OBSTRUCTION SETS FOR OUTER–PROJECTIVE–PLANAR GRAPHS

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Abstract. A graph G is *outer-projective-planar* if it can be embedded in the projective plane so that every vertex appears on the boundary of a single face. We exhibit obstruction sets for outer-projective-planar graphs with respect to the subdivision, minor, and $Y\Delta$ orderings. Equivalently, we find the minimal non-outer-projective-planar graphs under these orderings.

§1 Introduction

The most frequently cited [B] result in graph theory is Kuratowski's Theorem [K], which states that a graph is planar if and only if it does not contain a subdivision of either K_5 or $K_{3,3}$. This is an example of an *obstruction theorem*; a characterization of graphs with a particular property in terms of excluded subgraphs.

Obstruction theorems may involve other properties besides planarity and other orderings besides the subgraph order. Let $\mathcal{P}$ be a property of graphs, formally, $\mathcal{P}$ is some collection of graphs. Let $\preceq$ be a partial ordering on all graphs. We say that $\mathcal{P}$ is *hereditary under* $\preceq$ if $G \in \mathcal{P}$ and $H \preceq G$ implies that $H \in \mathcal{P}$. For example, the collection of all planar graphs is hereditary under the subgraph ordering. An *obstruction for* $\mathcal{P}$ *under* $\preceq$ is a graph G such that $G \notin \mathcal{P}$, but $H \in \mathcal{P}$ for all $H \prec G$ (here "$\prec$" means "$\preceq$" but not equal). In words, an obstruction is a minimal graph not having the given property. When the partial ordering does not have infinite strictly descending chains, which is the case with the orders considered herein, then a graph G has a hereditary property $\mathcal{P}$ if and only if there does not exist an obstruction $H \preceq G$.

We consider the following partial orders on graphs. The first is the *subgraph ordering*: $H \preceq G$ if and only if H is a subgraph of G. A graph K is a *subdivision* of H if it is formed by replacing some edges of H by vertex-disjoint paths with the same endpoints. The *subdivision ordering* has $H \preceq G$ if and only if H has as a subdivision some subgraph K of G. A graph H is a *contraction* of K if it is formed from K by deleting an edge (or edges) and identifying its endpoints (or pairwise identifying their endpoints). The *minor ordering* has $H \preceq G$ if and only if H is a contraction of some subgraph K of G. A graph H is a $Y\Delta$-*transformation* of K if it is formed from K by deleting

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a vertex v of degree 3 (or pairwise nonadjacent vertices) and inserting a triangle (or triangles) on the three vertices adjacent to v. The $Y\Delta$ *ordering* has $H \preceq G$ if and only if H is formed from a minor K of G by $Y\Delta$-transformations.

In summary: the subgraph ordering allows the deletion of edges and of isolated vertices, the subdivision ordering also allows the contraction of an edge incident with a vertex of degree 2, while the minor ordering allows arbitrary edge contractions, and the $Y\Delta$ ordering also includes $Y\Delta$-transformations. Each of these partial orderings is finer than the one before. Observe that an obstruction under a finer partial order is also an obstruction under a coarser partial order.

We examine the following properties of graphs. A graph is *planar* if it can be embedded in the real plane $\mathbb{R}^2$. (For background material on embeddings we refer the reader to [GT].) The graph is *outer-planar* if it can be embedded in the real plane so that every vertex lies on the boundary of a distinguished face. By convention this is the unbounded face. The graph is *projective-planar* if it can be embedded in the projective plane. The graph is *outer-projective-planar* if it embeds in the projective plane such that every vertex lies on the boundary of a distinguished face. By analogy we could embed the graph on the Möbius strip with every vertex on the "outside" or non-cellular face. We define an *S-embeddable* and *outer-S-embeddable* graph for an arbitrary surface S in a similar manner.

Observe that S-embeddable and outer-S-embeddable are hereditary under each of the four partial orders under consideration. Instead of referring to the obstruction sets for these properties, it is more convenient to refer to the subgraph-, subdivision-, minor-, or $Y\Delta$-minimal graphs not possessing these properties. To illustrate the concepts we briefly review known results about these minimal graphs. Archdeacon and Huneke [AH] contains a more detailed history.

As mentioned earlier, Kuratowski's Theorem [K] states that the subgraph-minimal nonplanar graphs are all subdivisions of K_5 and $K_{3,3}$. Equivalently, the subdivision-minimal nonplanar graphs are exactly K_5 and $K_{3,3}$. Wagner [Wag] showed that K_5 and $K_{3,3}$ are also the minor-minimal nonplanar graphs.

Outer-planar graphs were investigated in [CH]. The subdivision-minimal non-outer-planar graphs are K_4 and $K_{2,3}$. These two graphs are also the minor-minimal non-outer-planar graphs. Note that K_4 is a $Y\Delta$-transformation of $K_{2,3}$, so that the obstruction set for outer-planar under the $Y\Delta$ ordering is the single graph K_4.

The subdivision-minimal non-projective-planar graphs were originally found by Glover, Huneke, and Wang [Wan,GHW]; their list was proven complete by Archdeacon [A1,A2]. There are 103 such graphs. Mahader [M] showed that exactly 35 of these graphs are minor-minimal; this work is also implicit in [A1,A2].

Robertson and Seymour [RS3] have proven Wagner's Conjecture: that there does not exist an infinite set of graphs which are pairwise noncomparable under the minor order. It follows that for any hereditary property the set of obstructions for the minor order is finite. In particular, the obstruction sets for S-embeddable and outer-S-embeddable are finite for any surface S (see also [RS1,RS2]). Their work gives no bounds on the sizes of these sets.

The main result of this paper is the following.

The Main Theorem. *There are exactly 32 minor-minimal non-outer-projective-planar graphs.*

From the Main Theorem we derive the following.

Corollary. (1) *There are exactly 45 subdivision-minimal non-outer-projective-planar graphs.*

(2) *There are exactly 9 $Y\Delta$-minimal non-outer-projective-planar graphs.*

The graphs from the Main Theorem and its corollary are shown in Section 3.

We close the introduction with a related topic. Say that a graph is *k-outer-planar* if there exists an embedding in the plane such that every vertex is on the boundary of one of k distinguished faces. So an outer-planar graph is 1-outer-planar. These graphs were investigated by Bienstock and Dean [BD], who gave some explicit bounds on the maximum size of a k-outer-planar graph. Schrijver [S] points out that some NP-problems, such as the Steiner Tree Problem, have polynomial-time algorithms on the class of k-outer-planar graphs.

A 2-outer-planar graph could be called *outer-cylindrical*, as it can be embedded on the sides of a cylinder such that every vertex lies on the upper or lower rim. The authors have a list of 34 minor-minimal non-outer-cylindrical graphs. Bienstock and Dean [BD] claim there are at least 40.

§2 The Proofs

Recall that the minor-minimal non-planar graphs are K_5 and $K_{3,3}$. The minor-minimal non-outer-planar graphs are K_4 and $K_{2,3}$. Observe that the latter are created by deleting a vertex from the former. Does a similar relationship hold for other surfaces? It does, as we shall soon show. Although the following proposition is stated for the projective plane, the result holds for an arbitrary surface S.

Let $H + v$ denote the graph formed from H by adding a new vertex v adjacent to each existing vertex of H. Let $H\backslash v$ denote the graph formed by deleting the vertex v and its incident edges. Let $H\backslash e$ denote edge deletion, and H/e denote edge contraction.

Proposition. *If H is a minor-minimal non-outer-projective-planar graph, then there exists a minor-minimal non-projective-planar graph G and a vertex v of G such that $H = G\backslash v$.*

Proof. We begin with the observation that a graph K is outer-projective-planar if and only if $K + v$ is projective-planar. One direction follows from adding v and its incident edges in the distinguished face of an outer-projective-planar embedding of K; the other direction follows from deleting v from a projective-planar embedding of $K + v$.

Let H be minor-minimal non-outer-projective-planar. By the preceding, $H + v$ is non-projective-planar. Which edges of $H + v$ can be deleted or contracted and still have the resulting graph non-projective-planar?

Let $e \in E(H)$. Then $(H+v)\backslash e = (H\backslash e)+v$. Since H is minor-minimal non-outer-projective-planar, $H\backslash e$ is outer-projective-planar and hence $(H+v)\backslash e$ is projective-planar. Similarly, $(H+v)/e$ is $(H/e)+v$, except for an additional parallel edge. By minimality H/e is outer-projective-planar, and so $(H+v)/e$ is projective-planar.

Next let e be an edge of $H+v$ incident with v. Let u be the other endpoint of e. Then $(H+v)/e$ is $(H\backslash u)+v$, except for parallel edges. By minimality $H\backslash u$ is outer-projective-planar, hence $(H+v)/e$ is projective-planar.

We have shown that deleting any edge not incident with v or contracting any edge makes $H+v$ projective-planar. Hence a minor-minimal non-projective-planar graph is formed by deleting some set (possibly empty) of edges incident with v. The proposition follows. $\square$

Proof of the Main Theorem—There are 32 minor-minimal non-outer-projective-planar graphs. We use the 35 minor-minimal non-projective-planar graphs of [A1,M]. These graphs are listed in Section 3. For each graph G among these 35 and each vertex v in G we form the graph $G\backslash v$. (We need not then consider $G\backslash u$ if u is a vertex similar to v under the action of G's automorphism group.) We call the resulting (multi)set $\mathcal{G}$. Not all graphs in $\mathcal{G}$ are non-isomorphic, as the same graph may arise as a vertex-deleted subgraph of several minor-minimal non-projective-planar graphs. Using these conventions $\mathcal{G}$ has exactly 118 graphs. By the observation beginning the proof of the Proposition, each of these graphs is non-outer-projective-planar. By the same Proposition, the minor-minimal non-outer-projective-planar graphs are included in this list. So the Main Theorem is reduced to finding the minor-minimal graphs in $\mathcal{G}$; that is, checking the 118 to see which are contained in others as minors. This task is not as arduous as it appears, as many of the 118 graphs contain either disjoint non-outer-planar graphs, or non-outer-planar graphs sharing a single vertex. These graphs, the α's and β's of Section 3, are easily recognizable.

For example, let G be the graph formed from K_4 by first replacing each edge with a path of length two, then adding a new vertex ∞ adjacent to each of the six new degree 2 vertices. This G is one of the minor-minimal non-projective-planar graphs; E_2 using the notation of the next section. There are three types of vertices up to isomorphism, so G contributes three graphs to $\mathcal{G}$. If we delete the vertex ∞, then we get the minor-minimal non-outer-projective-planar graph ζ_6 shown in Section 3. If we delete a degree 3 vertex not adjacent to ∞, then the resulting graph has ζ_6 as a subgraph and hence as a minor. Likewise, if we delete a vertex adjacent to ∞, then the graph has γ_5 of Section 3 as a subgraph.

A similar through and careful check reveals that the graphs in Figures 3.1 through 3.7 in Section 3 are the minor-minimal members of $\mathcal{G}$. $\square$

Our proof clearly depends heavily on the list of 35 minor-minimal non-projective-planar graphs given by Archdeacon and by Mahader [A1,M]. We note that this list has been independently verified by Vollmerhaus [V1,V2]. Also, Bodendiek, Schumacher, and Wagner [BSW] show that there are exactly 12 minimal non-projective-planar graphs under a finer order than we are considering; from these it is possible to reconstruct the 35 minor-minimal graphs.

We have also left a great deal of work for the reader in constructing the set $\mathcal{G}$ and checking for the minor-minimal members. At least two authors independently checked each of the 35 non-projective-planar graphs for the vertex-deleted subgraphs. Similarly, the reductions showing members of $\mathcal{G}$ were not minor-minimal were checked by at least three authors. Finally, each edge e in each of the 32 graphs G was checked to ensure that both G/e and $G\backslash e$ were outer-projective-planar. Again, this task was not as ardous as it seems. The check of minimality is helped greatly by the fact that if a planar graph has all vertices on the boundary of a distinguished pair of faces with at least one vertex in common, then it is outer-projective-planar. Finally, several classes of minor-minimal non-projective-planar graphs (such as those with connectivity 0 or 1, or the nonplanar graphs) can be found independently; these classes agree with the ones on our list.

We next describe the proof of the Corollaries.

Proof of Corollary (1)—There are 45 subdivision-minimal non-outer-projective-planar graphs. Again, we describe the proof but avoid the excruciating details. Let K be a subdivision-minimal non-outer-projective-planar graph. Then by some sequence of edge contractions we can form a minor-minimal non-outer-projective-planar graph G. Alternatively, we can recover K from G by a sequence of vertex-splittings, the inverse operation to edge contraction. Note that when making G from K we contract only edges with both endpoints of degree at least 3. Hence when splitting G we must have both new vertices of degree at least 3. In particular, we need to split only vertices in G of degree at least 4.

An examination of the graphs in Section 3 reveals that there are exactly 45 vertices of degree 4 or more. Note that a degree 4 vertex can be split into two vertices of degree at least 3 in three ways. A degree 5 vertex can be split in ten ways, and a degree 6 vertex in 25 ways. The authors checked each vertex and each possible splitting to see if the resulting graph was subdivision-minimal. If it was, then the remaining vertices of degree 4 or more were split and checked. As before, this task was helped by using the automorphisms of the graphs to reduce the casework. Also helpful is the observation that many splittings are non-planar and hence easily dealt with.

A through and careful check reveals that the graphs in Figures 3.8 through 3.10 in Section 3 are the additional subdivision-minimal graphs resulting from recursively splitting minor-minimal non-outer-projective-planar graphs. Part (1) of the Corollary follows. $\square$

As in the proof of the Main Theorem, each splitting, each reduction, and the subdivision-minimal properties of the resulting graphs were checked independently by at least two authors.

Proof of Corollary (2)—There are 9 $Y\Delta$-minimal non-outer-projective-planar graphs. Again we start with the 32 minor-minimal non-outer-projective-planar graphs given by the Main Theorem. This time we need to check which of the graphs are obtainable from others by $Y\Delta$-transformations. Specifically, the $Y\Delta$-minimal graphs are the 9 graphs in Figures 3.1 to 3.7

with subscripts 1. These figures also show how to construct the remaining 23 graphs by ΔY-transformations (the inverse to $Y\Delta$-transformations). It can be checked that none of the 9 graphs are $Y\Delta$-transformations of one another. □

We close this section by noting that the subgraph-minimal non-outer-projective-planar graphs are precisely those which are subdivisions of subdivision-minimal non-outer-projective-planar graphs.

§3 The Graphs

In this section we give the graphs mentioned in the Main Theorem and its Corollary.

We begin with the 35 minor-minimal non-projective-planar graphs. These are given in [A1,M], which are not available in most libraries. For completeness we give them here using the notation of [GHW,Wan]. These references give pictures of the graphs. Also, for convenience in forming the vertex-deleted subgraphs, the number of vertex orbits under the action of the automorphism group is given in parenthesis following the name.

The minor-minimal non-projective-planar graphs are $\{A_1(2)$, $A_2(2)$, $A_5(1)$, $B_1(2)$, $B_3(2)$, $B_7(4)$, $C_1(4)$, $C_2(4)$, $C_3(5)$, $C_4(3)$, $C_7(3)$, $C_{11}(2)$, $D_1(3)$, $D_2(4)$, $D_3(5)$, $D_4(3)$, $D_9(4)$, $D_{12}(6)$, $D_{17}(1)$, $E_1(3)$, $E_2(3)$, $E_3(2)$, $E_5(4)$, $E_6(4)$, $E_{11}(7)$, $E_{18}(2)$, $E_{19}(5)$, $E_{20}(5)$, $E_{22}(3)$, $E_{27}(7)$, $E_{42}(1)$, $F_1(4)$, $F_4(4)$, $F_6(2)$, $G_1(2)\}$. The graphs with letter A have Betti number 12, those with B have Betti number 11, and so forth.

In Figures 3.1-3.7 we give the 32 graphs of the Main Theorem. In anticipation of the Corollary, these are grouped into families by their $Y\Delta$ relations. For example, in Figure 3.1 we give the α family which contains the disconnected minor-minimal non-outer-projective-planar graphs. The arrow from α_1 to α_2 shows that the former is a $Y\Delta$-transformation of the latter. Consequently, α_1 is the only $Y\Delta$-minimal member in this family.

The three non-planar graphs, ϵ_4, ϵ_6, and κ_1, are drawn as if embedded in the projective plane. In particular, pairs of antipodal "half-edges" are to be identified to a single edge.

Figure 3.1: The α family

Figure 3.2: The β family

Figure 3.3: The γ family

Figure 3.4: The δ family

Figure 3.5: The ϵ family

Figure 3.6: The ζ family

Figure 3.7: The η, θ, and κ families

Each graph in these figures comes from one of the 35 minor-minimal non-projective-planar graphs. Using our notation for the 32 minor-minimal non-outer-projective-planar graphs, and [GHW]'s for the 35 minor-minimal non-projective-planar graphs, our α_1 is a vertex-deleted subgraph of their A_1. We denote this by $\alpha_1 < A_1$. Similarly, $\alpha_2 < C_1$, and $\alpha_3 < E_1$. Continuing, $\beta_1 < B_3$, $\beta_2 < D_4$, $\beta_3 < C_2$, $\beta_4 < F_6$, $\beta_5 < E_6$, and $\beta_6 < D_1$. Furthermore $\gamma_1 < B_1$, $\gamma_2 < D_3$, $\gamma_3 < D_3$, $\gamma_4 < F_1$, $\gamma_5 < E_5$, and $\gamma_6 < F_1$. Next $\delta_1 < D_{12}$ and $\delta_2 < E_{11}$. And $\epsilon_1 < D_{17}$, $\epsilon_2 < E_{20}$, $\epsilon_3 < E_{20}$, $\epsilon_4 < E_{20}$, $\epsilon_5 < F_4$, $\epsilon_6 < G_1$. Moreover, $\zeta_1 < A_2$, $\zeta_2 < B_7$, $\zeta_3 < C_4$, $\zeta_4 < C_3$, $\zeta_5 < D_2$, $\zeta_6 < E_2$; $\eta_1 < E_{22}$; $\theta_1 < E_3$; and $\kappa_1 < E_{18}$.

Figures 3.8-3.10 give the subdivision-minimal graphs which are not minor-minimal. We explain the notation by example. The graph γ_{1a} of Figure 3.8 is a vertex-splitting of γ_1. Specifically, we can recover γ_1 from γ_{1a} by contracting the edge whose ends are squares rather than dots. The graphs γ_{3a} and γ_{3b} arise from two different splittings of γ_3. In Figure 3.9 we need to recursively split graphs; θ_{1a1} is formed by splitting a vertex of θ_{1a}.

Finally, the authors have checked that the 45 subdivision-minimal graphs herein are pairwise non-isomorphic.

FIGURE 3.8: THE γ SPLITTINGS

Figure 3.9: The θ splittings

Figure 3.10: The β and η splittings

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