Elimination of local bridges*

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Abstract

Let K be a subgraph of G. It is shown that if G is 3–connected modulo K then it is possible to replace branches of K by other branches joining same pairs of main vertices of K such that G has no local bridges with respect to the new subgraph K. A linear time algorithm is presented that either performs such a task, or finds a Kuratowski subgraph K_5 or $K_{3,3}$ in a subgraph of G formed by a branch e and local bridges on e. This result is needed in linear time algorithms for embedding graphs in surfaces.

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1 Introduction

Let K be a subgraph of a simple graph G. A *relative K-component* or a *K-bridge* is a subgraph of G which is either an edge $e \in E(G)\backslash E(K)$ (together with its endpoints) with both endpoints in K, or it is a connected component of $G-V(K)$ together with all edges (and their endpoints) between this component and K. Each edge of a relative K-component R having an endpoint in K is a *foot* of R. The vertices of $R \cap K$ are the *vertices of attachment* of R. A vertex of K of degree in K different from 2 is a *main vertex* of K. For convenience, if a connected component of K is a cycle, then we choose an arbitrary vertex of it and declare it to be a main vertex of K as well. A *branch* of K is any path (possibly a closed path) in K whose endpoints are main vertices but no internal vertex on this path is a main vertex. If a relative K-component is attached at a single branch of K, it is said to be *local*. Otherwise, it is *global*.

G is 3-*connected modulo* K if for every vertex set $X \subset V(G)$ with at most 2 elements, every connected component of $G - X$ contains a main vertex of K. This is obviously equivalent to the following condition: If $G^+(K)$ is the graph obtained from G by adding three mutually adjacent new vertices whose additional neighbours are the main vertices of K, then $G^+(K)$ is 3-connected. On the other hand, if K is homeomorphic to a 3-connected graph, then G is 3-connected modulo K if and only if it is 3-connected.

This paper is part of a larger project [8, 9] which shows that there is a linear time algorithm to construct embeddings of graphs in an arbitrary (fixed) surface, generalizing the well–known Hopcroft–Tarjan algorithm [6] testing planarity in linear time. These algorithms rely on the theory of bridges: a subgraph K of G is embedded in the surface and then this embedding is either extended to an embedding of G, or an obstruction for such extensions is found. One of the difficulties in this approach are local bridges. In this paper it is shown how to overcome this problem.

We believe that our results can also be used in some other problems involving bridges (see, e.g., [12]).

In our algorithm, we need plane embeddings of graphs. These can be described combinatorially [5] by specifying a *rotation system*: for each vertex v of the graph G we have the cyclic permutation π_v of its neighbours, representing their circular order around v on the surface. In order to make a clear presentation of our algorithm, we have decided to use this description only implicitly. Whenever we say that we have an embedding, we mean such a combinatorial description.

There are very efficient (linear time) algorithms which for a given graph determine whether the graph is planar or not. The first such algorithm was obtained by Hopcroft and Tarjan [6] back in 1974. There are several other linear time planarity algorithms (Booth and Lueker [1], Fraysseix and Rosenstiehl [4], Williamson [13, 14]). The extensions of original algorithms produce also an embedding (rotation system) whenever the given graph is found to be planar [2], or find a small obstruction — a subgraph homeomorphic to K_5 or $K_{3,3}$ — if the graph is non-planar [13, 14].

Concerning the time complexity of our algorithms, we assume a random-access machine (RAM) model with unit cost for basic operations. This model was introduced by Cook and Reckhow [3]. More precisely, our model is the *unit-cost* RAM where operations on integers, whose value is $O(n)$, need only constant time (n is the size of the given graph).

2 Elimination of local bridges

In many cases, it is useful to know that a subgraph K of G has no local bridges. There are 2–connected graphs G, $K \subseteq G$, such that it is not possible to find a subgraph $K' \subseteq G$ homeomorphic to K without local bridges. Suppose, for example, that K contains a branch e with at least one local bridge and no global bridge attached to it. Then it is not possible to eliminate local bridges on e by replacing e with another branch. However, if G is 3–connected, such an anomaly can not appear. A strengthening of this fact is exposed by the next result.

Proposition 2.1 *Let $K \subseteq G$ and let e be a branch of K joining main vertices x, y of K. Suppose that G is 3–connected modulo K. Then e can be replaced by a branch e' joining x and y, which is internally disjoint from $K-e$ such that there are no local bridges of $K-e+e'$ attached to e'. Consequently, it is possible to replace K by a subgraph K' of G homeomorphic to K having the same set of main vertices and such that there are no local K'-bridges.*

Proof. Traversing e from x toward y induces a linear order $\leq$, with $p \leq q$ iff in the above traversal, p is encountered before q.

Let $B_1, \ldots, B_k$ be all the K-bridges local on e. For every bridge B_i, let p_i and q_i be its endmost vertices of attachment, i.e. closest to x and y, respectively. Let H be the graph consisting of the branch e together with $B_1, \ldots, B_k$. Our proof is based on induction on the number of edges $|E(H)|$.

The base case is $|E(H)| = |E(e)|$. Since H is connected and $e \subseteq H$, it follows that $H = e$ and there are no local bridges on e.

Otherwise, suppose that $|E(H)| > |E(e)|$. We select a subsequence $B_{j_1}, \ldots, B_{j_m}$ of bridges $B_1, \ldots, B_k$ in the following way. Let $j_1$be such that p_{j_1} is the minimal element of the set $\{p_1, \ldots, p_k\}$ and let B_{j_1} be the corresponding bridge. Among all bridges B_i with $p_{j_1} \leq p_i < q_{j_1}$, select one with the maximal q_i. If $q_i \leq q_{j_1}$, let $m := 1$ and stop. If $q_i > q_{j_1}$, let $j_2 := i$ and proceed, looking for the maximal q_i of the bridges B_i with $p_{j_1} \leq p_i < q_{j_2}$, and so forth. If, after the selection of m bridges, the maximal vertex q_i equals to q_{j_m}, we stop. Since G is 3-connected mod K, it follows that there must be a vertex of attachment $r \in V(e)$ of a global bridge B strictly between p_{j_1} and q_{j_m}, or those two vertices would be cutvertices in the graph G yielding a component with no main vertices of K. From the above construction of the subsequence $B_{j_1}, \ldots, B_{j_m}$ it follows that there is a bridge B_{j_s}, $1 \leq s \leq m$, such that $p_{j_s} < r < q_{j_s}$. Now we construct a new path e'. We take e' to be the part of e between x and p_{j_s} and the part of e between q_{j_s} and y (one or both possibly trivial), joined together with a path from p_{j_s} to q_{j_s} in B_{j_s}, internally disjoint from e. Let $B'_1, \ldots, B'_\ell$ be the local bridges on e', and H' be the graph consisting of e' together with all the bridges $B'_1, \ldots, B'_\ell$. The new local bridges are either ones local on e (attached to $e \cap e'$), or are parts of the local bridge B_{j_s}. It follows that $H' \subseteq H$. However, the part of e between p_{j_s} and q_{j_s} becomes a part of a global bridge B' containing B. Note that the local bridges with a vertex of attachment strictly between p_{j_s} and q_{j_s} are merged into the global bridge B'. The part of e between p_{j_s} and r contains at least one edge, since $p_{j_s} < r$. Similarly, the part between r and q_{j_s} also contains at least one edge. Therefore, those two edges are not the part of H' and it follows that $|E(H')| < |E(H)|$.

We can repeat the above steps several times and since the original H contains only finitely many edges, it follows that the procedure reaches the base case. Finally we replace the original path e with the path e' from the last step. □

3 A linear time algorithm

Unfortunately, the proof of Proposition 2.1 yields a quadratic time algorithm for the local bridges elimination. It is possible to improve it into an $\mathcal{O}(n \log n)$ algorithm by some additional more sophisticated methods [7]. However, in various applications (e.g., [8, 9]), a linear time procedure is desired. A solution that is suitable for the applications in surface embedding algorithms is presented in this section. If L is a subgraph of G homeomorphic to K_5 or to $K_{3,3}$, we say that L is a *Kuratowski subgraph* of G. If H is a graph and $x, y \in V(H)$, denote by $H + xy$ the graph obtained from H by adding a new edge between x and y.

Lemma 3.1 *Let $K \subseteq G$ and let e be a branch of K joining main vertices x, y of K. Suppose that G is 3–connected modulo K. There is a linear time algorithm that performs one of the following:*

(1) Replaces e by a branch e' joining x and y which is internally disjoint from $K - e$ such that there are no local bridges of $K - e + e'$ attached to e'.

(2) Finds a subgraph L of G which is a Kuratowski subgraph of $G + xy$ such that $L \cap K \subseteq e$.

Proof. Let N be the graph obtained from the branch e by adding all local bridges attached to it. If the graph $N + xy$ is planar, consider one of its plane embeddings. Let W be the facial walk of one of the faces containing xy. Since G is 3–connected modulo K it follows easily that $N + xy$ is 2–connected and hence W is a (simple) cycle. Now we replace e by $e'' := W - xy$. The set of local bridges is modified accordingly. Some of the previous local bridges might merge together into a new local bridge, others might become global with respect to the changed subgraph K of G (and are therefore removed from consideration). But since the graph G is 3–connected modulo K, no new local bridges arise. Let N' be the modified graph of local bridges. By the above, $N' + xy \subseteq N + xy$. Using the induced plane embedding of $N' + xy$, we repeat the above procedure by selecting the "other" facial walk W' of the face containing xy on its boundary. Let $e' := W' - xy$ be the new branch replacing e'. One can show that e' has no local bridges attached to it (see [11] for details).

Otherwise, let L be a Kuratowski subgraph from the planarity test for $N+xy$. Note that L' can be obtained in linear time by the algorithm of Williamson [13, 14]. It is clear that L fits (2). □

Now we are ready for our main result.

Theorem 3.2 *Let $K \subseteq G$ and let e be a branch of K joining main vertices x, y of K. Suppose that G is 3–connected modulo K. There is a linear time algorithm that either replaces e by a branch e' joining x and y such that e' is internally disjoint from $K - e$ and there are no local bridges of $K - e + e'$ attached to e', or finds a Kuratowski subgraph L of G such that $L \cap K \subseteq e$.*

Proof. Let N be the graph obtained from the branch e by adding all local bridges attached to it. If N is not planar, its Kuratowski subgraph L, obtained by the algorithm of [13, 14] in linear time, has the property stated in the theorem.

Suppose now that N is planar. Traversing e from x towards y we get the first vertex x_1 with a local bridge attached to it. (If there is no such vertex, then we can stop.) Among all local bridges at x_1 we select a subset containing those bridges whose "rightmost" attachment on e is as close to y as possible. Denote this other extreme attachment by y_1. If among the selected bridges there is an edge x_1y_1, then let B_1 be this edge. Otherwise, let B_1 be any of the selected bridges.

Suppose now that we have constructed the sequence $B_1, \ldots, B_k$ of local bridges at e with the following property. If x_j and y_j are the "leftmost" (i.e., closest to x) and the "rightmost" (i.e., closest to y) attachments of B_j $(1 \leq j \leq i)$, then $x_1 < x_2 < y_1 \leq x_3 < y_2 \leq \cdots \leq x_i < y_{i-1} < y_i$, where the relation $<$ (and $\leq$) stands for "being closer to x on e". Moreover, every bridge of K attached strictly between x_1 and y_{i-1} has all its attachments on the closed segment $[x_1, y_i]$ of e. (Case $i = 1$ with B_1, x_1, y_1 defined as above is assumed to fulfil these conditions.) If some global bridge is attached strictly between x_i and y_i, then we terminate the construction of the sequence B_1, $B_2, \ldots, B_i$. The obtained sequence will be used later. Let us remark that we reach this point sooner or later for G is 3–connected modulo K. Suppose now that no global bridge is attached between x_i and y_i. By the 3–connectivity modulo K of the graph G and the properties of $B_1, \ldots, B_i$, there is a local bridge attached strictly between x_i and y_i which has an attachment closer to y

than y_i. Among all such bridges, let B_{i+1} be the bridge attached between x_i and y_i obtained as follows. We first determine the "rightmost" vertex y_{i+1} that is an attachment of such a bridge, among the candidates attached at y_{i+1} we select those which have an attachment x_{i+1} as close as possible to x, and in the obtained subset we choose as B_{i+1} the edge $x_{i+1}y_{i+1}$ if possible, and otherwise we choose as B_{i+1} any of these candidates. By the properties of the sequence $B_1, \ldots, B_i$, x_{i+1} cannot precede y_{i-1} on e. Now it is easy to see that the bridges $B_1, \ldots, B_{i+1}$ fulfil the "inductive" requirements for the sequence $B_1, \ldots, B_{i+1}$.

Upon terminating, the time spent in the above procedure is proportional to the number of edges of G in the segment of e from x_1 to the last vertex, say y_k, plus the number of edges in the local bridges attached to this segment. After changing the segment from x_1 to the last vertex y_k, we will not use the new segment in the above procedure any more. Therefore the overall time spent by this part of the algorithm is linear.

Suppose that we obtained the sequence $B_1, \ldots, B_k$ by the above procedure. Our goal is to replace the segment from x_1 to y_k by a path in $B_1 \cup \cdots \cup B_k \cup e$ such that the new segment will have no local bridges attached to it. This will be done in two steps. In the first step we define a path f from x_1 to y_k and replace the corresponding segment of e by f. In the second step we remove the remaining local bridges by applying the algorithm of Lemma 3.1.

For $i = 1, \ldots, k$, let f_i be a path in B_i from x_i to y_i which is internally disjoint from e. Let f be the path composed of f_k, f_{k-2}, f_{k-4}, … together with segments on e between y_{k-2} and x_k, y_{k-4} and x_{k-2}, etc. (together with the segment of e from x_1 to x_2 if k is even). Recall that there is a global bridge B attached between y_{k-1} and y_k (possibly at y_{k-1}). By the property of our sequence $B_1, \ldots, B_k$ it follows that after the above replacement of the segment of e from x_1 to y_k by f, the bridges B_{k-1}, B_{k-3}, B_{k-5}, … are all merged with B into a single global bridge.

Consider the local bridges with respect to the new graph that are attached to f. Since f and all considered local bridges are contained in N, we can take the induced plane embedding of the graph H consisting of f together with its local bridges. We claim that there exists a plane embedding of H such that every local bridge at f is attached to f from one side only (with respect to our embedding). The local bridges are of two types. They are either local at e as well (in which case they are attached to some of the segments of $e \cap f$), or they emerge as subgraphs of bridges B_k, B_{k-2}, B_{k-4},... By our choice of B_{i+1}, when constructing the sequence $B_1, \ldots, B_k$, a local bridge attached at the segment from y_{i-2} to x_i ($i \equiv k \pmod 2$) has all its attachments on this segment and it is easy to see that under the plane embedding of N, all the attachments are on the same side of e (otherwise, the path f_{i-1} in B_{i-1} and the local bridge would intersect). The new local bridges that are contained in $B_i (i \equiv k \pmod 2)$ may attach to f from both sides. But if this is the case, then either $i = k$, or $i = 1$, and the other sides can be attained only in x_1 or y_k. (To see this, consider the simple closed curve C consisting of a path P in our bridge Q joining the feet q_1, q_2 of Q attached at different sides of f together with the corresponding segment on f. This curve either separates in the plane x_{i-1} and y_{i-1}, or separates x_{i+1} and y_{i+1}.) Therefore these bridges can be re-embedded in such a way that each of them attaches to f from one side only. The obtained plane embeddings of local bridges at f enable us to use Lemma 3.1 (since the addition of the edge x_1y_k will not destroy the plane embedding) to replace f by a path f' without local bridges. Note that the actual re–embeddings will be done automatically by the planarity testing of the corresponding graph in the algorithm of Lemma 3.1. Again, the overall time spent for this purpose is linear.

If there are additional local bridges attached to e at the segment from y_k to y, we repeat the whole procedure. □

Corollary 3.3 *Let $K \subset G$ and suppose that G is 3–connected modulo K. There is a linear time algorithm that either replaces every branch of K by another branch joining the same pair of main vertices and such that G has no local bridges in respect to the new subgraph, or finds a Kuratowski subgraph L of G such that $L \cap K$ is contained in a single branch of K.*

Proof. Apply Theorem 3.2 to every branch of K separately. After changing one of the branches, the local bridges attached to other branches do not change. Therefore, the total time is linear. □

References

[1] K. S. Booth, G. S. Lueker, Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-trees, J. Comput. System Sci. 13 (1976) 335–379.

[2] N. Chiba, T. Nishizeki, S. Abe, T. Ozawa, A linear algorithm for embedding planar graphs using PQ-trees, J. Comput. System Sci. 30 (1985) 54–76.

[3] S. A. Cook, R. A. Reckhow, Time bounded random access machines, J. Comput. Syst. Sci. 7 (1976) 354–375.

[4] H. de Fraysseix, P. Rosenstiehl, A depth-first search characterization of planarity, Ann. Discr. Math. 13 (1982) 75–80.

[5] J. L. Gross, T. W. Tucker, Topological graph theory, Wiley-Interscience, New York, 1987.

[6] J. E. Hopcroft, R. E. Tarjan, Efficient planarity testing, J. ACM 21 (1974) 549–568.

[7] M. Juvan, J. Marinček, B. Mohar, Efficient elimination of local bridges, unpublished.

[8] M. Juvan, J. Marinček, B. Mohar, A linear time algorithm for embedding graphs in the torus, in preparation.

[9] M. Juvan, J. Marinček, B. Mohar, Efficient algorithm for embedding graphs in arbitrary surfaces, in preparation.

[10] A. Karabeg, Classification and detection of obstructions to planarity, Lin. Multilin. Algebra 26 (1990) 15–38.

[11] B. Mohar, Projective planarity in linear time, J. Algorithms, in press.

[12] H.–J. Voss, Cycles and bridges in graphs, Kluwer, 1991.

[13] S. G. Williamson, Embedding graphs in the plane — algorithmic aspects, Ann. Discrete Math. 6 (1980) 349–384.

[14] S. G. Williamson, Depth-first search and Kuratowski subgraphs, J. ACM 31 (1984) 681–693.