

On the Minimal Genus of 2-Complexes

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ABSTRACT

Gross and Rosen asked if the genus of a 2-dimensional complex K embeddable in some (orientable) surface is equal to the genus of the graph of appropriate barycentric subdivision of K . We answer the nonorientable genus and the Euler genus versions of Gross and Rosen's question in affirmative. We show that this is not the case for the orientable genus by proving that taking $\lceil \log_2 g \rceil$ th barycentric subdivision is not sufficient, where g is the genus of K . On the other hand, $(1 + \lceil \log_2(g+2) \rceil)$ th subdivision is proved to be sufficient. © 1997 John Wiley & Sons, Inc.

1. INTRODUCTION

Let K be a 2-dimensional cell complex (or a CW-complex). We say that K is (*surface*) *extendible* if K admits an embedding in some surface. It is easy to find local combinatorial conditions on K characterizing extendible complexes [3, 5, 9]. If K is extendible, then we define its *Euler genus* $\gamma_E(K)$ as the number $2 - \chi$ where χ is the maximal Euler characteristic of a surface in which K can be embedded. Similarly, K is *orientably extendible* if it has an embedding into an orientable surface. Then we define its genus $\gamma(K)$ as the smallest genus of an orientable surface into which K can be embedded. In the same way we define *nonorientably extendible complexes* and their *nonorientable genus*.

Let K^1 (respectively $K^{(r)}$) denote the *barycentric subdivision* (respectively, the r th barycentric subdivision) of K . The *graph* of a 2-complex K is defined as the 1-skeleton of K . The graph of $K^{(r)}$ will be denoted by $K_{(r)}$. Gross and Rosen [3] have shown that an arbitrary 2-complex

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K is planar (has genus 0) if and only if $K_{(3)}$ is a planar graph. Moreover, if K is simplicial, then already the first barycentric subdivision suffices. They asked if the same holds in general: Is the genus of an arbitrary orientably extendible 2-complex equal to the genus of the graph of its (third or some other) barycentric subdivision. In this note we prove that for each r there exist orientably extendible 2-complexes K such that the difference between the genus of K and the genus of the graph of $K^{(r)}$ is arbitrarily large. This settles the question of Gross and Rosen in negative. On the other hand, we show that for every g and every orientably extendible 2-complex K of genus g , the graph of the $(1 + \lceil \log_2(g + 2) \rceil)$ th barycentric subdivision has the same genus as K . This result is best possible in the sense that $\lfloor \log_2 g \rfloor$ th subdivisions do not suffice. This settles the question of Gross and Rosen in affirmative. Moreover, the nonorientable genus and the Euler genus behave differently. We prove that the Euler genus of an arbitrary 2-complex (or CW-complex) K is equal to the genus of the graph of $K^{(7)}$. (In case of simplicial complexes, already the fifth subdivision suffices.) A similar result holds for the nonorientable genus. As a corollary we get that the genus problems for graphs and extendible 2-dimensional complexes are polynomially equivalent from the computational complexity point of view.

2. EXTENDIBLE COMPLEXES AND 2-CELL EMBEDDINGS

2-cell embeddings of a connected graph can be defined in a purely combinatorial setting by specifying:

- (1) A *rotation system* $\pi = (\pi_v; v \in V(G))$; for each vertex v of the given graph G we have a cyclic permutation π_v of edges incident with v , representing their circular order around v on the surface.
- (2) A *signature* $\lambda : E(G) \rightarrow \{-1, 1\}$. Suppose that $e = uv$. Following the edge e on the surface, we see if the local rotations π_v and π_u are chosen consistently or not. If yes, then we have $\lambda(e) = 1$, otherwise we have $\lambda(e) = -1$. We may assume that λ is positive on an arbitrary spanning tree of G .

The reader is referred to [4] or [8] for more details. We will use this description as a definition: An *embedding* of a connected graph G is a pair $\Pi = (\pi, \lambda)$ where π is a rotation system and λ is a signature. Having an embedding Π of G , we say that G is Π -embedded. Every embedding determines a unique (up to homeomorphisms) topological 2-cell embedding in some closed surface. We define the Π -*facial walks* as closed walks in the graph which correspond to faces of the corresponding topological embedding.

Having a Π -embedded graph G , we shall use the prefix Π to refer to concepts depending on the embedding. For example, we say that a cycle of G is Π -*contractible* if it bounds a disk on the surface of Π ; the Π -*interior* of a Π -contractible cycle C is defined as the subgraph of G contained in the disk bounded by C , etc. A cycle of G with an odd number of edges e having $\lambda(e) = -1$ is Π -*onesided*.

A similar description can be given for 2-cell embeddings of 2-dimensional complexes in closed surfaces where parts of the rotation system and partial signature are prescribed. This description was already considered (in a slightly more general setting) by Širáň and Škoviera [9, 10].

If K is a simplicial 2-complex, its 0-simplices and 1-simplices are also called *vertices* and *edges* of K , respectively. If v is a vertex of K , its *link* in K is a subcomplex consisting of all vertices of K adjacent to v and all edges uw of K such that vuw is a 2-simplex in K .

The following results can be verified using the rotation system description, and we omit their proofs.

Lemma 2.1. *A connected simplicial 2-complex K is extendible if and only if the link in K of each vertex is either a simple cycle or union of disjoint paths in the graph of K .*

Lemma 2.2. *Suppose that a simplicial complex K is extendible. Then K is orientably extendible if and only if K does not contain a Möbius band.*

To use Lemmas 2.1 and 2.2 on a nonsimplicial 2-complex K , we first triangulate K and then apply the lemmas.

Lemma 2.3. *Suppose that a 2-complex K is surface extendible. Then K is nonorientably extendible if and only if K is not homeomorphic to an orientable closed surface.*

If one is interested in 2-cell embeddings, then an extendible complex whose graph is 2-connected has a nonorientable 2-cell embedding if and only if it is not homeomorphic to an orientable (bordered) surface.

Given K , we shall also consider complexes M that can be (topologically) embedded in K where M is not necessarily isomorphic to a subcomplex of K . It is clear that if K is (orientably or nonorientably) extendible, so is M . Moreover, the (Euler) genus of K is always larger than or equal to the (Euler) genus of M .

3. MÖBIUS OBSTRUCTIONS

In this section we will describe examples for which the graphs of rather fine subdivisions have smaller genus than the complex itself.

Suppose that we have a subcomplex D of K that is homeomorphic to a disk. Let C be its boundary cycle. Suppose that $K \setminus \text{int}D$ contains disjoint paths P_1, \dots, P_k (not necessarily in its graph) whose ends a_i, b_i (respectively), $i = 1, \dots, k$, appear on the boundary of D in interlaced order: $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_k$. Let M be the complex composed of D and the paths P_1, \dots, P_k . Then M can be embedded in K . We say that M is a *Möbius obstruction of order k* in K .

Suppose that G is a Π -embedded graph. The smallest integer $k \geq 1$ such that there are Π -facial walks F_1, \dots, F_k whose union (as a subgraph of G) contains a Π -noncontractible cycle is called the *face-width* (or *representativity*) of Π and is denoted by $\text{fw}(\Pi)$.

Theorem 3.1. *Let K be an orientably extendible 2-complex that contains a Möbius obstruction of order $k \geq 3$. Then the genus of K is at least $\lfloor k/2 \rfloor$ and the genus of the graph of its r th barycentric subdivision ($r \geq 1$) is at least $\min\{\lfloor k/2 \rfloor, 2^{r-1}\}$.*

Proof. Denote by M that Möbius obstruction in K . Then $M^{(r)}$ can be embedded in $K^{(r)}$. Therefore it suffices to show that the genus of the graph $G = M_{(r)}$ is as claimed. Note that G has a natural embedding in the projective plane. It is easy to see that the face-width ρ of the embedding of G in the projective plane is at least $\min\{k, 1 + 2^r\}$. By a theorem of Fiedler, Huneke, Richter, and Robertson [2], the genus of G is equal to $\lfloor \rho/2 \rfloor$ (if $\rho \neq 2$ which clearly holds in our case since $k \geq 3$ and $r \geq 1$). This proves the claim about the genus of G . Since the genus of K cannot be smaller than the genus of any of its barycentric subdivisions, also the claim about the genus of K is verified. ■

Graphs similar to the Möbius obstructions have been used by Auslander, Brown, and Youngs [1] to show existence of projective planar graphs whose orientable genus is arbitrarily large.

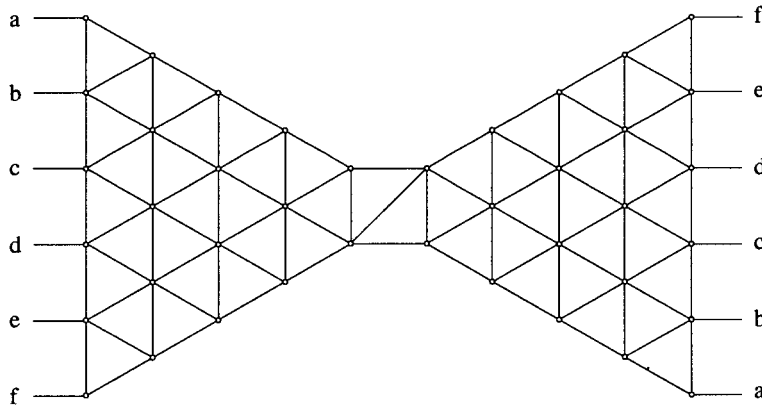


FIGURE 1. A Möbius obstruction of order 6.

The above proof gives more accurate result about the genus of $M_{(r)}$ than just a lower bound. For any specific M , it yields exact formula for its genus in terms of the face-width. In particular, we have:

Corollary 3.2. *For every $g > 0$ there exists an orientably extendible 2-complex of genus g such that the genus of the graph of its r th barycentric subdivision is equal to $\min\{g, 2^{r-1}\}$.*

Proof. Let K be the Möbius obstruction of order $2g$ as represented in Figure 1 for order 6 (with shown triangles being 2-cells of the complex). Using the notation from the proof of Theorem 3.1, the face-width of G is equal to $\rho = \min\{2g, 1 + 2^r\}$. By [2], the genus of G is equal to $\lfloor \rho/2 \rfloor = \min\{g, 2^{r-1}\}$ as claimed. By Theorem 3.1, the genus of K is g . The proof is complete. ■

The bound of Theorem 3.1 and the examples of Corollary 3.2 show that there are 2-complexes of genus $g \geq 2$ such that the graph of the r th barycentric subdivision has genus strictly smaller than g , $r = 0, 1, \dots, \lfloor \log_2 g \rfloor$. On the other hand, we shall prove in the last section that this is the worst case, since the same cannot happen for the r th barycentric subdivision when $r \geq 1 + \log_2(g + 2)$, g being the genus of K .

It is worth mentioning that the Möbius obstruction is no longer an obstruction if we minimize the nonorientable or the Euler genus of K .

The Möbius obstruction can be generalized as follows. Suppose that we have an embedding of a 2-complex K in a nonorientable surface S . A set $\{Q_1, \dots, Q_t\}$ of simple closed curves in S is a *blocker* if every 1-sided curve in S crosses at least one of the curves $Q_i, 1 \leq i \leq t$. The *order* of the blocker $\{Q_1, \dots, Q_t\}$ is equal to $\sum_{i=1}^t cr(Q_i, K)$ where $cr(Q_i, K)$ denotes the number of times Q_i intersects K , i.e., $cr(Q_i, K) = |Q_i \cap K|$. The minimum order of a blocker and the genus of K are closely related in the same way as the face-width of the Möbius obstructions in the projective plane and the genus are related by [2]. This can be derived from results of Mohar and Schrijver [7]. After replacing the complex K by the graph of its subdivision, one may get a blocker with smaller order than in K since some curves can now pass through the former 2-cells while intersecting the graph only finitely many times. In such a case we get another type of obstruction: a 2-complex whose genus is larger than the genus of the graphs of its subdivisions.

The above generalization again uses nonorientability and does not give rise to obstructions for the Euler or the nonorientable genus case. We shall see in the sequel what are the reasons for that.

4. THE EULER GENUS

In this section we shall prove that the Euler genus behaves differently from the orientable genus by proving that the Euler genus of an arbitrary 2-complex (or CW-complex) K is equal to the genus of the graph of $K^{(7)}$.

We shall need three lemmas.

Lemma 4.1. *Let G be a Π -embedded graph, let $v \in V(G)$ be a vertex of degree three in G , and let Π' be the embedding of G obtained from Π by reversing the local rotation at v . If v is contained in α Π -facial and in β Π' -facial walks, then*

$$\gamma_E(\Pi') = \gamma_E(\Pi) + \alpha - \beta.$$

In particular, $\gamma_E(\Pi') \leq \gamma_E(\Pi) + 2$.

Proof. Obvious by comparing the number of Π - and Π' -facial walks. ■

Lemma 4.2. *Let $C = v_1v_2 \cdots v_dv_1$ be a d -cycle in a Π -embedded graph G such that each vertex $v_i, 1 \leq i \leq d$, has degree three. Then G has an embedding Π' whose restriction to $G - C$ is the same as the restriction of Π to $G - C, C$ is a Π' -facial cycle, and*

$$\gamma_E(\Pi') \leq \gamma_E(\Pi) + 2\lfloor d/2 \rfloor - 1. \tag{1}$$

Proof. We may assume that the signature λ of Π is positive on edges $e_i = v_iv_{i+1}, i = 1, \dots, d - 1$. By following C from v_1 towards v_d , we meet at most $\lfloor d/2 \rfloor$ edges on one of the sides, say on the left. By reversing the local rotation at those vertices of C that have edges on the left, the cycle C either becomes facial (if $\lambda(v_1v_d) = 1$) or a segment of a facial walk. In the former case, Lemma 4.1 shows that the resulting embedding Π' satisfies

$$\gamma_E(\Pi') \leq \gamma_E(\Pi) + 2\lfloor d/2 \rfloor - 2 \tag{2}$$

since after each reversal of a local rotation the Euler genus increases at most by 2, and at the very last change we have a vertex contained in at most two facial walks ($\alpha \leq 2$) and in at least two new facial walks ($\beta \geq 2$). In the latter case ($\lambda(v_1v_d) = -1$), we can change the signature of the edge v_1v_d to positive and the Euler genus will increase by 1. Lemma 4.1 now implies that the resulting embedding Π' satisfies (1). ■

Lemma 4.3. *Suppose that a graph G can be written as $G = G_1 \cup \Delta$ where Δ is the graph from Figure 2 (isomorphic to the Cartesian product $C_9 \square P_9$) such that $Q = G_1 \cap \Delta$ is the outer cycle of Δ . If Π is a minimum Euler genus embedding of G , then G has an embedding Π' such that:*

- (a) $\gamma_E(\Pi') = \gamma_E(\Pi)$.
- (b) The restrictions of Π' and of Π to $G_1 - E(Q)$ are the same.
- (c) Every cycle of Δ is Π' -contractible.

Proof. We may assume that $\gamma_E(\Pi) > 0$. (Otherwise we take $\Pi' = \Pi$.) Let $Q_1 = Q, Q_2, \dots, Q_9$ be the ‘‘nested’’ 9-cycles of Δ . Suppose that for some $i, 2 \leq i \leq 9, Q_i$ is Π -contractible. Then Q_i either bounds a face, or contains in its interior the cycles Q_{i+1}, \dots, Q_9 . By contracting the paths from each vertex of Q_1 to the corresponding vertex of Q_i and replacing the embedded cycle Q_i by Q , we get an embedding Π_1 of G_1 such that Q is a Π_1 -facial

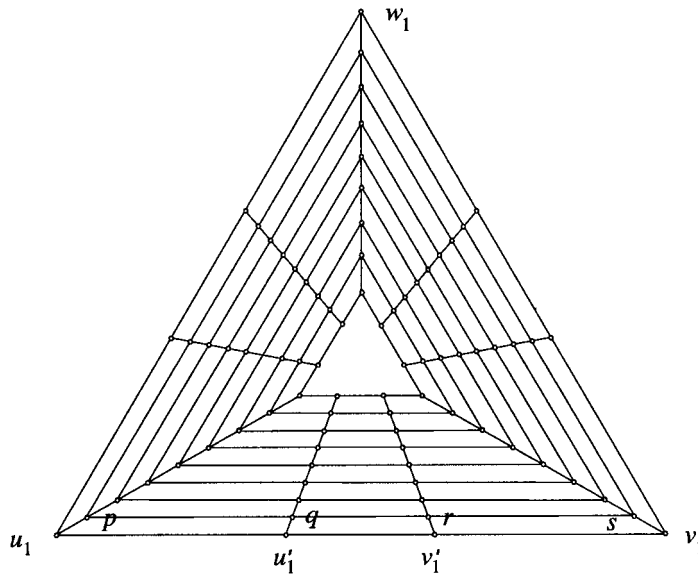


FIGURE 2. The graph $\Delta = C_9 \square P_9$.

cycle. Now we can draw Δ inside the Π_1 -face bounded by Q . The resulting embedding Π' of G satisfies (a)–(c).

A similar approach works if a subset \mathcal{C} of $\{Q_2, \dots, Q_9\}$ separates the surface of the embedding Π . In that case we take as Q_i the cycle from \mathcal{C} with the minimal index i . If no subset of \mathcal{C} separates, then the induced embedding Π'' of $G'' = G - (Q_3 \cup \dots \cup Q_9)$ satisfies $\gamma_E(\Pi'') \leq \gamma_E(\Pi) - 7$. (A more detailed treatment of these facts can be found in [6].) By Lemma 4.2, Π'' can be changed to an embedding Π''' of G'' such that the cycle Q_2 is Π''' -facial and such that $\gamma_E(\Pi''') \leq \gamma_E(\Pi'') + 7 \leq \gamma_E(\Pi)$. Clearly, Π''' can be extended to an embedding of G of the same Euler genus such that Q_2 is contractible. By the above, we get an embedding Π' satisfying (a)–(c). ■

The reader is referred to Section 6 for a discussion on how to use the similar approach in the orientable case. In that case, Δ is replaced by other products $C_m \square P_k$.

Now we are able to prove the main result of this section.

Theorem 4.4. *Suppose that K is a surface extendible simplicial 2-complex. Let G be the graph obtained from K by replacing each 2-simplex $T = uvw$ and its edges by a copy of the graph Δ from Figure 2 such that u, v, w are identified with vertices u_1, v_1, w_1 of Δ (respectively) and such that for any pair of 2-simplices sharing an edge uv , the paths $u_1 u'_1 v'_1 v_1$ of the corresponding copies of Δ are identified. Then the Euler genus of the complex K is equal to the Euler genus of the graph G .*

Proof. If T is a 2-simplex of K , we denote by $\Delta(T)$ the corresponding copy of Δ in G . Also, let $Q_2(T)$ be the cycle Q_2 of $\Delta(T)$ (the outer cycle in $\Delta - Q_1$). Let Π_0 be a minimum Euler genus embedding of G . By Lemma 4.3 and since the cycles $Q_2(T)$ are pairwise disjoint for distinct 2-simplices T , we may assume that for each 2-simplex T , the cycle $Q_2(T)$ is Π_0 -contractable and that the inner cycles $Q_3(T), \dots, Q_9(T)$ are embedded in the Π_0 -interior of the cycle $Q_2(T)$.

Let T, T' be distinct 2-simplices of K sharing an edge uv . Let G' be the subgraph of G obtained by removing the common edges of $\Delta(T)$ and $\Delta(T')$. Denote by Π' the induced embedding of

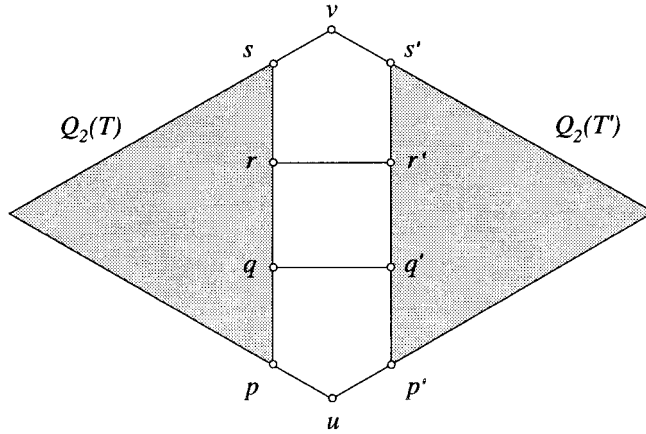


FIGURE 3. A part of the graph G' .

G' . Consider the subgraph of G' “between” $Q_2(T)$ and $Q_2(T')$ as shown in Figure 3. We shall suppress the two vertices of degree two and shall consider rr' and qq' as being edges of G' . Our goal is to change the embedding Π' (without increasing the Euler genus) to an embedding where the cycles $upq q' p' u$, $qrr' q' q$, and $vs' r' r s v$ are all facial. After doing so, we shall be able to embed the omitted edges of $\Delta(T) \cap \Delta(T')$ into these faces. If we succeed to perform the same changes everywhere, the resulting embedding of G gives rise to an embedding of K in the same surface, and we are done. So, we shall assume that for several pairs T, T' , the above change was already made, and by performing the next one, we shall try not to spoil our previous work.

Returning to Figure 3, we may assume that the local rotations at vertices $p, q, r, s, p', q', r', s'$ are as shown in the figure. The signature λ of Π' is then positive on each of $Q_2(T)$ and $Q_2(T')$, while we cannot tell its values on edges between $Q_2(T)$ and $Q_2(T')$. If λ is negative on one or both edges qq' and rr' , we change it to positive. After this change, the cycle $qrr' q' q$ becomes facial, and it is easy to see that the number of facial walks does not decrease. Therefore, the Euler genus does not increase. Now, there is a facial walk F containing the segment $W = upq q' p' u$ as a subwalk. If $F = W$, our goal of making W facial is achieved. Otherwise we change the local rotation at u as described below.

We may assume that λ is positive on the edge up . To make W facial, it is (necessary and) sufficient that the embedding is locally changed at u so that the signature is positive also on up' and such that up is followed by up' in the local rotation at u . Let us first assume that $\lambda(up') = 1$. Suppose that $\pi_u = (e_1 e_2 \cdots e_d)$ is the local rotation at u where $e_1 = up, e_t = up', 2 \leq t \leq d$. Denote by F_i the facial walk using consecutive edges e_i, e_{i+1} (indices modulo d). Observe that $F_1 = F_{t-1} = F$. Since K is surface extendible, there is an index $i, t \leq i \leq d$, such that the face F_i is not one of the previously fixed faces (or else we have $t = 2$ in which case we are done). Now we replace π_u by $\pi'_u = (e_1 e_t e_{t+1} \cdots e_i e_2 \cdots e_{t-1} e_{i+1} \cdots e_d)$. The only facial walks that are changed are $F_1 (= F_{t-1})$ and F_i . Clearly, W becomes a new facial cycle, and we get another one (or possibly two if $F_1 = F_i$) by concatenating F_i and a part of F_1 . Hence the Euler genus of the new embedding does not increase.

The other possibility is that $\lambda(up') = -1$. In that case we replace π_u by $\pi'_u = (e_1 e_t e_{t-1} \cdots e_2 e_{t+1} \cdots e_d)$ and for $j = 2, \dots, t$, we replace $\lambda(e_j)$ by $-\lambda(e_j)$. Then W becomes a facial cycle and the only Π' -facial walks that are changed are F_1 and $F_t (= F_1)$. Hence the Euler genus does not increase. (In fact, it is decreased by 1 which shows that this case indeed does not occur.)

The same procedure is then applied to the other cycle $W' = vs'r'sv$. Our goal is then achieved, and the proof is complete. ■

Theorem 4.4 implies a result on barycentric subdivisions.

Corollary 4.5. *If K is a surface extendible simplicial 2-complex, then the Euler genus $\gamma_E(K)$ is equal to the Euler genus of the graph $K_{(5)}$ of its fifth barycentric subdivision. If K is an arbitrary surface extendible 2-complex (or CW-complex), then the Euler genus $\gamma_E(K)$ is equal to the Euler genus of $K_{(7)}$.*

Proof. By Theorem 4.4, it suffices to prove that every subdivided 2-simplex of K contains a graph homeomorphic to Δ in the subdivision graph $K_{(5)}$ of K . Denote by Δ_q the graph $C_9 \square P_q$ (so that $\Delta_9 = \Delta$). Then it is easy to see that a subdivided 2-simplex in $K^{(2)}$ contains Δ_2 , in $K^{(3)}$ it contains Δ_4 , in $K^{(4)}$ we have Δ_8 , and in $K^{(5)}$ there is Δ_{16} (and so also Δ_9), where the outer cycle of Δ_q always corresponds to the boundary of the simplex.

The second part of the corollary follows from the above since $K^{(2)}$ is simplicial. ■

It is very likely that a slightly more detailed proof would allow us to replace $K_{(5)}$ in Corollary 4.5 by the graph of the fourth subdivision.

5. THE NONORIENTABLE GENUS

By using some additional nested cycles in Δ , e.g., by taking $\Delta = C_{12} \square P_{14}$, Lemma 4.3 can be formulated for the nonorientable genus instead of the Euler genus. Let us note that we may need to work with embeddings that are not 2-cell. (If K has a nonorientable 2-cell embedding, this trouble can be superseded.) Also, the proof of Theorem 4.4 works. In Figure 3 we now have three edges instead of two between $Q_2(T)$ and $Q_2(T')$. This makes it possible to maintain nonorientable embeddings. Since a subdivision of $C_{12} \square P_{14}$ is also contained in the fifth barycentric subdivision of a 2-simplex, we get:

Corollary 5.1. *If K is a nonorientably extendible simplicial 2-complex, then the nonorientable genus of K is equal to the nonorientable genus of the graph $K_{(5)}$ of its fifth barycentric subdivision. If K is an arbitrary nonorientably extendible 2-complex (or CW-complex), then its nonorientable genus is equal to the nonorientable genus of $K_{(7)}$.*

6. THE ORIENTABLE GENUS

We have shown in Section 3 that no fixed depth (barycentric) subdivision yields the same statement for the orientable genus as we have for the Euler or the nonorientable genera (Theorem 4.4 and Corollaries 4.5 and 5.1). The proof of Corollary 3.2 also shows that by choosing $\Delta = C_{3p} \square P_s$ with either $p < 2g$ or $s < g$, where g is the genus of K , will not suffice. On the other hand, we shall prove that taking $\Delta = C_{6g+6} \square P_{g+2}$ is sufficient to get the same statement as for the Euler and the nonorientable genera.

Theorem 6.1. *Suppose that K is an orientably extendible simplicial 2-complex of genus g . Let G be the graph obtained from K by replacing each 2-simplex $T = uvw$ and its edges by a copy of the graph $\Delta = C_{6g+6} \square P_{g+2}$ as in Theorem 4.4. Then the genus of the graph G is equal to the genus of the complex K .*

Proof. The proof is basically the same as the proof of Theorem 4.4, and we shall use the notation introduced in that proof. First observe that the orientable genus of an embedding Π is equal to one half of the Euler genus. Lemma 4.3 holds also for orientable embeddings if we replace $C_9 \square P_9$ by $\Delta = C_{6g+6} \square P_{g+2}$. Its proof in this case is even simpler (cf. [6]) since we can exclude the case where Lemma 4.2 is applied. All cycles $Q_2(T)$ (T a 2-simplex) are Π_0 -contractible. We define G' as before except that now we have $2g + 1$ edges $q_1q'_1, q_2q'_2, \dots, q_{2g+1}q'_{2g+1}$ (instead of the two edges qq' and rr') between $Q_2(T)$ and $Q_2(T')$. If all cycles $Q_i = q_iq_{i+1}q'_{i+1}q'_iq_i$ are Π_0 -facial, we can make W and W' facial and conclude the proof in the same way as we did in the proof of Theorem 4.4.

Suppose now that some Q_i is not Π_0 -facial. Then none of Q_1, \dots, Q_{2g} is facial. Assuming that Π_0 has local rotations as shown in Figure 3, the signature λ of Π_0 is negative on $q_iq'_i, i = 1, \dots, 2g + 1$, and we may also assume, since Π_0 is orientable, that $\lambda(up) = \lambda(vs) = 1$ and $\lambda(up') = \lambda(vs') = -1$. By changing the signature on the edges $q_iq'_i$ to positive, $i = 1, \dots, 2g + 1$, an easy Euler characteristic count shows that the resulting embedding Π' is an embedding in the projective plane. However, the cycles $upq_1q'_1p'u$ and $vs'q'_{2g+1}q_{2g+1}sv$ are disjoint and both Π' -onesided. This is a contradiction, and the proof is complete. ■

Corollary 6.2. *Let K be an orientably extendible simplicial 2-complex and $r \geq 1 + \log_2(\gamma(K) + 2)$. Then the genus $\gamma(K)$ is equal to the genus of the graph $K_{(r)}$ of its r th barycentric subdivision. If K is an arbitrary surface extendible 2-complex (or CW-complex), then the genus $\gamma(K)$ is equal to the genus of $K_{(r+2)}$.*

Proof. By Theorem 6.1, it suffices to see that the graph of the r th barycentric subdivision of a 2-simplex contains a subdivision of the graph $C_{6g+6} \square P_{g+2}$ ($g = \gamma(K)$) as a subgraph. The verification of this fact is left to the reader. ■

The results obtained above yield the following interesting corollary about the computational complexity of the minimal genus problem for extendible 2-dimensional complexes.

Corollary 6.3. *The problem of determining the orientable (respectively nonorientable or Euler) genus of a graph and the problem of determining the orientable (respectively the nonorientable or the Euler) genus of a 2-complex are polynomially equivalent.*

Proof. The genus problem for complexes contains the graph genus problem as a special case. Conversely, a 2-complex K with m edges has the genus bounded by m . Let G be the graph obtained from the simplicial complex $K^{(2)}$ by replacing each simplex with $\Delta = C_{6m+6} \square P_{m+2}$ as in Theorem 6.1. By Theorem 6.1, the genus of G and the genus of K coincide. The same reduction works for the nonorientable and for the Euler genus. ■

Let us recall that Thomassen proved that the genus and the nonorientable genus problems for graphs are NP-complete [11, 12].

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