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On the orientable genus of graphs with bounded nonorientable genus

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Abstract

A conjecture of Robertson and Thomas on the orientable genus of graphs with a given nonorientable embedding is disproved.

1. Introduction

Suppose that a graph G is embedded into a *nonplanar* surface Σ, i.e. a closed surface distinct from the 2-sphere. Then we define the *face-width* $\mathrm{fw}(G)$ (also called the *representativity*) of the embedded graph G as the smallest number of (closed) faces of G in Σ whose union contains a noncontractible curve. Alternatively, $\mathrm{fw}(G)$ is the largest number k such that every noncontractible closed curve in Σ intersects the graph G in at least k points. By a *closed curve* in Σ we mean a continuous mapping $\gamma : S^1 \to \Sigma$ (where S^1 is the 1-sphere), and by the number of intersections of γ with the graph we mean the number

$$\mathrm{cr}(\gamma, G) = |\{s \in S^1 \mid \gamma(s) \in G\}|.$$

The curve γ is 1-*sided* (or *orientation reversing*) if after traversing its image $\gamma(S^1)$ on Σ, the 'left' and the 'right' interchange. Otherwise γ is 2-*sided* (or *orientation preserving*).

If γ_1 and γ_2 are closed curves that intersect in only finitely many points, then they are said to *cross* at the point $u \in \Sigma$ if $u \in \gamma_1(S^1) \cap \gamma_2(S^1)$ and there is an open neighborhood U of u in Σ and a homeomorphism $U \to \mathbf{R}^2$ such that each of $\gamma_i(S^1) \cap U$ $(i = 1,2)$ is mapped to one of the axes in $\mathbf{R}^2$.

We shall denote by N_k the nonorientable surface of *genus* (or the *crosscap number*) k. If G is a graph, its *nonorientable genus* $\tilde{\gamma}(G)$ is the smallest k such that G has an

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embedding in N_k. Similarly, the *genus* $\gamma(G)$ of G is the smallest number k such that G has an embedding into an orientable surface of genus k. Let us recall that 2-cell embeddings in orientable surfaces can be described combinatorially by specifying a *local rotation* π_v for each vertex v of the graph where π_v is a cyclic permutation of edges incident with v, representing their circular order around v on the surface [3]. If Σ is a surface, we denote by $g(\Sigma)$ its genus (or the nonorientable genus if Σ is nonorientable).

It is easy to see that the nonorientable genus of every graph G is bounded by a linear function of its (orientable) genus. More precisely, $\tilde{\gamma}(G) \leqslant 2\gamma(G)+1$. On the other hand, Auslander et al. [1] proved that there are graphs embeddable in the projective plane whose orientable genus is arbitrarily large. This phenomenon is now appropriately understood since Fiedler et al. [2] proved the following result.

Theorem 1.1 (Fiedler et al. [2]). *Let G be a graph that is embedded in the projective plane. If* $\mathrm{fw}(G) \neq 2$, *then the genus of G is*

$$\gamma(G) = \left\lfloor \frac{\mathrm{fw}(G)}{2} \right\rfloor . \tag{1}$$

Theorem 1.1 has been generalized to the next nonorientable surface by Robertson and Thomas [4] as follows. Let G be a graph embedded in the Klein bottle N_2. We denote by $\mathrm{ord}_2(G)$ the minimum of $\lceil \mathrm{cr}(\gamma, G)/2 \rceil$ taken over all noncontractible and nonseparating 2-sided simple closed curves γ. Similarly, we denote by $\mathrm{ord}_1(G)$ the minimum of $\lfloor \mathrm{cr}(\gamma_1, G)/2 \rfloor + \lfloor \mathrm{cr}(\gamma_2, G)/2 \rfloor$ taken over all pairs γ_1, γ_2 of nonhomotopic 1-sided simple closed curves. The latter minimum restricted to all noncrossing pairs γ_1, γ_2 of 1-sided simple closed curves is denoted by $\mathrm{ord}_1'(G)$.

Theorem 1.2 (Robertson and Thomas [4]). *Let G be a graph that is embedded in the Klein bottle. Let*

$$g = \min\{\mathrm{ord}_1(G), \mathrm{ord}_2(G)\}. \tag{2}$$

If $g \geqslant 4$, then $g = \gamma(G)$. Moreover, g can be determined in polynomial time.

Robertson and Thomas also proved that

$$\gamma(G) = \min\{\mathrm{ord}_1'(G), \mathrm{ord}_2(G)\}. \tag{3}$$

Theorems 1.1 and 1.2 imply that the genus of graphs that can be embedded in the projective plane or the Klein bottle can be computed in polynomial time. By [6], the genus testing is NP-complete for general graphs. Therefore, it is interesting that classes of the projective graphs and graphs embeddable in the Klein bottle admit a polynomial time genus testing algorithm. Very likely the genus problem for graphs with bounded nonorientable genus is solvable in polynomial time as suggested in [4].

Robertson and Thomas [4] conjectured that Theorems 1.1 and 1.2 can be generalized as explained below. Suppose that $\Gamma = \{\gamma_1, \ldots, \gamma_p\}$ is a set of closed curves in the surface N_k. Then Γ is *crossing-free* if the following holds:
(a) No γ_i crosses itself.
(b) For $1 \leqslant i < j \leqslant p$, the curves γ_i and γ_j do not cross each other.
If there exist simple closed curves $\gamma'_1, \ldots, \gamma'_p$ with pairwise disjoint images in the surface such that γ'_i is homotopic to γ_i ($i = 1, \ldots, p$) and such that every 1-sided closed curve in N_k crosses at least one of the curves $\gamma'_1, \ldots, \gamma'_p$, then we say that the family Γ is a *blockage* and that Γ *blocks 1-sided curves* in the surface. Suppose that we have a graph G embedded in the same surface N_k. Robertson and Thomas define the *order* of the blockage Γ as

$$\operatorname{ord}(\Gamma) = \frac{1}{2}(k - s) + \sum_{i=1}^{p} \operatorname{ord}(\gamma_i), \tag{4}$$

where s is the number of 1-sided closed curves in Γ and

$$\operatorname{ord}(\gamma_i) = \begin{cases} \lfloor \operatorname{cr}(\gamma_i, G)/2 \rfloor & \text{if } \gamma_i \text{ is 1-sided,} \\ \lfloor (\operatorname{cr}(\gamma_i, G) - 1)/2 \rfloor & \text{if } \gamma_i \text{ is 2-sided.} \end{cases}$$

The conjecture of Robertson and Thomas [4] based on (1)–(3) is that if G is embedded in N_k with sufficiently large face-width, the following statements are equivalent for every integer g:
(RT1) The genus of G is at least g.
(RT2) Every crossing-free blockage has order at least g.
(RT3) Every blockage has order at least g.

In the next section we give examples that disprove this conjecture, and in the last section we present an improved version of the conjecture.

2. A counterexample

Let H be a graph without isolated vertices that is embedded (not necessarily 2-cell) in an orientable surface Σ_0. Suppose that its faces (the connected components of $\Sigma_0 \backslash H$) can be partitioned into two classes $\mathscr{F}$ and $\mathscr{F}'$ such that each edge of H lies on the boundary of a face from $\mathscr{F}$ and on the boundary of a face from $\mathscr{F}'$. (If the embedding is 2-cell, this condition is equivalent to bipartiteness of the geometric dual graph of H.) Suppose that $\mathscr{F} = \{F_1, \ldots, F_p\}$ and $\mathscr{F}' = \{F'_1, \ldots, F'_q\}$. Take an arbitrary tree T with vertices $u_1, \ldots, u_p$ in one bipartition class and vertices $u'_1, \ldots, u'_q$ in the other bipartition class. Now, for each edge $u_i u'_j \in E(T)$ ($1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$), we add a nonorientable handle between the faces F_i and F'_j. (*Adding a nonorientable handle* means that we cut out disjoint open disks D_{ij} from $\operatorname{int} F_i$ and D'_{ji} from $\operatorname{int} F'_j$, and identify the boundaries of these two disks such that each curve in $\Sigma_0 \backslash (D_{ij} \cup D'_{ji})$ from a point $x \in \partial \overline{D}_{ij}$ to its identified point $x' \in \partial \overline{D}'_{ji}$ becomes a 1-sided closed curve after the identification.) Call the resulting surface Σ. Since $E(T) \neq \emptyset$, Σ is nonorientable. Clearly, Σ has genus

$$g(\Sigma) = 2g(\Sigma_0) + 2|E(T)| = 2g(\Sigma_0) + 2(p + q - 1). \tag{5}$$

If the original embedding of H in Σ_0 is 2-cell, then Euler's formula and (5) imply that

$$g(\Sigma) = p + q + |E(H)| - |V(H)|. \tag{6}$$

Let $t > 0$ be an integer. Let us subdivide each edge of H by inserting $2t$ vertices of degree 2 and call the resulting graph H'. By adding a very dense graph in Σ, the embedding of H' can be extended to a triangulation G of Σ with the following properties:

(P1) H' is an induced subgraph of G.

(P2) Every noncontractible cycle of G with at most one vertex in H' has length more than $g(\Sigma) + (2t + 1)|E(H)| + 3$.

(P3) If P is a path in $G - E(H')$ joining distinct vertices of H', say u and v, and if the length of P is at most $g(\Sigma) + (2t + 1)|E(H)| + 3$, then there is a path P' from u to v in H' that is homotopic to P (relative its endpoints). Moreover, P' is shorter than P.

The face width of every triangulation is equal to the length of the shortest noncontractible cycle in the graph. Therefore, (P2) and (P3) imply that $\mathrm{fw}(G)$ is at least the length of a shortest noncontractible cycle of H' in the surface Σ. In particular,

$$\mathrm{fw}(G) \geqslant 2t + 1. \tag{7}$$

Now we show that blockages in Σ of minimal order (with respect to the graph G) have a special structure.

Proposition 2.1. *Let $\Gamma = \{\gamma_1, \dots, \gamma_r\}$ be a minimum blockage for the graph G in Σ. Then*

(a) *Each curve $\gamma_i \in \Gamma$ is 2-sided and intersects G only in vertices of H'. Each vertex of H' lies on some curve from Γ.*

(b) *For each curve $\gamma_i \in \Gamma$, the sequence of vertices of H' intersected by γ_i determines a closed walk C_i' in the graph H'. The walks $C_1', \dots, C_r'$ cover all edges of H', each exactly once.*

(c) *If s is the number of closed walks C_i' $(1 \leqslant i \leqslant r)$ of even length, then*

$$\mathrm{ord}(\Gamma) = \tfrac{1}{2}(g(\Sigma) + (2t + 1)|E(H)| - (r + s)). \tag{8}$$

Proof. Let $\delta_1, \dots, \delta_p$ be closed curves in Σ corresponding to the facial walks of the faces $F_1, \dots, F_p$ of the graph H in Σ_0. Then $\Delta = \{\delta_1, \dots, \delta_p\}$ is a blockage as the reader will easily verify. The curves δ_i are 2-sided. Therefore, the order of Γ is bounded as follows:

$$\begin{aligned} 2\,\mathrm{ord}(\Gamma) \leqslant 2\,\mathrm{ord}(\Delta) &= g(\Sigma) + 2\sum_{i=1}^{p} \mathrm{ord}(\delta_i) \\ &\leqslant g(\Sigma) + \sum_{i=1}^{p} \mathrm{cr}(\delta_i, G) \\ &= g(\Sigma) + (2t+1)|E(H)|. \end{aligned} \tag{9}$$

Since G is a triangulation, we may assume that each curve $\gamma_i \in \Gamma$ intersects G only in vertices and that there is a closed walk C_i' in G whose length $\ell(C_i')$ is equal to $\mathrm{cr}(\gamma_i, G)$ and such that the order of vertices on C_i' coincides with the order of vertices intersected by γ_i. Properties (P2) and (P3) of G and the inequality (9) imply that the closed walks C_i' are contained in H'. In particular, each γ_i is 2-sided. This proves the first claim of (a).

Suppose now that an edge of H' is contained in no walk C_i'. Then our construction of Σ shows that the surface obtained by cutting Σ along the curves of Γ is nonorientable. This contradicts the fact that Γ is a blockage and, hence, proves the other statement of (a). Since $\mathrm{ord}(\Gamma) \leqslant \mathrm{ord}(\Delta)$, no edge of H' belongs to more than one walk C_i'. This proves (b).

Let us now compute the order of Γ. Since $\mathrm{cr}(\gamma_i, G) = \ell(C_i')$, we have

$$\mathrm{ord}(\gamma_i) = \left\lfloor \frac{\mathrm{cr}(\gamma_i, G) - 1}{2} \right\rfloor = \frac{\ell(C_i')}{2} - \tfrac{1}{2} - z, \tag{10}$$

where $z = \frac{1}{2}$ if $\ell(C_i')$ is even and $z = 0$ otherwise. Since the sum of the lengths of the walks C_i' $(1 \leqslant i \leqslant r)$ is equal to $(2t+1)|E(H)|$, the above equality implies

$$2\,\mathrm{ord}(\Gamma) = g(\Sigma) + 2\sum_{i=1}^{p} \mathrm{ord}(\gamma_i) = g(\Sigma) + (2t+1)|E(H)| - (r+s).$$

This completes the proof. $\square$

We shall now consider a special case of the construction described above. Let H be the complete graph K_7 embedded in the torus (the surface Σ_0) as shown in Fig. 1. Then $p = q = 7$. By (6) we have

$$g(\Sigma) = 28. \tag{11}$$

Suppose that $\Gamma = \{\gamma_1, \ldots, \gamma_r\}$ is a minimum blockage for the graph G corresponding to the above example. By Proposition 2.1, the closed walks $C_1', \ldots, C_r'$ of H' corresponding to $\gamma_1, \ldots, \gamma_r$, respectively, maximize the sum $r+s$ taken over all sets of closed walks that contain all edges of H', each exactly once. Since the parity of lengths of closed walks in H' and corresponding closed walks in H is the same, we can as well work with the corresponding closed walks $C_1, \ldots, C_r$ in H. Since $H = K_7$ has 21 edges, $r+s$ is always odd. In particular, $r \geqslant s+1$. Clearly, $r \leqslant |E(H)|/3 = 7$. If $r = 7$, then all walks are triangles, hence, $s = 0$. If $r = 6$, then we similarly get $s \leqslant 3$ and, hence,

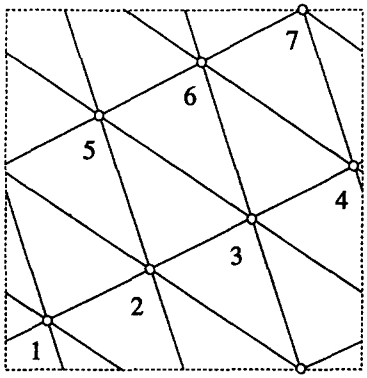

Fig. 1. K_7 in the torus.

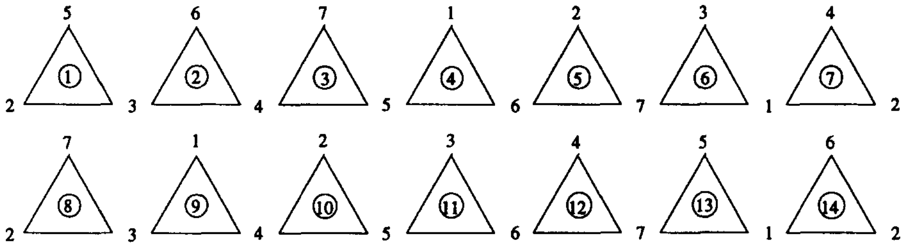

Fig. 2. The new faces of $\tilde{G}$.

$r+s\leqslant 9$. If $r\leqslant 5$, then $r+s\leqslant 9$ as well (since $s\leqslant r-1$). This implies that $r+s$ cannot be larger than 9. Using this fact, (8) gives

$$\mathrm{ord}(\Gamma)\geqslant \tfrac{1}{2}(28+(2t+1)21-9)=21t+20. \tag{12}$$

Let us now cut the surface Σ along the edges of H' so that we get a surface with $p+q$ boundary components (corresponding to the facial walks of H' in Σ_0) and such that each edge of H' gives rise to two new edges on the boundary, and each vertex v of H' gives rise to $\deg_{H'}(v)$ new vertices (called *copies* of v). After pasting disks to these boundary components, we get an embedding of a graph $\tilde{G}$ in the 2-sphere. Under this embedding there are 14 faces corresponding to the (triangular) faces of K_7 in Σ_0. All other faces of $\tilde{G}$ correspond to the facial triangles of G in Σ. The exceptional faces are oriented as shown in Fig. 2. Each vertex v of H' has all its copies on the boundaries of these faces.

We now extend the graph $\tilde{G}$ as follows. We first add seven new vertices $v_1,\ldots,v_7$ and join each v_i $(1\leqslant i\leqslant 7)$ to all copies of the vertex i of H (see Figs. 1 and 2 for notation). Next, we construct an orientable embedding Π of the resulting graph G' such that the restriction of Π to $\tilde{G}$ coincides with the embedding of $\tilde{G}$ in the 2-sphere. Moreover, the local rotation around each copy of the vertex $i\in V(H)$ is the same as

Table 1

v_1	4	14	13	9	6	7
v_2	1	8	14	7	5	10
v_3	1	11	2	6	8	9
v_4	2	3	9	10	12	7
v_5	1	13	4	11	3	10
v_6	2	4	14	12	11	5
v_7	3	5	13	6	8	12

Note. The numbers in the row of v_i represent the copies of the vertex i in the faces with these numbers in Fig. 2.

Table 2

f_1	v_1	6	v_7	8	v_3	9						
f_2	v_1	7	v_4	2	v_3	6						
f_3	v_1	4	v_6	14	v_2	7						
f_4	v_3	2	v_6	4	v_5	11						
f_5	v_4	3	v_7	5	v_6	2						
f_6	v_4	12	v_7	3	v_5	10						
f_7	v_1	9	v_4	10	v_2	1	v_5	13				
f_8	v_1	13	v_7	6	v_3	8	v_2	14				
f_9	v_2	10	v_5	1	v_3	11	v_6	5				
f_{10}	v_1	14	v_6	12	v_4	7	v_2	5	v_7	13	v_5	4
f_{11}	v_2	8	v_7	12	v_6	11	v_5	3	v_4	9	v_3	1

in $\tilde{G}$ except that the new edge from i to v_i is placed between the two edges in the exceptional face containing that copy of the vertex. For the new vertices we use the local rotations as given in Table 1.

The chosen local rotations determines an embedding of G' whose faces coincide with the faces of $\tilde{G}$ except that the 14 exceptional faces are replaced by 11 new faces $f_1, \ldots, f_{11}$ given in Table 2.

In Table 2, numbers 1–14 again represent the numbers of faces from Fig. 2. For example, the above encoding of the facial walk f_1 means that the walk starts with v_1, proceeds to the copy of the vertex 1 in the face 6, then goes through the copy of vertex 7 in face 6, continues to v_7, to the copies of 7 and 3 in the face 8, uses v_3, and finally visits copies of vertices 3 and 1 in face 9.

Since G' has 7 new vertices and 42 new edges, the Euler characteristic of the constructed embedding Π is

$$\chi(\Pi) = 2 + 7 - 42 + (11 - 14) = -36.$$

The first term 2 in the above equation comes, of course, from the Euler characteristic of the embedding of $\tilde{G}$ in the 2-sphere. This implies that the genus of the embedding Π is 19.

We now contract the 42 edges of $E(G')\backslash E(\tilde{G})$ and get an embedding in the same surface. Finally, by adding $t|E(H)| = 21t$ handles we can identify all copies of vertices of $V(H')\backslash V(H)$ (two identifications made across each handle) to get an orientable embedding of G whose genus is $21t+19$. By (12), this is smaller than $\mathrm{ord}(\Gamma)$. By (7) we can achieve the face width of G being arbitrarily large. Therefore, our example disproves the conjecture of Robertson and Thomas.

By taking the connected sum of k copies of toroidal embeddings of K_7, we get examples where the difference between the orientable genus and the value conjectured by (RT2) or (RT3) is at least k.

For crossing-free blockages in our example one can prove that $r+s\leqslant 7$, and, hence (12) can be strengthened to

$$\mathrm{ord}(\Gamma)\geqslant 21t+21.$$

The proof of this fact needs more care and since it does not essentially enlighten the problem, we omit it. However, it would be interesting to know if this yields also an example showing that the minimal values of (RT2) and (RT3) are not always the same.

3. A new conjecture

Based on the counterexamples from the previous section and on some further insight into the problem of determining the orientable genus of a graph with the given nonorientable embedding in N_k, we propose a slightly different conjecture that might be off by a constant (depending on k) from the conjectured values (RT2) and (RT3) of Robertson and Thomas.

Suppose that G is embedded in N_k. Consider a crossing-free blockage $\Gamma=\{\gamma_1,\ldots,\gamma_p\}$ and cut the surface N_k along $\gamma_1,\ldots,\gamma_p$. This results in a graph $\tilde{G}$ embedded in an orientable surface. Now each vertex $a\in V(G)\cap(\bigcup_{i=1}^{p}\gamma_i(S^1))$ has two or more copies in $\tilde{G}$, and we add a new vertex v_a and join it to all copies of a in $\tilde{G}$. Call the resulting graph G' and note that contraction of the new edges results in the original graph G. Now, the orientable embedding of $\tilde{G}$ defines local rotations of all vertices of G' except for the new vertices v'. The minimum genus of an orientable embedding of G' extending this partial embedding is called the *genus order* of the blockage Γ. We note that in case when no vertex of G is split into more than two vertices of $\tilde{G}$, the genus order coincides with (4), and that in general it is majorized by (4).

Conjecture 3.1. *If G is embedded in a nonorientable surface with sufficiently large face-width, then the orientable genus of G is equal to the minimal genus order of a crossing-free blockage.*

Unfortunately, it is not clear how one can find in polynomial time a (crossing-free) blockage of minimum genus order.

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