to the same solution. This is justified by the following theorems.

Theorem 5.2 *Process A is convergent, i.e., the radii converge to a positive limit. In particular, a map admits a primal-dual circle packing representation on a surface with constant curvature if and only if it is reduced.*

Proof. Assume that the map is reduced. For some vertex v we have $\vartheta_v > 0$ all the time (or else the procedure stops with an exact solution in a finite number of steps). Then r_v is always changed in such a way that it is multiplied by $\beta \geq 1$. Since r_v is bounded (Lemma 5.1), the product of β's (values of β in the consecutive steps of Process A) converges. The product of γ's is decreasing and bounded below by 0. Hence it is convergent. But then also any mixed product of β's and γ's converges, so we get the convergence of all the radii. The limiting radii are positive by Lemma 5.1, and hence they determine a primal-dual circle packing representation by Proposition 2.1.

For the proof of the converse, let us remark that the reducubility is only needed in proving the lower bound on r_v in Lemma 5.1. If the map is not reduced, the radii may still converge, but the limit may be 0. The proof of Lemma 5.1 shows that this must indeed happen. The arguments used in the proof of Theorem 5.3 below then show that a primal-dual circle packing cannot exist. □

Theorem 5.3 *Let $r = (r_v \mid v \in V)$ be an approximation for the primal-dual circle packing radii of a reduced map M_0 on a surface Σ with negative Euler characteristic. Suppose that $r^\circ = (r_v^\circ \mid v \in V)$ is an exact solution for the primal-dual circle packing radii. If $\mu(r) < 2^{-4n}m^{-4n-16}\varepsilon$, where $\varepsilon < 1/4$, then for every $v \in V$ we have*

$$1 - \sqrt{\varepsilon} < \frac{r_v}{r_v^\circ} < 1 + \sqrt{\varepsilon}\,. \tag{28}$$

Proof. We know by Theorem 5.2 that there is an exact solution $r^\circ = (r_v^\circ \mid v \in V)$. Note that we do not assume that r°, or r is obtained by our algorithm.

Let $\kappa = \max\{r_v/r_v^\circ \mid v \in V\}$ and let v be a vertex where the maximum is attained. Suppose that $\kappa > 1$. For every neighbour u of v we have $r_u \le \kappa r_u^\circ$. Therefore

$$\alpha(r_u, r_v) = \alpha(r_u, \kappa r_v^\circ) \le \alpha(\kappa r_u^\circ, \kappa r_v^\circ). \tag{29}$$

By Lemma B.6, if $\kappa > 2$, then (29) still holds if we replace κ by 2. Thus we may assume that $\kappa \le 2$. Lemma B.6 and (29) imply that

$$\alpha(r_u^\circ, r_v^\circ) - \alpha(r_u, r_v) \ge \alpha(r_u^\circ, r_v^\circ) - \alpha(\kappa r_u^\circ, \kappa r_v^\circ) \ge \frac{\omega^2\Omega^2}{24\,\mathrm{ch}^8\,\Omega}(\kappa - 1)\,, \tag{30}$$

where ω and Ω are a lower and an upper bound, respectively, on r_v°, r_u°. By (22) and (21) we see that a good choice is $\omega = (2m)^{-n}$, $\Omega = \log m$. Then (30) and (36) imply:

$$\begin{aligned} -\vartheta_v &= -\vartheta_v + \vartheta_v^\circ = \sum_{vu \in E(G)} (\alpha(r_u^\circ, r_v^\circ) - \alpha(r_u, r_v)) \\ &\ge \frac{(\log m)^2}{24(2m)^{2n}\,\mathrm{ch}^8(\log m)}(\kappa - 1) \ge \frac{\kappa - 1}{2^{2n}m^{2n+8}}\,. \end{aligned} \tag{31}$$

Then $\mu(r) \ge \vartheta_v^2 \ge 2^{-4n}m^{-4n-16}(\kappa - 1)^2$, and with the assumed bound on $\mu(r)$ we get $\kappa - 1 \le \sqrt{\varepsilon}$. Since $\varepsilon < 1$, the validity of this inequality carries over to the case when the initial value of κ was greater than 2.

If $\tau = \min\{r_v/r_v^\circ \mid v \in V\}$ and v is a vertex where the minimum is attained, then we get by the same arguments as in the first part that $\vartheta_v \ge 2^{-2n}m^{-2n-8}(1 - \tau)$, assuming that $\tau \ge 1/2$. This implies that $\tau \ge 1 - \sqrt{\varepsilon}$. Since $\varepsilon < 1/4$, this bound also takes care of the case when the initial value of τ is smaller than 1/2. □

The following corollary is an immediate consequence of Theorem 5.3.

Corollary 5.4 *Primal-dual circle packing radii of a reduced map M_0 on a surface Σ with negative Euler characteristic are uniquely determined.*

Algorithm B in Appendix A shows how to use the radii obtained by Process A to determine centres of circles of a circle packing. The results of the last two sections can now be summarized in the main theorem of this paper:

Theorem 5.5 *Given a reduced map M_0 on a surface with negative Euler characteristic and an $\varepsilon > 0$, one can find in polynomial time ε-approximations for the centres and the radii of a primal-dual circle packing representation of M_0 on a surface with constant curvature –1.*

This result holds also in the spherical and the Euclidean case. A proof is essentially the same as in the hyperbolic case.

A simple but interesting consequence of Theorem 5.5 is a characterization of maps that admit circle packings.

Corollary 5.6 *For a map M on a surface Σ with non-positive Euler characteristic the following conditions are equivalent:*

(a) M admits a circle packing representation on a surface with constant curvature.

(b) M admits a straight-line representation on a surface with constant curvature.

(c) M does not contain contractible loops or pairs of edges (possibly loops) with the same endpoint(s) that are homotopic relative their endpoint(s).

To show equivalence of (a)–(c), one should note that by properly triangulating every face of a map satisfying (c), a reduced map is obtained. On the other hand, if a map does not satisfy (c), then it has no straight-line representation on a surface with constant curvature by an easy application of the Gauss-Bonnet Theorem.

A Appendix: The algorithm

In this section we present circle packing algorithms in more detail. We will use notation introduced in Section 4. First we describe an algorithm for the following problem:

Instance: A reduced map M_0 on a surface Σ with negative Euler characteristic and a rational number $\varepsilon > 0$.

Task: Find normalized positive numbers $r = (r_v \mid v \in V)$ for the vertex-face graph G of M_0 such that $\mu(r) \leq \varepsilon$.

ALGORITHM A:

1. Construct G, $n := |V(G)|$, $m := |E(G)|$.
2. Let $p = 20n\lceil \log_2 m \rceil + \lceil \log_2(1/\varepsilon) \rceil$ be the number of binary digits used in all the computations in the following steps.
3. Set $r_v := \operatorname{Ar}\operatorname{ch}(\operatorname{ctg}(\frac{n\pi}{2m}))$, $v \in V(G)$.
4. **while** $\mu(r) > \varepsilon/2$ **do**
   - 4.1 Determine $\sigma = \sigma(r)$ and the set $S = S(r) \subset V$.
   - 4.2 Find a suitable pair (β, γ). The search is performed by bisection as described by Lemma A.1.
   - 4.3 $r := r'$, where r' is defined by (11).
5. Output r and $(\vartheta_v \mid v \in V(G))$.

In determining the radii we cannot guarantee that the arithmetic with precision p will give the exact value. Instead, we only require the radii to be close enough to the normalized values, i.e., if τr is normalized, then τ is close enough to 1. It is clear that an error cannot accumulate during the algorithm since we "normalize" r' at each step and the error does not depend on errors in previous steps.

To find a suitable pair (β, γ) in Step 4.2 of Algorithm A we use a method commonly known as *bisection.* To be precise, we need a slightly different version of bisection than the usual one. It will solve the following problem:

Instance: Rational numbers $\varepsilon > 0$, a, b, $0 \le a \le b \le \infty$, and two properties $\mathcal{L}(x)$ and $\mathcal{R}(x)$ of real numbers given by oracles and such that

$$I(\mathcal{L}) = \{x \in \boldsymbol{R}^+ \mid \mathcal{L}(x)\}$$

and

$$I(\mathcal{R}) = \{x \in \boldsymbol{R}^+ \mid \mathcal{R}(x)\}$$

are intervals on $\boldsymbol{R}^+ = \{x \in \boldsymbol{R} \mid x \ge 0\}$ and such that $a \in I(\mathcal{L})$, $b \in I(\mathcal{R})$. (If $b = \infty$, the last condition is replaced by $I(\mathcal{R})$ being unbounded.)

Task: Give one of the following answers:

(a) Return a rational number $x \in I(\mathcal{L}) \cap I(\mathcal{R})$.

(b) Conclude that the intersection $I(\mathcal{L}) \cap I(\mathcal{R}) \cap [a, b]$ is an interval (possibly empty) with diameter at most ε.

(c) Conclude that the intersection $I(\mathcal{L}) \cap I(\mathcal{R}) \cap [a, b]$ is contained in the interval $[\frac{1}{\varepsilon}, \infty]$.

If $\mathcal{L}(x)$ and $\mathcal{R}(x)$ are given as oracles, the following algorithm solves the above problem in oracle-polynomial time (polynomial in the sizes of a, b, ε, where each oracle call is assumed to take constant time):

BISECTION(a, b, ε)

1. **If** $\mathcal{L}(b)$ **then** STOP (return $x := b$, Case (a)).
2. **If** $\mathcal{R}(a)$ **then** STOP (return $x := a$, Case (a)).
3. **If** $b - a < \varepsilon$ **then** STOP (conclude Case (b)).
4. **If** $a > \frac{1}{\varepsilon}$ **then** STOP (conclude Case (c)).
5. **If** $b < \infty$ **then** $c := \frac{a+b}{2}$ **else if** $a \neq 0$ **then** $c := 2a$ **else** $c := 1$.
6. **If** $\mathcal{L}(c)$ **then** $a := c$ **and goto** Step 3.
7. **If** $\mathcal{R}(c)$ **then** $b := c$ **and goto** Step 3.
8. STOP (Case (b), the intersection is empty).

It will be shown in the sequel that the bisection needs only polynomial time to discover a suitable pair (β, γ) in Step 4.2 of Algorithm A. However, by running Algorithm A in practice, it may be more appropriate to use some other techniques, for example a version of Newton's method.

Let us define predicates $\mathcal{L}_1(x), \mathcal{R}_1(x), \mathcal{L}_2(x)$, and $\mathcal{R}_2(x)$. $\mathcal{L}_1(x)$ and $\mathcal{R}_1(x)$ are determined by the following procedure. Let $\gamma := x$ and find β $(1 \le \beta < \infty)$ by a classical bisection so that r' defined by (11) is normalized (up to the given precision p). Here we assume that the appropriate β really exists. If not, $\mathcal{L}_1(x)$ and $\mathcal{R}_1(x)$ are undefined. Then $\mathcal{L}_1(x) = \mathcal{L}_1(\gamma)$ holds if and only if $f(\beta, \gamma) \geq \sigma(r)/6$. Similarly, $\mathcal{R}_1(x) = \mathcal{R}_1(\gamma)$ holds if and only if for every $v \in S$, $u \notin S$, $\vartheta'_v \geq \vartheta'_u$. Similarly, $\mathcal{L}_2(x)$ and $\mathcal{R}_2(x)$ check properties of β. Having $\beta = x$, find γ so that r' is normalized. Then let $\mathcal{L}_2(x) = \mathcal{R}_1(\gamma)$ and $\mathcal{R}_2(x) = \mathcal{L}_1(\gamma)$. We remark that $\mathcal{L}_i(x)$ and $\mathcal{R}_i(x)$, $i = 1, 2$, can be described as oracles with time bounded by a polynomial in x, the size of x, m and p (the precision). This is easy to see since the bisection used to compute β (given γ), or γ (given β), needs at most p steps to compute the best resulting β (or γ), and the computation of the "angles" ϑ'_v is also polynomial since the Taylor series of $\mathrm{arctg}(x)$, $\mathrm{sh}(x)$, and $\mathrm{th}(x)$ converge fast enough. We note that the computed β (or γ) is not exact for r' to be normalized but since p is large enough, it is sufficiently close to the exact value.

Lemma A.1 *Let $\omega = m^{-n}$, $\Omega = \log(m)$, and $\eta \leq \omega^7 \sigma(r)/(400 m^3 \Omega^2)$. The search for a suitable pair (β, γ) in Step 4.2 of Algorithm A can be performed as follows. If $|E(S)| + 2\chi(\Sigma) \geq 0$, then BISECTION$(0, 1, \eta)$ for $\mathcal{L}_1(x)$ and $\mathcal{R}_1(x)$ (the resulting x is γ). Otherwise BISECTION$(1, \infty, \eta)$ for $\mathcal{L}_2(x)$ and $\mathcal{R}_2(x)$ (the resulting x is β).*

Proof. Suppose that $|E(S)| + 2\chi(\Sigma) \geq 0$. We need to prove that the procedure BISECTION$(0, 1, \eta)$ ends up with Case (a) of the bisection problem given above, i.e., it finds $\gamma = x \in I(\mathcal{L}_1) \cap I(\mathcal{R}_1)$. By the definition of $\mathcal{L}_1$ and $\mathcal{R}_1$, the pair (β, γ) is suitable. It suffices to see that:

(a) $\mathcal{L}_1(\gamma)$ and $\mathcal{R}_1(\gamma)$ are well-defined for every γ, $0 < \gamma \leq 1$,

(b) $0 \in I(\mathcal{L}_1)$, $1 \in I(\mathcal{R}_1)$,

(c) $I(\mathcal{L}_1)$ and $I(\mathcal{R}_1)$ are intervals, and

(d) $I(\mathcal{L}_1) \cap I(\mathcal{R}_1) \cap [0, 1]$ is an interval of length at least η.

It is obvious that $1 \in I(\mathcal{R}_1)$ and (c) is easy to see. The proof that $0 \in I(\mathcal{L}_1)$ is contained in the sequel where we show (d). But let us start with (a).

Let $0 < \gamma \leq 1$. If $\beta \geq 1$ and r' is defined by (11), let $g(\beta) = \sum_{v \in V} \vartheta'_v$. It is easy to see that $g(1) > 0$ and that $g(\beta)$ is a strictly decreasing function of β (cf. Lemmas 4.2 and B.6). To prove that $\mathcal{L}_1(\gamma)$ and $\mathcal{R}_1(\gamma)$ are well-defined, we need to prove that there is a β such that r' is normalized, i.e., $g(\beta) = 0$. It suffices to see that $g(\infty) = \lim_{\beta \to \infty} g(\beta) < 0$. By (13) we easily see that

$$\begin{aligned} g(\infty) &= -\tfrac{\pi}{2}(|E(S)| + 2\chi(\Sigma)) \\ &\quad - \sum_{vu \in E(T)} (\tfrac{\pi}{2} - \overline{\alpha}(\gamma r_u, \gamma r_v)) - \sum_{vu \in E(T,S)} (\tfrac{\pi}{2} - \alpha(\infty, \gamma r_v)) < 0 \end{aligned}$$

where $T = V \backslash S$. This completes the proof of (a).

It remains to prove (d). Note that $I(\mathcal{L}_1) \cap I(\mathcal{R}_1)$ contains all those γ (and possibly some others) for which the corresponding pair (β, γ) is suitable. By Lemma 4.6, (β, γ) is suitable if $f(\beta, \gamma)$ is between $\sigma/6$ and $\sigma/2$. By Lemma 4.5, there exist (β_1, γ_1) and (β_2, γ_2) such that $f(\beta_1, \gamma_1) = \sigma/6$ and $f(\beta_2, \gamma_2) = \sigma/2$. We are done by using Lemma B.7.

The case when $|E(S)| + 2\chi(\Sigma) < 0$ is similar. The details are left to the reader. $\square$

The next lemma shows that the precision p used in the calculations in Algorithm A suffices in order to obtain the desired result.

Lemma A.2 *In Algorithm A it suffices to take*

$$p = 20n\lceil \log_2 m \rceil + \lceil \log_2(1/\varepsilon) \rceil$$

as the number of binary digits used in all the calculations.

Proof. The number p given in the lemma is a generous upper bound on the number of binary digits necessary to encode η in Lemma A.1, while the actual precision required for the intermediate results is much smaller. This is established in some more detail by the following.

During the repetitions of Step 4 in Algorithm A, the errors because of r not being exact (and so not really being normalized) do not accumulate since we do the normalization of r' independently of the previous results. It is easy to see that a small change of r does not change ϑ_v ($v \in V(G)$) too much. Moreover, computing ϑ_v using p binary digits, gives a result which is exact on almost p digits. Since p is much larger than the size of ε, the final result of Algorithm A is a set of radii with $\mu(r) < \varepsilon$ as required.

The next question to be raised is about the computation of a suitable pair (β, γ). Let us only consider the case when $|E(S)| + 2\chi(\Sigma) \geq 0$ which was treated in detail when proving Lemma A.1. Given a γ, we have to compute (by bisection, for instance) the corresponding β. A similar calculation as used in (46) shows that β can be chosen in such a way that $|\sum_{v \in V} \vartheta'_v| < \tau$ if we use the bisection so long that the interval containing the candidates for β is smaller than $\omega^2\tau/(2m\Omega)$. For our purpose, it suffices to take τ of size comparable to the size of ε. So, if we choose τ to be of size $15n\lceil \log_2 m \rceil + \lceil \log_2(1/\varepsilon) \rceil$, it will be more than enough.

We may assume that $\varepsilon < 1$. From (20) it is easy to see that the sufficient number of binary digits to encode η in Lemma A.1 is $\lceil 7n \log_2 m + \log_2(400m^3\Omega^2) + \log_2(1/\varepsilon) \rceil \leq 13n\lceil \log_2 m \rceil + \lceil \log_2(1/\varepsilon) \rceil < p$.

Other details are left to the reader. $\square$

By combining the results in Lemmas A.1, 4.7, 4.8, and A.2 we conclude that: *The time used by Algorithm A is polynomial in $|E(M_0)|$ and the size of ε.*

In establishing the algorithm which computes the centers of the primal-dual circle packing we will need an additional geometric lemma.

Lemma A.3 *Fix a line ℓ and a point $P \in \ell$ in the hyperbolic plane. Suppose that for $i = 1, 2$, a point P_i in the hyperbolic plane is given. Let d_i be the distance of P_i from P, and let the angle between ℓ and the line segment from P to P_i be α_i, $i = 1, 2$. If $|\alpha_1 - \alpha_2| \leq \min\{2, 1/(2 \operatorname{sh} d_1)\}$, then*

$$\operatorname{dist}(P_1, P_2) \leq |d_1 - d_2| + 4|\alpha_1 - \alpha_2| \operatorname{sh} d_1 \,. \tag{32}$$

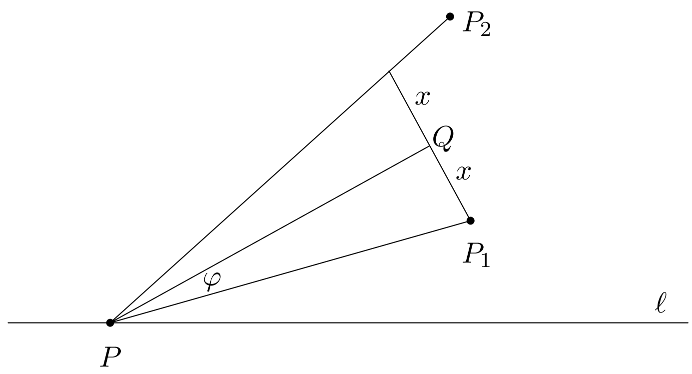

Figure 3:

Proof. Using the notation of Figure 3, $\mathrm{dist}(P_1, P_2) \leq |d_1 - d_2| + 2x$. The required estimate on x is obtained by considering the right triangle PP_1Q and its angle $\varphi = |\alpha_1 - \alpha_2|/2$. We have $\tan\varphi = \mathrm{th}\, x/\,\mathrm{sh}\, \overline{PQ} \geq \mathrm{th}\, x/\,\mathrm{sh}\, d_1$. This implies that $\mathrm{th}\, x \leq 2\varphi\, \mathrm{sh}\, d_1 \leq 1/2$. Since in this case $\mathrm{th}\, x \geq x/2$, we have $x \leq 4\varphi\, \mathrm{sh}\, d_1$ which implies (32). □

The centers of circles in a primal-dual circle packing with given radii can be computed as follows.

Instance: A reduced map M_0 and a rational number $\delta > 0$.

Task: For the vertex-face map M of M_0 find the radii $r = (r_v;\, v \in V)$ and points P_v, $v \in V(M)$, in the fundamental polygon of the universal cover of M_0 (a polygon in the hyperbolic plane) such that there is a primal-dual circle packing of M_0 with radii $r^\circ = (r_v^\circ;\, v \in V(M))$ and centres P_v°, $v \in V(M)$, and for each vertex v of M we have $|r_v^\circ - r_v| \leq \delta$ and $\mathrm{dist}(P_v^\circ, P_v) \leq \delta$.

ALGORITHM B:

1. Construct M.
2. Set $\delta_1 = 2^{-2n-4} m^{-2n-5} \delta$ and $\varepsilon = 2^{-4n} m^{-4n-18} \delta_1^2$.
3. Using Algorithm A determine radii $r = (r_v \mid v \in V)$ such that $\mu(r) < \varepsilon$.
4. Compute P_v, $v \in V$.
5. For $v \in V$ output r_v and P_v.

The choice of ε implies by Theorem 5.3 and (21) that for every $v \in V$ we have $|r_v - r_v^\circ| \leq \delta_1$. Let us describe how to obtain the centres P_v. Choose an arbitrary vertex $v_0 \in V$ and put it in the origin of the hyperbolic plane. By using the elementary hyperbolic geometry we can calculate the coordinates P_v for all vertices v that are adjacent to v_0 in G. The error in the calculations of the angles is estimated as follows. If α is the angle obtained by using (7), and α° is the exact value, then

$$\alpha - \alpha^\circ = \sum_v (\alpha(r_v, r_{v_0}) - \alpha(r_v^\circ, r_{v_0}^\circ))$$

where the sum is over some of the neighbours v of v_0. By using Lemma B.4 we see that

$$|\alpha - \alpha^\circ| \le m(\frac{1}{\omega}|r_v - r_v^\circ| + \frac{1}{\omega \operatorname{th} \omega}|r_{v_0} - r_{v_0}^\circ|) \le \frac{4m}{\omega^2}\delta_1 \,, \tag{33}$$

where $\omega = (2m)^{-n}$. It is easy to see that $\operatorname{dist}(P_v, P_{v_0}) \le r_v + r_{v_0} \le 2\log m$ and that $|\operatorname{dist}(P_{v_0}, P_v) - \operatorname{dist}(P_{v_0}, P_v^\circ)| \le 2\delta_1$. By applying Lemma A.3 we get:

$$\operatorname{dist}(P_v, P_{v_0}) \le 2\delta_1 + 4\,\frac{4m}{\omega^2}\delta_1 \operatorname{sh}(2\log m) \le \frac{16m^3}{\omega^2}\delta_1 \,. \tag{34}$$

In proceeding to the remaining vertices, we use the obtained approximations P_v instead of the exact coordinates P_v°. The error because of the shifted coordinates accumulates linearly (by adding up). The same situation is with the angles. Fixing the initial direction from v to v_0 (in the general step from v to a neighbour u covered previously), we may have an error that was accumulated up until reaching the vertex v plus the new error at v. Since the diameter of M is bounded by n, we see that for every $v \in V$ we have the angle error at v (with respect to the choice of a reference direction at the initial vertex) bounded by $4m^2\omega^{-2}\delta_1$ and the error in coordinates bounded by

$$\operatorname{dist}(P_v, P_v^\circ) \le \frac{16m^5}{\omega^2}\delta_1 = \delta \,. \tag{35}$$

B Appendix: Some estimates

At several places we use, usually even without referring to them, the following well known (or easy provable) facts. If $x \ge 0$ then: $\operatorname{arctg} x + \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2}$, $\operatorname{arctg} x \le x$, $\operatorname{th} x \le 1$, $\operatorname{th} x \le \operatorname{sh} x$, $\operatorname{sh} x \le \frac{1}{2}\exp x$, and $\operatorname{sh} x \ge x$. If $0 \le x \le 1$ then: $\operatorname{arctg} x \ge x/2$, $\operatorname{th} x \ge x/2$, $\operatorname{sh} x \le 4x/3$, and $\operatorname{ch} x \le 1 + x$. If $x \ge 1$ then

$$\operatorname{ch} x \le \frac{3}{5}\exp x \,, \tag{36}$$

$$\operatorname{sh} x \ge \frac{1}{3}\exp x \,. \tag{37}$$

Let $\alpha(x, y)$ and $\overline{\alpha}(x, y)$ be defined by (2) (the hyperbolic case) and (12), respectively. We need estimates about the behaviour of $\alpha(x, y)$ and $\overline{\alpha}(x, y)$ when x, y are large, or small, respectively.

Lemma B.1 *If $x, y \in (0, \tau]$, where $\tau \le 1$, then $0 < \frac{\pi}{2} - \overline{\alpha}(x, y) \le 2\tau$.*

Proof. If we use the fact that for $X \ge Y$ we have $\operatorname{arctg} X - \operatorname{arctg} Y \le X - Y$, then we get:

$$\overline{\alpha}(x, y) \quad = \quad \tfrac{\pi}{2} + \operatorname{arctg}\frac{\operatorname{th} x}{\operatorname{sh} y} - \operatorname{arctg}\frac{\operatorname{sh} x}{\operatorname{th} y} \ge \tfrac{\pi}{2} - \left(\frac{\operatorname{sh} x}{\operatorname{th} y} - \frac{\operatorname{th} x}{\operatorname{sh} y}\right) .$$

Consequently, if $x \le y$, which we may assume,

$$\tfrac{\pi}{2} - \overline{\alpha}(x, y) \le \frac{\operatorname{sh} x}{\operatorname{sh} y}(\operatorname{ch} y - \frac{1}{\operatorname{ch} x}) \le \operatorname{ch}\tau - \frac{1}{\operatorname{ch}\tau} \le 1 + \tau - \frac{1}{1+\tau} = \frac{\tau(\tau+2)}{\tau+1} \le 2\tau \,.$$

□

Lemma B.2 *Let $0 < y \le x$. If $x \le 1$, then $\frac{\pi}{2} - \alpha(x,y) \le 1.55y/x$. If $x \ge 1$ and $y \le 1$, then $\frac{\pi}{2} - \alpha(x,y) \le 2y$.*

Proof. If $x \le 1$, then

$$\tfrac{\pi}{2} - \alpha(x,y) = \operatorname{arctg} \frac{\operatorname{sh} y}{\operatorname{th} x} \le \frac{\operatorname{sh} y}{\operatorname{th} x} \le \operatorname{ch} 1 \, \frac{\operatorname{sh} y}{\operatorname{sh} x} \le \frac{1.55y}{x} \,.$$

In the other case we get

$$\tfrac{\pi}{2} - \alpha(x,y) \le \frac{\operatorname{sh} y}{\operatorname{th} x} \le \frac{4y}{3 \operatorname{th} 1} \le 2y \,.$$

□

Lemma B.3 *For any $x \ge 0$, $y \ge 1$, we have $\alpha(x,y) \le 3\exp(-y)$.*

Proof. We have: $\alpha(x,y) = \operatorname{arctg}(\operatorname{th} x / \operatorname{sh} y) \le \operatorname{th} x / \operatorname{sh} y \le 1/\operatorname{sh} y \le 3\exp(-y)$. □

Lemma B.4 *Let ω and Ω be positive constants and $\omega \le x \le \Omega$, $\omega \le y \le \Omega$. Then*

$$\frac{\operatorname{th}\omega}{\operatorname{ch}^3 \Omega} \le \frac{\partial \alpha(x,y)}{\partial x} \le \frac{1}{\omega} \tag{38}$$

and

$$-\frac{1}{\omega \operatorname{th} \omega} \le \frac{\partial \alpha(x,y)}{\partial y} \le -\frac{\operatorname{th}\omega}{\operatorname{ch}\Omega} \,. \tag{39}$$

Proof.

$$\frac{\partial \alpha(x,y)}{\partial x} = \frac{1}{1 + (\operatorname{th} x / \operatorname{sh} y)^2} \cdot \frac{1}{\operatorname{sh} y \operatorname{ch}^2 x} \le \frac{1}{\operatorname{sh} y \operatorname{ch}^2 x} \le \frac{1}{\omega} \,.$$

Similarly,

$$\frac{\partial \alpha(x,y)}{\partial x} = \frac{\operatorname{sh} y}{\operatorname{sh}^2 y + \operatorname{th}^2 x} \cdot \frac{1}{\operatorname{ch}^2 x} \ge \frac{\operatorname{sh} y}{(\operatorname{sh}^2 y + 1) \operatorname{ch}^2 x} = \frac{\operatorname{th} y}{\operatorname{ch} y \operatorname{ch}^2 x} \ge \frac{\operatorname{th}\omega}{\operatorname{ch}^3 \Omega} \,,$$

$$-\frac{\partial \alpha(x,y)}{\partial y} = \frac{1}{1 + (\operatorname{th} x / \operatorname{sh} y)^2} \cdot \frac{\operatorname{th} x \operatorname{ch} y}{\operatorname{sh}^2 y} \le \frac{\operatorname{ch} y}{\operatorname{sh}^2 y} \le \frac{1}{\omega \operatorname{th}\omega} \,,$$

and

$$-\frac{\partial \alpha(x,y)}{\partial y} = \frac{\operatorname{th} x \operatorname{ch} y}{\operatorname{sh}^2 y + \operatorname{th}^2 x} \ge \frac{\operatorname{th} x \operatorname{ch} y}{\operatorname{sh}^2 y + 1} \ge \frac{\operatorname{th}\omega}{\operatorname{ch}\Omega} \,.$$

□

Lemma B.5 *For every $x > 0$ and $y > 0$ we have*

$$\frac{\partial \overline{\alpha}(x,y)}{\partial x} = -\frac{(\operatorname{ch} x \operatorname{ch} y - 1) \operatorname{sh} y}{\operatorname{ch}^2 x (\operatorname{sh}^2 y + \operatorname{th}^2 x)} < 0. \tag{40}$$

If ω and Ω are positive constants and $\omega \le x \le \Omega$, $\omega \le y \le \Omega$, then

$$-\frac{1}{\operatorname{th}\omega} \le \frac{\partial \overline{\alpha}(x,y)}{\partial x} \le -\frac{\operatorname{th}^3 \omega}{\operatorname{ch}\Omega} \,. \tag{41}$$

Proof. It is routine to show (40). The lower bound of (41) can be established as follows:

$$-\frac{\partial\overline{\alpha}(x,y)}{\partial x} \le \frac{\operatorname{ch} x \operatorname{ch} y \operatorname{sh} y}{\operatorname{ch}^2 x \operatorname{sh}^2 y} \le \frac{\operatorname{ch} y}{\operatorname{sh} y} \le \frac{1}{\operatorname{th}\omega}.$$

The upper bound:

$$-\frac{\partial\overline{\alpha}(x,y)}{\partial x} \ge \frac{(\operatorname{ch} x \operatorname{ch} y - 1)\operatorname{sh} y}{\operatorname{ch}^2 x(\operatorname{sh}^2 y + 1)} \ge \frac{(1-\frac{1}{\operatorname{ch} x \operatorname{ch} y})\operatorname{th} y}{\operatorname{ch} x} \ge \frac{(1-\frac{1}{\operatorname{ch}^2\omega})\operatorname{th}\omega}{\operatorname{ch}\Omega} = \frac{\operatorname{th}^3\omega}{\operatorname{ch}\Omega}.$$

□

Lemma B.6 *Let $\varphi(k) = \operatorname{th}(ka)/\operatorname{sh}(kb)$, where a and b are given positive constants. Then $\frac{d\varphi}{dk} < 0$ for $k > 0$. Moreover, if $0 < \omega \le a \le \Omega$ and $0 < \omega \le b \le \Omega$, then for every k, $\frac{1}{2} \le k \le 2$, we have*

$$\frac{d\varphi}{dk} \le -\frac{\omega^2\Omega^2}{24\operatorname{ch}^8\Omega}.$$

Proof. We will use the notation $a' = ka, b' = kb$. Then

$$\frac{d\varphi}{dk} = \frac{a\operatorname{sh} b' - b\operatorname{sh} a' \operatorname{ch} a' \operatorname{ch} b'}{\operatorname{sh}^2 b' \operatorname{ch}^2 a'} \le \frac{a}{\operatorname{sh}^2 b' \operatorname{ch}^2 a'}(\operatorname{sh} b' - b'\operatorname{ch} b') \le -\frac{ab'^3}{3\operatorname{sh}^2 b' \operatorname{ch}^2 a'}.$$

At the end we have used the fact that $\operatorname{sh} x - x\operatorname{ch} x \le -x^3/3$ for $x \ge 0$. This proves that $d\varphi/dk < 0$. Since $\operatorname{sh} x/x$ is monotone increasing for $x \ge 0$, we also have (using that $k \le 2$)

$$\frac{b'}{\operatorname{sh} b'} \ge \frac{2\Omega}{\operatorname{sh}(2\Omega)} \ge \frac{\Omega}{\operatorname{ch}^2\Omega}.$$

This implies that

$$-\frac{d\varphi}{dk} \ge \frac{ab'}{3\operatorname{ch}^2 a'}\frac{\Omega^2}{\operatorname{ch}^4\Omega} \ge \frac{\omega^2\Omega^2}{24\operatorname{ch}^8\Omega}.$$

The proof is complete. □

Lemma B.7 *Let $\omega = m^{-n}$ and $\Omega = 2\log m$. If $f(\beta_1,\gamma_1) \le \sigma(r)/6$, $\sigma(r)/3 \le f(\beta_2,\gamma_2) \le \sigma(r)/2$, then*

$$\beta_2 - \beta_1 \ge \frac{\omega^7}{200m^3\Omega^2}\sigma(r) \tag{42}$$

and

$$\gamma_1 - \gamma_2 \ge \frac{\omega^7}{200m^3\Omega^2}\sigma(r). \tag{43}$$

Proof. By Lemma 4.6, the pair (β_2,γ_2) is suitable. By Lemma 4.4, $\omega \le \beta_2 r_v \le \Omega$ (if $v \in S$) and $\omega \le \gamma_2 r_u \le \Omega$ (if $u \notin S$). Since f is monotone in β, we also have $1 \le \beta_1 < \beta_2$ and $1 \ge \gamma_1 > \gamma_2$. Hence the above bounds on $\beta_2 r_v$ and $\gamma_2 r_u$ also hold for $\beta_1 r_v$ and $\gamma_1 r_u$, respectively. Using Lemma B.5 we get the following estimate when $u, v \in S$:

$$\begin{aligned}
&\overline{\alpha}(\beta_1 r_u, \beta_1 r_v) - \overline{\alpha}(\beta_2 r_u, \beta_2 r_v) \\
&= \overline{\alpha}(\beta_1 r_u, \beta_1 r_v) - \overline{\alpha}(\beta_1 r_u, \beta_2 r_v) + \overline{\alpha}(\beta_1 r_u, \beta_2 r_v) - \overline{\alpha}(\beta_2 r_u, \beta_2 r_v) \\
&\le r_v(\beta_2 - \beta_1)\frac{1}{\operatorname{th}\omega} + r_u(\beta_2 - \beta_1)\frac{1}{\operatorname{th}\omega} \le \frac{4\Omega}{\omega}(\beta_2 - \beta_1).
\end{aligned} \tag{44}$$

For mixed terms ($v \in S, u \notin S$) we get by Lemma B.4:

$$
\begin{aligned}
&\alpha(\gamma_1 r_u, \beta_1 r_v) - \alpha(\gamma_2 r_u, \beta_2 r_v) \\
&= \alpha(\gamma_1 r_u, \beta_1 r_v) - \alpha(\gamma_1 r_u, \beta_2 r_v) + \alpha(\gamma_1 r_u, \beta_2 r_v) - \alpha(\gamma_2 r_u, \beta_2 r_v) \\
&\le r_v(\beta_2 - \beta_1)\frac{1}{\omega \operatorname{th}\omega} + r_u(\gamma_1 - \gamma_2)\frac{1}{\omega} \le \frac{2\Omega}{\omega^2}[(\beta_2 - \beta_1) + (\gamma_1 - \gamma_2)]\,. \qquad (45)
\end{aligned}
$$

The above bounds (44) and (45) imply:

$$
\begin{aligned}
\frac{\sigma(r)}{6} &\le f(\beta_2, \gamma_2) - f(\beta_1, \gamma_1) \\
&= \sum_{vu \in E(S)} (\overline{\alpha}(\beta_1 r_u, \beta_1 r_v) - \overline{\alpha}(\beta_2 r_u, \beta_2 r_v)) + \\
&\quad \sum_{vu \in E(S,T)} (\alpha(\gamma_1 r_u, \beta_1 r_v) - \alpha(\gamma_2 r_u, \beta_2 r_v)) \\
&\le |E(S)|\frac{4\Omega}{\omega}(\beta_2 - \beta_1) + |E(S,T)|\frac{2\Omega}{\omega^2}(\beta_2 - \beta_1 + \gamma_1 - \gamma_2) \\
&\le \frac{2m\Omega}{\omega^2}(\beta_2 - \beta_1) + \frac{2m\Omega}{\omega^2}(\gamma_1 - \gamma_2)\,. \qquad (46)
\end{aligned}
$$

Similarly as in (44) we get:

$$
\begin{aligned}
&\overline{\alpha}(\gamma_1 r_u, \beta_1 r_v) - \overline{\alpha}(\gamma_2 r_u, \beta_2 r_v) \\
&= \overline{\alpha}(\gamma_1 r_u, \beta_1 r_v) - \overline{\alpha}(\gamma_1 r_u, \beta_2 r_v) + \overline{\alpha}(\gamma_1 r_u, \beta_2 r_v) - \overline{\alpha}(\gamma_2 r_u, \beta_2 r_v) \\
&\le r_v(\beta_2 - \beta_1)\frac{1}{\operatorname{th}\omega} - r_u(\gamma_1 - \gamma_2)\frac{\operatorname{th}^3\omega}{\operatorname{ch}\Omega} \le \frac{\Omega}{\operatorname{th}\omega}(\beta_2 - \beta_1) - \frac{\omega \operatorname{th}^3\omega}{\operatorname{ch}\Omega}(\gamma_1 - \gamma_2) \\
&\le \frac{2\Omega}{\omega}(\beta_2 - \beta_1) - \frac{\omega^4}{8 \operatorname{ch}\Omega}(\gamma_1 - \gamma_2)\,. \qquad (47)
\end{aligned}
$$

Since (β_1, γ_1) and (β_2, γ_2) give rise to normalized radii, we obtain by using (44), (47), and Lemma B.5 the following bound:

$$
\begin{aligned}
0 &= \sum_{vu \in E(S)} (\overline{\alpha}(\beta_1 r_u, \beta_1 r_v) - \overline{\alpha}(\beta_2 r_u, \beta_2 r_v)) + \\
&\quad \sum_{vu \in E(S,T)} (\overline{\alpha}(\gamma_1 r_u, \beta_1 r_v) - \overline{\alpha}(\gamma_2 r_u, \beta_2 r_v)) + \\
&\quad \sum_{vu \in E(T)} (\overline{\alpha}(\gamma_1 r_u, \gamma_1 r_v) - \overline{\alpha}(\gamma_2 r_u, \gamma_2 r_v)) \\
&\le |E(S)|\frac{2\Omega}{\omega}(\beta_2 - \beta_1) + |E(S,T)|\frac{2\Omega}{\omega}(\beta_2 - \beta_1) - |E(S,T)|\frac{\omega^4}{8 \operatorname{ch}\Omega}(\gamma_1 - \gamma_2) \\
&\le \frac{2m\Omega}{\omega}(\beta_2 - \beta_1) - \frac{\omega^4}{8 \operatorname{ch}\Omega}(\gamma_1 - \gamma_2)\,. \qquad (48)
\end{aligned}
$$

It follows that $\gamma_1 - \gamma_2 \le 16m^2\Omega(\beta_2 - \beta_1)/\omega^5$. Combining (48) with (46) we easily prove (42).

Note that in the above proof β and γ could exchange their role. Therefore, using similar calculation as in obtaining (48) we get $\beta_2 - \beta_1 \le 16m^2\Omega(\gamma_1 - \gamma_2)/\omega^5$, and the combination with (46) yields (43). □

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