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DIRAC'S MAP-COLOR THEOREM FOR CHOOSABILITY

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Abstract

It is proved that the choice number of every graph G embedded on a surface of Euler genus $\varepsilon \geq 1$ and $\varepsilon \neq 3$ is at most the Heawood number $H(\varepsilon) = \lfloor (7 + \sqrt{24\varepsilon + 1})/2 \rfloor$ and that the equality holds if and only if G contains the complete graph $K_{H(\varepsilon)}$ as a subgraph.

1 Introduction

1.1 History and results

This paper is concerned with the choice number of graphs embedded on a given (closed) surface. Surfaces can be classified according to their genus and orientability. The *orientable surfaces* are the sphere with g handles Σ_g, where $g \geq 0$. The *non-orientable surfaces* are the surfaces Π_h ($h \geq 1$) obtained by taking the sphere with h holes and attaching h Möbius bands along their boundary to the boundaries of the holes. Π_1 is the projective plane, Π_2 is the Klein bottle, etc. The *Euler genus* $\varepsilon(\Sigma)$ of the surface $\Sigma = \Sigma_g$ is $2g$, and the *Euler genus* of $\Sigma = \Pi_h$ is h. Then $2 - \varepsilon(\Sigma)$ is the *Euler characteristic* of Σ.

Consider a simple graph G with vertex set V and edge set E that is embedded on a surface Σ of Euler genus $\varepsilon = \varepsilon(\Sigma)$. Euler's Formula tells us that $|V| - |E| + |F| \geq 2 - \varepsilon$, where F is the set of faces and with equality holding if and only if every face is a 2-cell. Therefore, if $|V| \geq 3$, then $|E| \leq 3|V| - 6 + 3\varepsilon$. For $\varepsilon \geq 1$, this implies that G is $(H(\varepsilon) - 1)$-degenerate, that is every subgraph of G has a vertex of degree at most $H(\varepsilon) - 1$, where

$$H(\varepsilon) = \left\lfloor \frac{7 + \sqrt{24\varepsilon + 1}}{2} \right\rfloor .$$

Consequently, if $\varepsilon \geq 1$, then

$$\chi(G) \leq \chi_l(G) \leq H(\varepsilon),$$

where $\chi(G)$ denotes the *chromatic number* of G and $\chi_l(G)$ denotes the *choice number* of G. For every surface Σ distinct from the Klein bottle, the Heawood number $H(\varepsilon)$ is, in fact, the maximum chromatic number of graphs embeddable on Σ where the maximum is attained by the complete graph on $H(h)$ vertices. This landmark result, that was conjectured by Heawood [9], is due to Ringel [13] and Ringel & Youngs [14]. Conversely, every graph with chromatic number $H(\varepsilon)$ embedded on Σ contains a complete graph on $H(\varepsilon)$ vertices as a subgraph. This result was proved by Dirac [3, 5] for the torus and $\varepsilon \geq 4$ and by Albertson and Hutchinson [1] for $\varepsilon = 1, 3$.

Franklin [7] proved that the coloring problem for the Klein bottle has not the answer $H(2) = 7$ but 6. Furthermore, there are 6-chromatic graphs on the Klein bottle without a K_6. One example of such a graph is given in [1]. Brooks' theorem for the choice number implies that if G is a graph on the Klein bottle, then $\chi_l(G) \leq 6$. For graphs on the sphere the maximum chromatic number is 4, however the maximum choice number is 5. The last statement follows from results of Thomassen [15] and Voigt [17].

The aim of this paper is to prove the following extension of Dirac's result.

Theorem 1 *Let Σ be a surface of Euler genus ε with $\varepsilon \geq 1$ and $\varepsilon \neq 3$. If G is a graph embedded on Σ, then $\chi_l(G) \leq H(\varepsilon)$ where equality holds if and only if G contains a complete subgraph on $H(\varepsilon)$ vertices.* □

The proof of Theorem 1 for $\varepsilon = 2$ and $\varepsilon \geq 4$ is given in Section 2 and resembles the proof of Dirac's result for the chromatic number. The proof for $\varepsilon = 1$, i.e. the projective plane, is given in Section 4.

1.2 Terminology

All graphs considered in this paper are finite, undirected and simple. For a graph G, we denote by $V(G)$ the *vertex set* and by $E(G)$ the *edge set* of G. The *subgraph* of G *induced* by $X \subseteq V(G)$ is denoted by $G[X]$; further, $G - X = G[V(G) - X]$. The *degree* of a vertex x in G is denoted by $d_G(x)$. As usual, let K_n denote the *complete graph* on n vertices.

Consider a graph G and assign to each vertex x of G a set $\Phi(x)$ of colors (positive integers). Such an assignment Φ of sets to vertices in G is referred to as a *color scheme* (or briefly, a *list*) for G. A Φ-*coloring* of G is a mapping φ of $V(G)$ into the set of colors such that $\varphi(x) \in \Phi(x)$ for all $x \in V(G)$ and $\varphi(x) \neq \varphi(y)$ whenever $xy \in E(G)$. If G admits a Φ-coloring, then G is called Φ-*colorable*. In case of $\Phi(x) = \{1, \ldots, k\}$ for all $x \in V(G)$, we also use the terms k-*coloring* and k-*colorable*, respectively. G is said to be k-*choosable* if G is Φ-colorable for every list Φ for G satisfying $|\Phi(x)| = k$ for all $x \in V(G)$. The *chromatic number* $\chi(G)$ (*choice number* $\chi_l(G)$) of G is the least integer k such that G is k-colorable (k-choosable). We say that G is Φ-*critical* if G is not Φ-colorable but every proper subgraph of G is Φ-colorable

1.3 Gallai trees and critical graphs

Let G be a graph. A vertex x of G is called a *separating vertex* of G if $G - x$ has more components than G. By a *block* of G we mean a maximal connected subgraph B of G such that no vertex of B is a separating vertex of B. Any two blocks of G have at most one vertex in common and, clearly, a vertex of G is a separating vertex of G iff it is contained in more than one block of G.

A connected graph G all of whose blocks are complete graphs and/or odd circuits is called a *Gallai tree*; a *Gallai forest* is a graph all of whose components are Gallai trees.

By a *bad pair* we mean a pair (G,Φ) consisting of a connected graph G and a list Φ for G such that $|\Phi(x)| \geq d_G(x)$ for all $x \in V(G)$ and G is not

Φ-colorable.

Lemma 2 *If (G,Φ) is a bad pair, then the following statements hold.*

(a) $|\Phi(x)| = d_G(x)$ *for all* $x \in V(G)$.

(b) *If G has no separating vertex, then $\Phi(x)$ is the same for all $x \in V(G)$.*

(c) *G is a Gallai tree.* □

Lemma 2 was proved independently by Borodin [2] and Erdős, Rubin and Taylor [6]. For a short proof of Lemma 2 based on the following simple reduction idea the reader is referred to [10].

Remark ([11]). Let G be a graph, Φ a list for G, $Y \subseteq V(G)$, and let φ be a Φ–coloring of $G[Y]$. For the graph $G' = G - Y$, we define a list Φ' by

$$\Phi'(x) = \Phi(x) - \{\varphi(y) \mid y \in Y \text{ and } xy \in E(G)\}$$

for every $x \in V(G')$. In what follows, we denote Φ' by $\Phi(Y, \varphi)$. Then it is straightforward to show that the following statements hold.

(a) If G' is Φ'–colorable, then G is Φ–colorable.

(b) If $|\Phi(x)| = d_G(x) + p$ for some $x \in V(G')$, then $|\Phi'(x)| \geq d_{G'}(x) + p$. □

Theorem 3 ([11]) *Assume that $k \geq 4$ and $G \neq K_k$ is a Φ-critical graph where Φ is a list for G satisfying $|\Phi(x)| = k - 1$ for every $x \in V(G)$. Let $H = \{y \in V(G) \mid d_G(y) \geq k\}$ and $L = V(G) - H$. Then the following statements hold.*

(a) *$G[L]$ is empty or a Gallai forest and $d_G(x) = k - 1$ for every $x \in L$.*

(b) *$G[L]$ does not contain a K_k.*

(c) $2|E(G)| \geq (k - 1 + (k - 3)/(k^2 - 3))|V(G)|$.

Proof. For the proof of (a), consider the vertex set X of some component of $G[L]$ and let $Y = V(G) - X$. Since G is Φ-critical, there is a Φ-coloring φ of $G[Y]$. Let $G' = G[X] = G - Y$ and $\Phi' = \Phi(Y, \varphi)$. By the above remark,

(G', Φ') is a bad pair and, therefore, Lemma 2 implies that G' is a Gallai tree and $d_G(x) = k-1$ for all $x \in X$. This proves (a).

To prove (b), suppose that $G[L]$ contains a K_k. Then, because of (a), K_k is a component of G. Since every Φ-critical graph is connected, this implies that $G = K_k$, a contradiction.

Statement (c) follows from (a), (b) and a result of Gallai. He proved in [8] that if G is a graph on n vertices and m edges such that the minimum degree is at least $k-1$ ($k \geq 4$) and the subgraph of G induced by the set of vertices of degree $k-1$ is empty or a Gallai forest not containing a K_k, then $2m \geq (k-1+(k-3)/(k^2-3))n$.

2 Proof of Theorem 1 for $\varepsilon = 2$ and $\varepsilon \geq 4$

Let Σ be a surface of Euler genus ε where $\varepsilon = 2$ or $\varepsilon \geq 4$ and let

$$k = H(\varepsilon) = \lfloor (7+\sqrt{24\varepsilon+1})/2) \rfloor. \tag{1}$$

Let G be an arbitrary graph embedded in Σ. Since G is $(k-1)$-degenerate, $\chi_l(G) \leq k$ and we need only to show that $\chi_l(G) \leq k-1$ provided that G does not contain a K_k.

Suppose that this is not true and let G be a minimal counterexample. Then G does not contain a K_k and there is a list Φ for G such that $|\Phi(x)| = k-1$ for all $x \in V(G)$ and G is Φ-critical. Let $n = |V(G)|$ and $m = |E(G)|$. Then, by Euler's Formula,

$$2m \leq 6n - 12 + 6\varepsilon. \tag{2}$$

Furthermore, $n \geq k+1$ and, by Theorem 3,

$$2m \geq \Big(k - 1 + \frac{k-3}{k^2-3}\Big)n. \tag{3}$$

First, assume $n \geq k+4$. Then it follows from (2) and (3) that

$$\Big(k - 7 + \frac{k-3}{k^2-3}\Big)(k+4) \leq 6\varepsilon - 12$$

and, therefore,

$$k^2 - 3k + \frac{(k-3)(k+4)}{k^2-3} \leq 6\varepsilon + 16. \tag{4}$$

It can be verified that (4) leads to a contradiction for $\varepsilon = 2$ and $4 \le \varepsilon \le 10$. If $\varepsilon \ge 11$, then $k \ge 11$ and, therefore,

$$\frac{(k-3)(k+4)}{k^2-3} > 1.$$

Consequently, because of (4), $k^2 - 3k - 15 - 6\varepsilon < 0$ implying that

$$k < \frac{3 + \sqrt{24\varepsilon + 69}}{2}.$$

On the other hand, because of (1),

$$k \ge \frac{5 + \sqrt{24\varepsilon + 1}}{2}.$$

Hence $2 + \sqrt{24\varepsilon + 1} < \sqrt{24\varepsilon + 69}$. This implies that $4\sqrt{24\varepsilon + 1} < 64$ and, therefore, $\varepsilon < 255/24 < 11$, a contradiction.

Now, assume $n \le k + 3$. Then $n \in \{k+1, k+2, k+3\}$. Let $H = \{x \in V(G) \mid d_G(x) \ge k\}$ and $L = \{x \in V(G) \mid d_G(x) = k-1\}$. By Theorem 3, $V(G) = H \cup L$ and $G' = G[L]$ is a Gallai forest.

If $n = k+1$, then G is obtained from a K_{k+1} by deleting the edges of some matching M. Obviously, L is the set of all vertices incident with some edge of M. Since $G[L]$ is a Gallai forest, this implies that $|M| = 1$ and, therefore, $|L| = 2$. Then G contains a K_k, a contradiction.

If $n \in \{k+2, k+3\}$, then we distinguish two cases. For the case when $2m \ge (k-1)n + k - 3$ we can use the same argument as Dirac in [5] to arrive at a contradiction. For the case when $2m \le (k-1)n + k - 4$, we argue as follows. First, we infer that

(a) $|H| \le k - 4$ and $|L| \ge 6$ if $n = k+2$ or $|L| \ge 7$ if $n = k+3$.

Next, suppose that $G' = G[L]$ is a complete graph. By (a), every vertex of H is adjacent to some vertex of L. Since $d_G(x) = k - 1$ for all $x \in L$, this implies that there are vertices $z \in H$ and $x, y \in L$ satisfying $zx \in E(G)$ and $zy \notin E(G)$. Because of (a) and $|\Phi(v)| = k - 1$ for all $v \in V(G)$, there are two Φ-colorings φ_1, φ_2 of $G[H]$ such that $\varphi_1(v) = \varphi_2(v)$ for all $v \in H - \{z\}$ and $\varphi_1(z) \ne \varphi_2(z)$. For $i = 1, 2$, let $\Phi_i = \Phi(H, \varphi_i)$ be the list for $G' = G[L] = G - H$. Then, see the remark in Section 1.3, (G', Φ_i) is a bad pair for $i = 1, 2$ and, moreover, either $\Phi_1(x) \neq \Phi_1(y)$ or $\Phi_2(x) \neq \Phi_2(y)$, a contradiction to statement (b) of Lemma 2.

Finally, assume that $G' = G[L]$ is not a complete graph. Since G' is a Gallai forest and every vertex of L has degree $k-1$ in G, we infer from (a) that G' has at least two blocks and, therefore, $n = k+3$, $|L| = 7$ and G' consists of exactly two blocks B_1, B_2 that are both complete graphs on four vertices and that have a vertex x in common. Consequently, $|H| = k-4$ and, since $2m \leq (k-1)n + k - 4$, every vertex of H has degree k in G. Moreover, since $d_G(y) = k-1$ for all $y \in L$, every vertex of $L - \{x\}$ is adjacent to all vertices of H in G. Then there are two vertices z, u in H such that $zu \notin E(G)$. Let y denote an arbitrary vertex of $B_1 - x$. Because of (a) and $|\Phi(v)| = k-1$ for all $v \in V(G)$, there is a Φ-coloring φ of $G[H]$ such that either $\varphi(z) = \varphi(u)$ or $\varphi(z) \notin \Phi(y)$ or $\varphi(u) \notin \Phi(y)$. Let $\Phi' = \Phi(H, \varphi)$ be the list for G'. Then (G', Φ') is a bad pair and, since $yz, yu \in E(G)$ and $|\Phi(y)| = d_G(y)$, $|\Phi'(y)| > d_{G'}(y)$, a contradiction to Lemma 2(a).

Thus Theorem 1 is proved for $\varepsilon = 2$ and $\varepsilon \geq 4$.

3 5-choosability of planar graphs

To prove Theorem 1 for the projective plane, some auxiliary results about list colorings of planar graphs are needed. A graph G is said to be a *near-triangulation* with *outer cycle* C if G is a plane graph that consists of the cycle C and vertices and edges inside C such that each bounded face is bounded by a triangle. Thomassen [15] proved that every planar graph is 5-choosable. His proof is based on the following stronger result.

Theorem 4 *Let G be a near-triangulation with outer cycle C and let Φ be a list for G such that $|\Phi(v)| \geq 3$ for all $v \in V(C)$ and $|\Phi(v)| \geq 5$ for all $v \in V(G) - V(C)$. Assume that xy is an edge of C, $\alpha \in \Phi(x)$ and $\beta \in \Phi(y)$. Then there is a Φ-coloring φ of G such that $\varphi(x) = \alpha$ and $\varphi(y) = \beta$.* □

The next result is an immediate consequence of Theorem 4, see also [16].

Theorem 5 (Thomassen [16]) *Let G be a plane graph, let W be the set of vertices on the outer face of G and let Φ be a list for G such that $|\Phi(v)| \geq 3$*

for all $v \in W$ *and* $|\Phi(v)| \geq 5$ *for all* $v \in V(G) - W$*. Assume that* xy *is an edge on the boundary of the outer face of* G*,* $\alpha \in \Phi(x)$ *and* $\beta \in \Phi(y)$*. Then there is a* Φ*-coloring* φ *of* G *such that* $\varphi(x) = \alpha$ *and* $\varphi(y) = \beta$*.* □

For the proof of Theorem 1 in case of $\varepsilon = 1$ we need the following extension of Thomassen's result.

Theorem 6 *Let* G *be a plane graph and let* W *be the set of vertices on the outer face of* G*. Let* $P = (v_1, \ldots, v_k)$ *be a path on the boundary of the outer face. Assume that* Φ *is a list for* G *satisfying* $|\Phi(v)| \geq 5$ *if* $v \in V(G) - W$*,* $|\Phi(v)| \geq 4$ *if* $v \in V(P) - \{v_1, v_k\}$*,* $|\Phi(v)| \geq 2$ *if* $v \in \{v_1, v_k\}$*, and* $|\Phi(v)| \geq 3$ *if* $v \in W - V(P)$*. Then* G *is* Φ*-colorable.*

Proof (by induction on the number of vertices of G). For $k \leq 2$, Theorem 6 follows by Theorem 5. Now assume $k \geq 3$.

If G is the union of two non-trivial subgraphs G_1, G_2 such that $|V(G_1) \cap V(G_2)| \leq 1$ and P is contained in G_1, then we argue as follows. By the induction hypothesis, there is a Φ-coloring φ_1 of G_1 and, by Theorem 5, there is a Φ-coloring φ_2 of G_2 where $\varphi_2(x) = \varphi_1(x)$ in case of $V(G_1) \cap V(G_2) = \{x\}$ (note that x is on the outer face of G_2). Then $\varphi_1 \cup \varphi_2$ is a Φ-coloring of G.

Otherwise, G is connected and every block of G is either an edge of P or a 2-connected plane subgraph with an outer cycle C' such that $C' \cap P$ is a subpath of P with at least two vertices where for distinct 2-connected blocks of G, these subpaths are edge-disjoint. Then there is a near-triangulation G' with an outer cycle C such that $V(G) = V(G')$, $E(G) \subseteq E(G')$, $W = V(C)$, and P is a subpath of C. Then $C = (v_1, \ldots, v_p)$ with $p \geq k \geq 3$. If $p = k$, then Theorem 5 implies that there is a Φ-coloring of G' and hence also of G. If $p \geq k + 1$, then we argue as follows.

First, we consider the case when C has a chord incident with v_k, say $v_k v_i$. If $1 \leq i \leq k - 2$, then we apply the induction hypothesis to the cycle $(v_1, \ldots, v_i, v_k, \ldots, v_p)$ and its interior and then we apply Theorem 4 to the cycle $(v_i, v_{i+1}, \ldots, v_k)$ and its interior where $v_k v_i$ is the precolored edge. If $k + 2 \leq i \leq p$, then we apply the induction hypothesis to the cycle $(v_1, \ldots, v_k, v_i, v_{i+1} \ldots, v_p)$ and its interior and then we apply Theorem 4 to the cycle $(v_k, v_{k+1}, \ldots, v_i)$ and its interior where $v_k v_i$ is the precolored edge.

Now, we consider the case when C has no chord incident with v_k. Let $v_{k-1}, u_1, \ldots, u_m, v_{k+1}$ be the neighbors of v_k in that clockwise order around

v_k. As the interior of C is triangulated, $P' = (v_{k-1}, u_1, \dots, u_m, v_{k+1})$ is a path and $C' = P' \cup (C - v_k)$ is a cycle of G'. Let α, β be distinct colors in $\Phi(v_k)$. Define a list Φ' for $G' - v_k$ by $\Phi'(v) = \Phi(v) - \{\alpha, \beta\}$ if $v \in \{v_{k-1}, u_1, \dots, u_m\}$ and $\Phi'(v) = \Phi(v)$ otherwise. Then we apply the induction hypothesis to C' and its interior with respect to the path $P - v_k$ and the list Φ'. We complete the coloring by assigning α or β to v_k such that v_k and v_{k+1} get distinct colors. Thus Theorem 6 is proved. □

The next result is crucial for the proof of Theorem 1 restricted to the case of the projective plane.

Theorem 7 *Let G be a plane graph with outer cycle C of length $p \leq 6$. Assume that Φ is a list for G satisfying $|\Phi(v)| \geq 5$ for all $v \in V(G)$ and φ is a Φ-coloring of $G[V(C)]$. Then φ can be extended to a Φ-coloring of G unless $p \geq 5$ and the notation may be chosen such that $C = (v_1, \dots, v_p)$, $\varphi(v_i) = \alpha_i$ for $1 \leq i \leq p$ and one of the following three conditions holds where all indices are computed modulo p.*

(a) *There is a vertex u inside C such that u is adjacent to $v_1, \dots, v_5$ and $\Phi(u) = \{\alpha_1, \dots, \alpha_5\}$.*

(b) *$p = 6$ and there is an edge u_0u_1 inside C such that, for $i = 0, 1$, the vertex u_i is adjacent to $v_{3i+1}, v_{3i+2}, v_{3i+3}, v_{3i+4}$ and $\Phi(u_i) = \{\alpha_{3i+1}, \alpha_{3i+2}, \alpha_{3i+3}, \alpha_{3i+4}, \beta\}$.*

(c) *$p = 6$ and there is a triangle (u_0, u_1, u_2) inside C such that, for $i = 0, 1, 2$, the vertex u_i is adjacent to $v_{2i+1}, v_{2i+2}, v_{2i+3}$ and $\Phi(u_i) = \{\alpha_{2i+1}, \alpha_{2i+2}, \alpha_{2i+3}, \beta, \gamma\}$.*

Proof (by induction on the number of vertices of G). If one of the conditions (a), (b) or (c) holds, we briefly say that (G, Φ, φ) is *bad*. For a subgraph H of G and a vertex $u \in V(G)$, let $d(u : H)$ denote the number of vertices in H that are adjacent to u in G. We consider two cases.

Case 1: There is an edge vw of C such that $d(u : C - v - w) \leq 2$ for all vertices u inside C. Then let $X = V(C - v - w)$ and define a list Φ' for the plane graph $G' = G - X$ by

$$\Phi'(u) = \Phi(u) - \{\varphi(v') \mid uv' \in E(G) \;\&\; v' \in X\}$$

if u is a vertex inside C and $\Phi'(u) = \Phi(u)$ for $u \in \{v, w\}$. Theorem 5 implies that there is a Φ'-coloring φ' of G' with $\varphi'(v) = \varphi(v)$ and $\varphi'(w) = \varphi(w)$. Hence φ can be extended to a Φ-coloring of G.

Case 2: For every edge vw of C, we have $d(u : C - v - w) \geq 3$ for some vertex u inside C. Then $p \geq 5$.

First, assume that G has a separating cycle C' (i.e. there are vertices inside and outside C') of length at most four. Then $C' \neq C$. Let G' be the graph obtained from G by deleting all vertices inside C'. If φ can be extended to a Φ-coloring of G', then, by Case 1, we can extend this coloring to the vertices inside C' and, therefore, φ can be extended to a Φ-coloring of G. Otherwise, we conclude from the induction hypothesis that (G', Φ, φ) is bad and, therefore, (G, Φ, φ) is bad, too.

Now, assume that G is *tough*, that is G has no separating cycle of length at most four. Let u denote a vertex inside C such that $d = d(u : C)$ is maximum. Then $3 \leq d \leq 6$. If $d \geq 5$, then u is the only vertex inside C, since otherwise G would not be tough. Then, clearly, φ can be extended to a Φ-coloring of G unless (a) holds.

If $d = 4$, then, since G is tough, the assumption of Case 2 implies that $p = 6$, $C = (v_1, \ldots, v_6)$, u is adjacent to, say, $v_1, \ldots, v_4$ but not to v_5 and v_6, and all vertices of $G - V(C) - \{u\}$ are inside the cycle $C' = (v_1, u, v_4, v_5, v_6)$. Clearly, there is a color $\alpha \in \Phi(u) - \{\varphi(v_1), \varphi(v_2), \varphi(v_3), \varphi(v_4)\}$. If there is no vertex w inside C' such that w is adjacent to all vertices of C' and $\Phi(w) = \{\alpha, \varphi(v_1), \varphi(v_4), \varphi(v_5), \varphi(v_6)\}$, then, by the induction hypothesis, φ can be extended to a Φ-coloring of G with $\varphi(w) = \alpha$. Otherwise, because of G is tough, this vertex w is the only vertex inside C' and we easily conclude that φ can be extended to a Φ-coloring of G unless (b) holds.

Finally, consider the case $d = 3$. Since G is tough, we infer from the assumption of Case 2 that if $d(u : C) = 3$ for some vertex u inside C, then u has three consecutive neighbors on C. Furthermore, we conclude that there are at least three vertices u_0, u_1, u_2 inside C such that $C = (v_1, \ldots, v_6)$ and, for $i = 0, 1, 2$, the neighbors of u_i on C are $v_{2i+1}, v_{2i+2}, v_{2i+3}$ with $v_7 = v_1$. G being tough, all vertices of $V(G) - V(C) - \{u_0, u_1, u_2\}$ are inside the cycle $C' = (v_1, u_0, v_3, u_1, v_5, u_2)$. If (u_0, u_1, u_2) is a triangle, then $V(G) = V(C) \cup \{u_0, u_1, u_2\}$ and, therefore, either φ can be extended to a Φ-coloring of G or, for $i = 0, 1, 2$, $\Phi(u_i) = \{\varphi(v_{2i+1}), \varphi(v_{2i+2}), \varphi(v_{2i+3}), \beta, \gamma\}$, that is

(c) holds. Hence, we may assume that $u_0u_2 \notin E(G)$. Let $G' = G - v_2$. The outer cycle of G' is $C' = (v_1, u_0, v_3, v_4, v_5, v_6)$. Let $\varphi'(v) = \varphi(v)$ for $v \in V(C') - \{u_0\}$, and let $\varphi'(u_0)$ be a color in $\Phi(u_0) - \{\varphi(v_1), \varphi(v_2), \varphi(v_3)\}$. If (G', Φ, φ') is not bad, then the induction hypothesis implies that φ' can be extended to a Φ-coloring of G' and we are done. If (G', Φ, φ') is bad, then there is a vertex inside C' distinct from u_1 and u_2 which has two neighbors in $\{v_1, v_3, v_5\}$ (since u_2 has only three neighbors in C'). Since G is tough, no vertex inside C' except u_1 and u_2 can have two neighbors in $\{v_1, v_3, v_5\}$. This contradiction completes the proof. □

4 List colorings on the projective plane

In this section we prove Theorem 1 for the projective plane. Let G denote an arbitrary graph embedded on the projective plane. Since G is 5-degenerate, $\chi_l(G) \leq 6$ and we need only to show that G is 5-choosable provided that G does not contain a K_6.

In the sequel, let Φ denote a list for G such that the following two conditions hold.

(a) $|\Phi(x)| = 5$ for all $x \in V(G)$.

(b) If K is a complete subgraph on 6 vertices of G, then $\Phi(x) \neq \Phi(y)$ for two vertices $x, y \in V(K)$.

By induction on the number of vertices of G, we prove that G is Φ-colorable.

If G contains a vertex x of degree at most 4, then, by the induction hypothesis, there is a Φ-coloring φ of $G - x$. Clearly, because of (a), φ can be extended to a Φ-coloring of G.

Next, consider the case when G contains a contractible cycle C of length three such that C is a nonfacial cycle of G. Let G_I denote the plane subgraph of G that consists of the cycle C and the vertices and edges inside C. Moreover, let $G_O = G - (V(G_I) - V(C))$. Then G_O has fewer vertices than G. Hence, by the induction hypothesis, there is a Φ-coloring φ_O of G_O. By Theorem 7, there is a Φ-coloring φ_I of G_I such that $\varphi_I(v) = \varphi_O(v)$ for all $v \in V(C)$. Clearly, $\varphi_I \cup \varphi_O$ is a Φ-coloring of G. Therefore, we may henceforth assume:

(c) The minimum degree of G is at least 5 and each contractible cycle of length three in G is a facial cycle of G.

If all cycles of G are contractible, then G is planar and, by Theorem 4, G is Φ-colorable. Hence we may assume that G contains a noncontractible cycle. Let $k \geq 3$ be the length of a shortest noncontractible cycle of G, and let $\mathcal{N}$ denote the set of all noncontractible cycles of G having length k. Our aim is to show that there is a cycle $C \in \mathcal{N}$ such that a certain Φ-coloring of C can be extended to a Φ-coloring of the plane graph $G - V(C)$.

Consider a noncontractible cycle $C = (v_1, \ldots, v_k) \in \mathcal{N}$. Then C has no chords and, by cutting Π_1 along C, we obtain a plane graph G_C with outer cycle $O_C = (v_1, \ldots, v_k, v'_1, \ldots, v'_k)$. The graph G_C can be considered as a representation of G on a closed disc where antipodal points on the boundary are identified. The plane graphs $G - V(C)$ and $G_C - V(O_C)$ are identical and, for $y \in V(G) - V(C)$, $yv_i \in E(G)$ if and only if yv_i or yv'_i belongs to $E(G_C)$, $i \in \{1, \ldots, k\}$. Furthermore, a path $P = (v_i, x_1, \ldots, x_m, v'_i)$ of G_C with $x_1, \ldots, x_m \in V(G) - V(C)$ corresponds to the noncontractible cycle $(v_i, x_1, \ldots, x_m)$ of G, implying that $m + 1 \geq k$. In particular, for every $y \in V(G) - V(C)$, the edges yv_i and yv'_i are not both in $E(G_C)$. Let W_C denote the set of all vertices of $G - V(C)$ that are in G adjacent to some vertex of C and, for $x \in W_C$, let $N_C(x)$ denote the set of all neighbors of x in G that belong to C.

First, assume $k = 3$. Let φ be a Φ-coloring of some cycle $C = (v_1, v_2, v_3) \in \mathcal{N}$ and let φ' be the Φ-coloring of $O_C = (v_1, v_2, v_3, v'_1, v'_2, v'_3)$ with $\varphi'(v_i) = \varphi'(v'_i) = \varphi(v_i)$ and $\Phi(v'_i) = \Phi(v_i)$ for $i = 1, 2, 3$. If φ' can be extended to some Φ-coloring of G_C, then this coloring determines a Φ-coloring of G. Otherwise, we conclude from Theorem 7 that in the plane graph G_C there is a triangle D inside O_C such that each vertex of D is adjacent with three vertices that are consecutive on O_C. Therefore, in G each vertex of D is adjacent to all vertices of C and thus $G[V(C) \cup V(D)]$ is a complete graph on 6 vertices. Since every noncontractible triangle of G is a facial triangle of G, this implies that $V(G) = V(C) \cup V(D)$ and, therefore, $G = K_6$. From (b) it then follows that G is Φ-colorable.

Now, assume $k \geq 4$. Let $C = (v_1, \ldots, v_k) \in \mathcal{N}$. First, we claim that $|N_C(x)| \leq 3$ for each $x \in W_C$. If some vertex $x \in W_C$ is adjacent in G to v_i and v_j with $i < j$, then exactly one of the two cycles $(v_i, v_{i+1}, \ldots, v_j, x)$ and

$(v_j, v_{j+1}, \ldots, v_i, x)$ is noncontractible, where all indices are computed modulo k. It follows that the claim is true in case $k \geq 5$, since otherwise there would exists a noncontractible cycle of length at most $k-1$, a contradiction. If $k = 4$ and some vertex x is adjacent to all vertices of C, then all four triangles (x, v_i, v_{i+1}), $i = 1, 2, 3, 4$, are contractible and hence facial triangles of G. Consequently, x is a vertex of degree four in G, a contradiction to (c). This proves the claim. Let T_C denote the set of all vertices $x \in W_C$ such that $|N_C(x)| = 3$ and, for $v \in V(C)$, let T_C^v denote the set of all vertices $x \in T_C$ such that $xv \in E(G)$.

Next, since C is a shortest noncontractible cycle of G and (c) holds, we conclude that the following holds:

(d) For each vertex $x \in T_C$, $N_C(x) = \{v_i, v_{i+1}, v_{i+2}\}$ for some i and the triangles (x, v_i, v_{i+1}) and (x, v_{i+1}, v_{i+2}) are contractible and hence facial (all indices are modulo k). Moreover, $N_C(y) \neq N_C(x)$ for all $y \in T_C - \{x\}$ (since otherwise v_{i+1} would be a vertex of degree 4 in G, contradicting (c)).

Consequently, $|T_C^v| \leq 3$ for all $v \in V(C)$. Moreover, the three neighbors of x in G_C that belong to O_C are consecutive on O_C, since otherwise either (x, v_i, v_{i+1}) or (x, v_{i+1}, v_{i+2}) would be noncontractible in G, a contradiction.

Two vertices $z, u \in T_C$ are said to be *C-conform* if there is a vertex in O_C adjacent to z and u in G_C. If $k \geq 5$ and $|T_C^v| = 3$ for some vertex $v \in V(C)$, then there are exactly two vertices $z, u \in T_C^v$ such that z, u are C-conform.

For a Φ-coloring φ of $C \in \mathcal{N}$ and a vertex $x \in W_C$, let $\varphi(C : x) = \{\varphi(v) \mid v \in N_C(x)\}$. Suppose that $X \subseteq T_C$ such that $|X| \leq 1$ or $k \geq 5$, $X = \{z, u\}$, and z, u are C-conform. A Φ-coloring φ of C is called *X-good* if $|\Phi(x) - \varphi(C : x)| \geq 3$ for all $x \in T_C - X$.

We claim that, if there is a cycle $C \in \mathcal{N}$, an appropriate $X \subseteq T_C$, and a Φ-coloring φ of C that is X-good, then φ can be extended to a Φ-coloring of G. For the proof of this claim, define a list Φ' for the plane graph $G' = G - V(C) = G_C - V(O_C)$ by $\Phi'(x) = \Phi(x) - \varphi(C : x)$ if $x \in W_C$ and $\Phi'(x) = \Phi(x)$ otherwise. We have to show that G' is Φ'-colorable. Since φ is X-good and each vertex of $W_C - T_C$ has in G at most two neighbors on C, we have $|\Phi'(x)| \geq 3$ for all $x \in W_C - X$, $|\Phi'(x)| \geq 5$ for all $x \in V(G') - W_C$ and, because of $X \subseteq T_C$, $|\Phi'(x)| \geq 2$ for all $x \in X$. Furthermore, each vertex of W_C belongs to the outer face of G'. If $|X| \leq 1$, then Theorem 5 implies that G' is Φ'-colorable. Otherwise, $k \geq 5$, $X = \{z, u\}$, and z, u are C-conform. Therefore, by (d), we conclude that the notation may be chosen so that $C = (v_1, \ldots, v_k)$, $N_C(z) = \{v_1, v_2, v_3\}$, $N_C(u) = \{v_3, v_4, v_5\}$, and in G_C the neighbors of z and u on O_C are v_1, v_2, v_3 and v_3, v_4, v_5, respectively. Let $z, x_1, \ldots, x_m, u$ be the neighbors of v_3 in G_C in that clockwise order around v_3. Then, by adding certain edges, we may assume that $P = (z, x_1, \ldots, x_m, u)$ is a path on the boundary of the outer face of $G' = G - V(O_C)$ where each vertex of W_C still belongs to the outer face of G'. Since $k \geq 5$ and C is a shortest noncontractible cycle of G, we conclude that, for all $x \in V(P) - \{z, u\}$, $N_C(x) = \{v_3\}$ and, therefore, $|\Phi'(x)| \geq 4$. Now, Theorem 6 implies that G' is Φ'-colorable. This proves the claim.

Therefore, to complete the proof of Theorem 1, it suffices to prove that, for some cycle $C \in \mathcal{N}$ and an appropriate $X \subseteq T_C$, there is an X-good Φ-coloring of C. For the proof of this statement, we consider the following procedure for a given cycle $C \in \mathcal{N}$. First, we choose a vertex $v_1 = v$ of C and a color $\alpha_1 \in \Phi(v_1)$. Next, we choose an orientation of C such that $C = (v_1, \ldots, v_k)$. Now, we choose a set $X \subseteq T_C^{v_k}$ such that $|T_C^{v_k} - X| \leq 1$. Recall that $|T_C^v| \leq 3$ for all $v \in V(C)$. Eventually, we define a mapping $\varphi = \varphi(C, v_1, \alpha_1, v_k, X)$ from $V(C)$ into the color set as follows. First, we set $\varphi(v_1) = \alpha_1$. Now, assume that $\varphi(v_1), \ldots, \varphi(v_{i-1})$ are already defined where $2 \leq i \leq k$. Because of (d) and $|T_C^{v_k} - X| \leq 1$, there is at most one vertex $x \in T_C^{v_i} - X$ such that $N = N_C(x) - \{v_i\}$ is a subset of $\{v_1, \ldots, v_{i-1}\}$. Then, because of (a) and $|N| = 2$, $M = \Phi(x) - \{\varphi(v) \mid v \in N\}$ is a set of at least three colors and, therefore, there is a color $\alpha \in \Phi(v_i) - \{\varphi(v_{i-1})\}$ such that $|M - \{\alpha\}| \geq 3$. We define $\varphi(v_i) = \alpha$. Clearly, φ is a Φ-coloring of $C - v_1 v_k$ and $|\Phi(x) - \varphi(C : x)| \geq 3$ for all $x \in T_C - X$. Therefore, φ is an X-good Φ-coloring of C provided that $\varphi(v_k) \neq \alpha_1$ and $|X| \leq 1$ or $k \geq 5$, $X = \{z, u\}$, and z, u are C-conform. If $|T_C^{v_k}| \leq 1$ or $k \geq 5$, $T_C^{v_k} = \{z, u\}$ and z, u are C-conform, then we choose $X = T_C^{v_k}$ and, in the last step of our procedure, we choose a color $\alpha \in \Phi(v_k) - \{\varphi(v_1), \varphi(v_{k-1})\}$ and define $\varphi(v_k) = \alpha$. This leads to an X-good Φ-coloring φ of C. Therefore, we assume henceforth that for every cycle $C \in \mathcal{N}$ the following two conditions hold.

(1) $|T_C^v| \geq 2$ for every $v \in V(C)$.

(2) If $T_C^v = \{z, u\}$ for some $v \in V(C)$, then $k = 4$ or $k \geq 5$ and z, u are not C-conform.

Now we distinguish two cases. First, we consider the case that there is a cycle $C \in \mathcal{N}$ such that two vertices of C have distinct lists. Then there is also an edge vw of C such that $\Phi(v) \neq \Phi(w)$. Because of (a), this implies that there are two colors $\alpha_w \in \Phi(w) - \Phi(v)$ and $\alpha_v \in \Phi(v) - \Phi(w)$. If for one of the two vertices v, w, say v, we have $T_C^v = \{x, y\}$, then $\varphi = \varphi(C, v_1, \alpha_1, v_k, X)$ with $v_1 = w$, $\alpha_1 = \alpha_w$, $v_k = v$ and $X = \{x\}$ is an X-good Φ-coloring of C. Otherwise, because of (1) and (d), we have $|T_C^v| = |T_C^w| = 3$ and, therefore, $k \geq 5$ and there are two vertices $u, z \in T_C^v$ such that u, z are C-conform. Then $\varphi = \varphi(C, v_1, \alpha_1, v_k, X)$ with $v_1 = w$, $\alpha_1 = \alpha_w$, $v_k = v$ and $X = \{u, z\}$ is an X-good Φ-coloring of C.

Finally, we consider the case when for every cycle $C \in \mathcal{N}$ there is a set F of five colors such that $\Phi(v) = F$ for all $v \in V(C)$. If k is even, then we choose an arbitrary cycle $C \in \mathcal{N}$. By the assumption of this case, there is a Φ-coloring φ of C such that φ uses only two colors. Then, for $X = \emptyset$, φ is an X-good Φ-coloring of C. Now assume that k is odd. In particular, $k \geq 5$. For a cycle $C \in \mathcal{N}$, let $t(C)$ denote the number of all vertices $v \in V(C)$ such that $|T_C^v| = 3$. Consider a cycle $C \in \mathcal{N}$ such that $t(C)$ is minimum.

If $t(C) \geq 1$, then there is a vertex $v \in V(C)$ such that T_C^v is a set of three vertices, say y, z, u and, since $k \geq 5$, two of these three vertices, say z, u, are C-conform. Therefore, because of (d), the notation may be chosen so that $C = (v_1, \ldots, v_k)$, where $v_1 = v$, $N_C(z) = \{v_{k-1}, v_k, v_1\}$, $N_C(u) = \{v_1, v_2, v_3\}$ and $N_C(y) = \{v_k, v_1, v_2\}$ where all indices are computed modulo k. Furthermore, there is a vertex $x \in W_C$ such that $N_C(x) = \{v_{k-2}, v_{k-1}, v_k\}$, since otherwise $C' = (v_1, u, v_3, \ldots, v_k)$ would be a noncontractible cycle of length k in G such that $T_{C'}^{v_k} = \{z\}$, a contradiction to (1). By symmetry, there is also a vertex $x' \in W_C$ such that $N_C(x') = \{v_2, v_3, v_4\}$. But then $\tilde{C} = (y, v_2, \ldots, v_k)$ is a noncontractible cycle of length k in G and, because of (d), $T_{\tilde{C}}^y = \{v_1\}$, a contradiction to (1).

Now assume $t(C) = 0$ where $C = (v_1, \ldots, v_k)$. Then we conclude from (1), (2) and (d) that, for $i = 1, \ldots, k$, $|T_C^{v_i}| = 2$ and, since $k \geq 5$, the two vertices of $T_C^{v_i}$ are not C-conform. This implies that in the plane graph G_C every vertex of the outer cycle $O_C = (v_1, \ldots, v_k, v_{k+1} = v'_1, \ldots, v_{2k} = v'_k)$ has exactly one neighbor in T_C. Consequently, $2k \equiv 0 (\operatorname{mod} 3)$ and, therefore, $k \equiv 0 (\operatorname{mod} 3)$. Furthermore, for every vertex $x \in T_C$, the three neighbors of x that belong to O_C are consecutive on O_C. Since $k \equiv 0 (\operatorname{mod} 3)$, we now see that if the vertices v_i, v_{i+1}, v_{i+2} have a common neighbor in G_C, then the vertices $v_{i+k}, v_{i+k+1}, v_{i+k+2}$ (indices modulo $2k$) have a common neighbor in G_C, too. Consequently, in G there are two vertices $x, y \in T_C$ such that $N_C(x) = N_C(y)$, a contradiction to (d).

This shows that, for some cycle $C \in \mathcal{N}$ and some subset X of T_C, there is an X-good Φ-coloring of C. Theorem 1 is proved. □

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