Graphs of Degree 4 are 5-Edge-Choosable

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Abstract: It is shown that every simple graph with maximal degree 4 is 5-edge-choosable. © 1999 John Wiley & Sons, Inc. J Graph Theory 32: 250–264, 1999

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1. INTRODUCTION

Graphs in this article are undirected and finite. They have no loops and no multiple edges, but may contain edges with only one end, called *halfedges*; other edges are called *proper edges*. The maximal degree of G is denoted by $\Delta(G)$. A *list assignment* of G is a function L that assigns to each edge $e \in E(G)$ a *list* $L(e) \subseteq N$. The elements of the list $L(e)$ are called the *admissible colors* for the edge e. An *L-edge-coloring* (also called *list coloring*) is a function λ: $E(G) \to N$ such that $\lambda(e) \in L(e)$ for $e \in E(G)$ and such that, for any pair of adjacent edges e, f in $G, \lambda(e) \neq \lambda(f)$. If G admits an L-edge-coloring, it is *L-edge-colorable*. For $k \in N$, the graph is *k-edge-choosable*, if it is L-edge-colorable for every list assignment L with $|L(e)| \geq k$ for each $e \in E(G)$.

List colorings were introduced by Vizing [15] and independently by Erdös, Rubin, and Taylor [1]. In 1976, Vizing [15] conjectured that every (multi)graph G is $\chi'(G)$-edge-choosable, where $\chi'(G)$ is the usual chromatic index of G (see also [4, Problem 12.20]). In 1979, Dinitz posed a question about a generalization of Latin

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squares, which is equivalent to the assertion that every complete bipartite graph $K_{n,n}$ is n-edge-choosable. This problem became known as the Dinitz conjecture and resisted proofs up to 1995, when Galvin [2] proved the conjecture in the affirmative. More generally, Galvin established Vizing's List Chromatic Index Conjecture for bipartite graphs by showing that every bipartite (multi)graph G is $\Delta(G)$-edge-choosable. A short self-contained proof of this result can also be found in [12]. Kahn [6] was the first to prove that, for simple graphs, $\chi_l'(G) = \Delta(G) + o(\Delta(G))$. Häggkvist and Janssen [3] improved this result by showing that every graph with maximal degree Δ is $(\Delta + \mathcal{O}(\Delta^{2/3}\sqrt{\log \Delta}))$-edge-choosable, and recently Molloy and Reed [9] improved the bound to $\Delta + \mathcal{O}(\Delta^{1/2} \log^6 \Delta)$.

In [5] it is proved that subcubic graphs are "$\frac{10}{3}$-edge-choosable." The precise meaning of this statement is that, no matter how we prescribe arbitrary lists of three colors to edges of a subgraph H of G such that $\Delta(H) \leq 2$, and prescribe lists of four colors on $E(G)\backslash E(H)$, the subcubic graph G will have an edge-coloring with the given colors.

In this article, the method of auxiliary graphs with halfedges from [5] is further developed in order to prove that every (simple) graph with maximal degree 4 is 5-edge-choosable.

Results of a similar flavor have been obtained for the closely related notion of total graph colorings. The total chromatic number of graphs of small maximal degree has been studied by Rosenfeld [10] and Vijayaditya [14]. They independently proved that total chromatic number for graphs of maximal degree 3 is 5. Kostochka in [7, 8] proved that the total chromatic number of multigraphs of maximal degree 4 (respectively, 5) is 6 (respectively, 7).

2. CRITICAL CHAINS

A sequence of distinct edges $P = e_1e_2 \cdots e_k$ $(k \geq 1)$ in a graph G is called an (*open*) *chain*. If among $e_2, \ldots, e_{k-1}$ there are p proper edges, then the chain P is called a *p-chain*. In our coloring procedures, a special role will be played by *critical chains*. Critical open chains are defined inductively as follows. Let L be a list assignment of G. The chain $P = e_1e_2$ is *critical with respect to* L, if $L(e_1) = L(e_2) = \{a, b\}$ (for some colors a, b). The chain $P = e_1e_2 \cdots e_k$ $(k \geq 3)$ is *critical with respect to* L, if there exists an index i $(1 < i < k)$ such that one of the following holds:

(O1) e_i, e_{i+1} are halfedges: Let $P_1 = e_1 \cdots e_i$ and $P_2 = e_{i+1} \cdots e_k$. There is a color $x \in L(e_i) \cap L(e_{i+1})$ such that, if L' is the list assignment that coincides with L except that $L'(e_i) = L(e_i)\backslash\{x\}$ and $L'(e_{i+1}) = L(e_{i+1})\backslash\{x\}$, then P_1 and P_2 are critical chains with respect to L'. We say that P is obtained by *combining* the critical chains P_1 and P_2 using color x. (See Fig. 1 for an example.)

(O2) e_i is a proper edge, $L(e_i)$ consists of four distinct colors, and $L(e_i)$ can be partitioned into $L(e_i) = \{a, b\} \cup \{c, d\}$ such that the following holds: Let

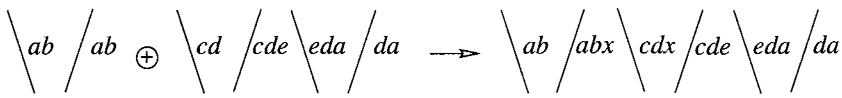

FIGURE 1. Combining critical chains using (O1).

$P_1 = e_1 \cdots e_i$ and $P_2 = e_i \cdots e_k$. If L' (and L'') is the list assignment that coincides with L except that $L'(e_i) = \{a, b\}$ (respectively, $L''(e_i) = \{c, d\}$), then P_1 and P_2 are critical with respect to L' and L'', respectively. Again, we say that P is obtained by *combining* P_1 and P_2 at e_i. (See Fig. 2, where proper edges are drawn horizontally.)

If $P = e_1 \cdots e_k$ is a critical p-chain, then $p \equiv k \pmod 2$. Moreover, $|L(e_1)| = |L(e_k)| = 2$, and $|L(e_i)| = 3$ or 4 (if e_i is a halfedge or a proper edge, respectively), $i = 2, \ldots, k-1$. Let $1 < j_1 < j_2 < \cdots < j_p < k$ be the indices such that e_{j_i} is a proper edge, $i = 1, \ldots, p$. Then j_1 is even, and $j_{i+1} - j_i$ is odd for $i = 1, \ldots, p-1$ (and $k - j_p$ is also odd). These properties are easy to prove by induction.

In the sequel, we need the following properties of critical chains. If e_i $(1 < i < k)$ is any proper edge of the critical chain $P = e_1 \cdots e_k$, then the subchains P_1 with respect to L' and P_2 with respect to L'' defined as in (O2) are both critical. Similarly, if e_i and e_{i+1} are halfedges $(1 < i < k)$, then the p_1-chain P_1 and the p_2-chain P_2 defined as in (O1) are critical chains if and only if $p_1 \equiv i \pmod 2$ (and $p_2 \equiv k - i \pmod 2$). The proof is left to the reader.

Lemma 2.1. *Let $P = e_1 \cdots e_k$ be a p-chain, and let L_1 and L_2 be list assignments for P such that P is critical with respect to L_1 and with respect to L_2.*

- (*a*) *Suppose that there is an index i $(1 \le i \le k)$ such that $L_1(e_j) = L_2(e_j)$ for each $j \neq i$. Then also $L_1(e_i) = L_2(e_i)$.*
- (*b*) *If $L_1(e_j) = L_2(e_j)$, $j = 2, \ldots, k-1$, and $L_1(e_1) = \{a, b\}$, $L_2(e_1) = \{b, c\}$, where $a \neq c$, then either $L_1(e_k) = \{a, d\}$ and $L_2(e_k) = \{c, d\}$ (if p is even), or $L_1(e_k) = \{c, d\}$ and $L_2(e_k) = \{a, d\}$ (if p is odd), for some color d.*

Proof. The proof of (a) is by induction on k using the remark above that P can be split into critical subchains P_1, P_2 using the same edge e_i for both list assignments. The details are left to the reader.

First we will prove (b) for 0-chains by induction on k, and afterwards for p-chains by induction on p.

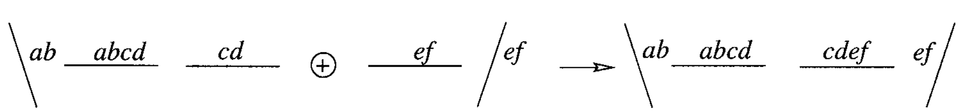

FIGURE 2. Combining critical chains using (O2).

For 0-chains, the case $k = 2$ is obvious (with $b = d$). Since P is critical with respect to L_1 and $L_2, \{a, b\} \subseteq L_1(e_2)$ and $\{b, c\} \subseteq L_2(e_2)$. Therefore, $L_1(e_2) = L_2(e_2) = \{a, b, c\}$. By the above remark, both chains can be split into critical subchains at e_2, e_3 as in (O1). In the case of $L_1, x = c$, and in the case of $L_2, x = a$. Hence, $\{a, c\} \subseteq L(e_3)$. Clearly, $P_2 = e_3 \cdots e_k$ is a critical chain for list assignments L_1' and L_2' obtained from L_1 and L_2, respectively, as described in (O1). Note that $L_1'(e_3) = \{a, f\}$ and $L_2'(e_3) = \{c, f\}$, where f is the third color in $L(e_3)$. Now the claim follows by induction.

Suppose now that P is a p-chain, where $p > 0$, and let e_i be the first proper edge in P. As before, we split P at e_i into two critical subchains P_1 and P_2. Let L_1', L_1'' and L_2', L_2'' be the list assignments for P_1 and P_2 obtained as in (O2) from L_1 and L_2, respectively. Applying induction on P_1 with list assignments L_1' and L_2', we get that $L_1'(e_i) = \{a, f\}$ and $L_2'(e_i) = \{c, f\}$ for some color f. Since $L_1'(e_i) \cup L_2'(e_i) \subseteq L_1(e_i) = L_2(e_i)$, we can assume that $L_1(e_i) = L_2(e_i) = \{a, c, f, g\}$. Hence, $L_1''(e_i) = \{c, g\}$ and $L_2''(e_i) = \{a, g\}$. Now, P_2 is a critical $(p-1)$-chain with respect to L_1'' and L_2'', and we complete the proof by induction. ∎

We will deal only with chains $P = e_1 \cdots e_k$ and list assignments L such that $|L(e_1)| \geq 2, |L(e_k)| \geq 2, |L(e_i)| \geq 3\ (1 < i < k)$, and $|L(e_i)| \geq 4$ if e_i is a proper edge $(1 < i < k)$. We say that such a list assignment L is *colorful*.

Lemma 2.2. *Let $P = e_1 \cdots e_k$ be a chain that is not critical with respect to a colorful list assignment L. Choose an integer r $(1 \leq r < k)$ and a color c. Suppose that e_1, e_k are halfedges and let $K_1 = L(e_r)\backslash\{c\}$ and $K_2 = L(e_{r+1})\backslash\{c\}$.*

- (*a*) *If e_r and e_{r+1} are halfedges, then there are distinct colors $d_1 \in K_1$ and $d_2 \in K_2$ such that the chains $P_1 = e_1 \cdots e_{r-1}$ and $P_2 = e_{r+2} \cdots e_k$ (if nonempty) are not critical with respect to the list assignment that agrees with L except that $L(e_{r-1})$ is replaced by $L(e_{r-1})\backslash\{d_1\}$ and $L(e_{r+2})$ is replaced by $L(e_{r+2})\backslash\{d_2\}$.*
- (*b*) *If e_r is a halfedge and e_{r+1} is a proper edge, then there is a color $d \in K_1$ such that the chains $P_1 = e_1 \cdots e_{r-1}$ (if nonempty) and $P_2 = e_{r+1} \cdots e_k$ are not critical with respect to the list assignments that agree with L, except that $L(e_{r-1})$ is replaced by $L(e_{r-1})\backslash\{d\}$ and $L(e_{r+1})$ is replaced by $L(e_{r+1})\backslash\{c, d\}$.*
- (*c*) *Suppose that e_r and e_{r+1} are proper edges. Suppose also that, if $|L(e_1)| = |L(e_k)| = 2$, then there is an even number of halfedges among $e_2, \ldots, e_{r-1}$. Then the chain P (where we interpret e_r and e_{r+1} as halfedges) is not critical with respect to the list assignment that agrees with L except that $L(e_r)$ is replaced by K_1 and $L(e_{r+1})$ is replaced by K_2.*
- (*d*) *Suppose that e_r and e_{r+1} are proper edges with $|L(e_r)| \geq 5$ and $|L(e_{r+1})| \geq 5$. Then $L(e_r)$ contains at least two colors x, y such that, if $c \in \{x, y\}$, then P is not critical with respect to L_1 (obtained from L by replacing $L(e_r)$ and $L(e_{r+1})$ by K_1 and K_2, respectively).*

Proof.

(a) By Lemma 2.1(a), there is at most one color $d'_1 \in K_1$ such that the chain P_1 is critical with respect to the new list assignment; similarly for P_2 (for the color d'_2). Suppose that $2 \leq r \leq k-2$. If the color d'_1 does not exist, take $d_2 \in K_2 \backslash \{d'_2\}$ (or $d_2 \in K_2$ if d'_2 does not exist) arbitrarily, and select $d_1 \in K_1 \backslash \{d_2\}$. So we assume that d'_1 and d'_2 exist. If $(K_1 \backslash \{d'_1\}) \cup (K_2 \backslash \{d'_2\})$ contains at least two colors, we can choose d_1 and d_2 to be distinct. Otherwise, there exists a color d such that $L(e_r) = \{c, d, d'_1\}$ and $L(e_{r+1}) = \{c, d, d'_2\}$. But then P, L can be obtained by using (O1) twice: first by combining chains P_1 and $e_r e_{r+1}$ (with colors $\{c, d\}$) using $x = d'_1$, and then by combining this chain with P_2 (where $x = d'_2$). This is a contradiction. The remaining case when $r = 1$ or $r = k-1$ is handled similarly.

(b) By Lemma 2.1(a), there is a color $d \in K_1$ such that P_1 is not critical for the new list assignment. If there is more than one such possibility for d, then again by Lemma 2.1(a), we can choose d such that P_1 and P_2 are not critical. So suppose that d is uniquely determined and that P_2 is critical. In that case, $\{c, d\} \subseteq L(e_r) \cap L(e_{r+1})$. But then we can construct P, L from critical chains as follows. By using (O2), we combine critical chains $e_r e_{r+1}$ (with colors $\{c, d\}$) and P_2. Let $P_3 = e_r \cdots e_k$. If $r \neq 1$, let d' be the third color in $L(e_r)$. Since P_1 is critical when we use d' instead of $d, d' \in L(e_{r-1})$. Now we get P by using (O1) to combine P_1 and P_3 (where $x = d'$). This shows that P is critical, a contradiction.

(c) Suppose that the new chain P is critical. Then $c \in L(e_r) \cap L(e_{r+1})$ and $|L(e_1)| = |L(e_k)| = 2$. Therefore, there is an even number of halfedges among $e_2, \ldots, e_{r-1}$ and, hence, by the remark before Lemma 2.1, we can split P at e_r, by using (O1), into two subchains P_1, P_2, which are critical with respect to the corresponding list assignment L'. Let $x \in K_1 \cap K_2$ be the color used in (O1). Then we can obtain P, L from P_1, P_2, L' as follows: we apply (O2) twice, first to combine P_1 with $e_r e_{r+1}$ (with colors $\{x, c\}$), and then to combine the resulting chain with P_2. Hence, P is critical with respect to L, a contradiction.

(d) Suppose that P is critical with respect to L_1. Then $P_1 = e_1 \cdots e_r$ is critical with respect to L'_1 (defined as in (O2)). By Lemma 2.1(a), $L'_1(e_r)$ contains a uniquely determined pair of colors, say x and y. Now, it is easy to check that the chains obtained by taking $c \in \{x, y\}$ are not critical.

∎

A *closed chain* $C = (e_1 \cdots e_k)$ is a cyclic sequence of distinct edges. We regard $(e_1 e_2 \cdots e_k), (e_2 \cdots e_k e_1)$, etc., as the same closed chain. The closed chain C is *critical at* e_r $(1 \leq r \leq k)$ (with respect to the list assignment L) *for the color* x, if one of the following holds:

(C1) e_r, e_{r+1} are halfedges and $P = e_{r+2} \cdots e_k e_1 \cdots e_{r-1}$ is a critical open chain for every L' obtained from L such that $L'(e_i) = L(e_i)$ $(i \neq r-1, r+$

$2), L'(e_{r-1}) = L(e_{r-1})\backslash\{d_1\}$, and $L'(e_{r+2}) = L(e_{r+2})\backslash\{d_2\}$, where $d_1 \in L(e_r)\backslash\{x\}, d_2 \in L(e_{r+1})\backslash\{x\}$, and $d_1 \neq d_2$.

(C2) e_r is a halfedge, e_{r+1} is a proper edge, $|L(e_{r+1})| \geq 4$, and $P = e_{r+1} \cdots e_k e_1 \cdots e_{r-1}$ is a critical open chain for every L' obtained from L such that $L'(e_i) = L(e_i)\ (i \neq r-1, r+1), L'(e_{r-1}) = L(e_{r-1})\backslash\{d\}$, and $L'(e_{r+1}) = L(e_{r+1})\backslash\{x, d\}$, where $d \in L(e_r)\backslash\{x\}$.

(C3) Same as (C2) except that the roles of e_{r-1} and e_{r+1} interchange.

One can construct arbitrarily long critical closed chains for either of the above cases.

We will deal only with *colorful* closed chains $C = (e_1 \cdots e_k)$ (with respect to L) for which $|L(e_i)| \geq 3$ if e_i is a halfedge and $|L(e_i)| \geq 4$ if e_i is a proper edge, $i = 1, \ldots, k$. The following lemma describes some basic properties of colorful critical closed chains.

Lemma 2.3. *Let $C = (e_1 \cdots e_k)$ be a closed chain, which is critical at e_r $(1 \leq r \leq k)$ for the color x, and let L be the corresponding list assignment. If C is colorful with respect to L, then the following holds*:

(*a*) $|L(e)| = 4$ *for each proper edge e in C*, $|L(e)| = 3$ *for each halfedge e in C, and $x \in L(e_r) \cap L(e_{r+1})$*.

(*b*) *If C is critical by* (C1), *then $L(e_r) = L(e_{r+1})$. If C is critical by* (C2), *then $L(e_r) \subset L(e_{r+1})$. If C is critical by* (C3), *then $L(e_{r+1}) \subset L(e_r)$*.

(*c*) *C contains an odd number of proper edges*.

(*d*) *If C is critical at e_r for a color y, then $y = x$*.

Proof. Part (a) easily follows from Lemma 2.1 and the remarks preceding it. To prove (b)–(d), suppose first that C is critical by (C1). Let $K_1 = L(e_r)\backslash\{x\}$ and $K_2 = L(e_{r+1})\backslash\{x\}$. Suppose that there exists a color $d_1 \in K_1$ (say) such that $K_2\backslash\{d_1\}$ contains at least two elements, d_2, d_2'. By taking d_1, d_2 and d_1, d_2' in (C1), we have two lists assignments for which P is critical. Since they differ only on e_{r+2}, we get a contradiction by Lemma 2.1(a). This proves that $K_1 = K_2 = \{a, b\}$. Now (a) implies that $L(e_r) = L(e_{r+1}) = \{a, b, x\}$. By (C1) for $d_1 = a, d_2 = b$ and $d_1 = b, d_2 = a$, respectively, we have two list assignments for which P is critical. Then Lemma 2.1(b) shows that P is a p-chain, where p is odd. Finally, suppose that C is critical at e_r also for the color $y \neq x$. We may assume that $y = b$. Taking $d_1 = a, d_2 = b$ (when critical for x) and $d_1 = a, d_2 = x$ (when critical for y) we get two list assignments for which P is critical. But these two assignments differ only at e_{r+2}, a contradiction with Lemma 2.1(a).

The second case is when C is critical by (C2). The proof of (b) is similar as above and is left to the reader. Let $L(e_r) = \{a, b, x\}$. By (C2) for $d = a$ and $d = b$, respectively, we have two list assignments for which P is critical. By Lemma 2.1(b), P contains an even number of proper edges. Together with e_{r+1}, this gives an odd number of proper edges in C. It remains to check (d). We may assume that $y = b$. By taking $d = a$ (for both x and y) in (C2), we get two list assignments for which P is critical. Since they differ only at e_{r+1}, this contradicts Lemma 2.1(a).

When C is critical by (C3), the proof follows the same steps as above and is, therefore, omitted. ∎

3. EDGE-COLORINGS OF GRAPHS WITH $\Delta \leq 4$

Let G be a graph with $\Delta(G) \leq 4$. Suppose that H is a 2-factor (i.e., a spanning 2-regular subgraph) of G. Let σ be an involution (i.e., $\sigma^2 = id$) on the set $E' = E(G) \backslash E(H)$ such that $\sigma(s) = s$ for each proper edge $s \in E'$. If $\sigma(s) = s$ ($s \in E'$), we say that s is *σ-free*. Otherwise, s is *σ-constrained*. For $e, f \in E'$, we write $e \sim f$ if either $e = f$, or $\sigma(e) = f$, or e and f are incident with the same vertex of G. Equivalence classes of the transitive closure of the relation $\sim$ on E' are called *σ-components*. Each σ-component determines a unique chain P or closed chain C (up to its direction) in which any two consecutive edges are either incident with the same vertex or σ-constrained with each other. The chain P (or C) is called a *σ-chain* (either *open* or *closed*).

An example of chains is shown in Fig. 3, where H consists of two thick cycles and action of σ on E' is represented by dotted lines (e.g., $\sigma(x_1) = x_1, \sigma(x_2) = x_3, \sigma(x_3) = x_2, \ldots$). Then $x_1 x_2 \cdots x_5$ and $y_1 y_2 \cdots y_8$ are σ-chains. There are three other σ-chains with 1, 2, and 3 edges, respectively.

Theorem 3.1. *Let G be a 4-regular graph with a 2-factor H. Let σ be an involution on E' as described above. Suppose that L is a list assignment of G such that*

$$|L(e)| \geq \begin{cases} 2, & e \text{ is a } \sigma\text{-free halfedge} \\ 3, & e \text{ is a } \sigma\text{-constrained halfedge} \\ 5, & e \text{ is a proper edge.} \end{cases} \tag{1}$$

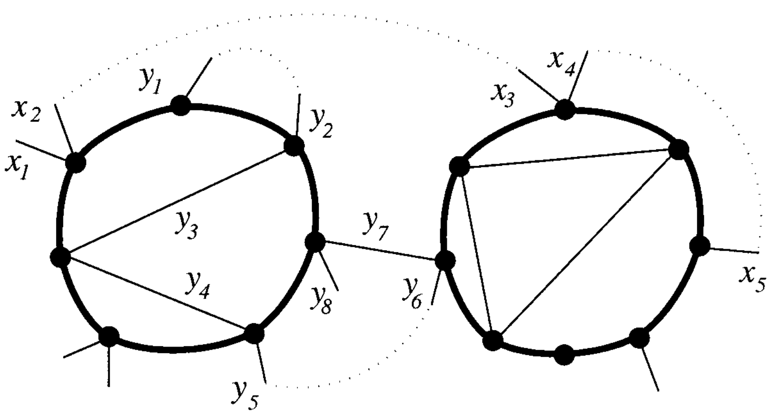

FIGURE 3. An example of σ-chains.

If no σ-chain in G is critical with respect to L, then G admits an L-edge-coloring λ such that, for each σ-constrained halfedge s, $\lambda(s) \neq \lambda(\sigma(s))$.

Proof. Since adjacent halfedges always receive distinct colors, we may assume that, for each halfedge s, either $s = \sigma(s)$, or s and $\sigma(s)$ are not adjacent. By Lemma 2.1(a), we may also assume that we have equalities in (1). Observe that all σ-chains are colorful with respect to L. By Lemma 2.3(a) and (c), there are no critical closed chains. Let q be the number of cycles in H. The proof is by induction on q. It runs as follows: (a) First, a general description is given how the proof will proceed. (b) Then seven cases are given that explain how to choose a first vertex along the cycle and an orientation along the cycle; in each case, it is shown that the lists of admissible colors are modified only in what turns out to be acceptable ways for the completion of (c), the general coloring step.

A. Outline of the Proof

We shall construct a coloring λ of G as follows. First, we choose a vertex $v_1 \in V(G)$. If C is the cycle of H that contains v_1, we also select an orientation of C. This selection determines the order $v_1, v_2, \ldots, v_n$ of vertices on C. Denote by s_i and s_i' the edges of E' incident with v_i. We shall color edges incident with C in the following order. First we color the edge $e_1 = v_1v_2 \in E(C)$. For $i = 2, \ldots, n$, having colored $e_{i-1} = v_{i-1}v_i$, we color s_i and s_i' if they are halfedges. If s_i (or s_i') is a proper edge whose other end v has not yet been considered, we change s_i (or s_i') into a halfedge incident with v and color it when the vertex v is encountered. Of course, we remove the colors used on e_{i-1} and e_i from its list. Hence, if one of s_i, s_i' is a halfedge and the other is a proper edge, the halfedge is colored and the proper edge is changed into a halfedge incident with its other end v. If s_i and s_i' are both changed into halfedges, then we also change σ so that $\sigma(s_i) = s_i'$ to assure that they will get distinct colors. After each such change, the lists of admissible colors satisfy (1) (except possibly the lists of s_1 and s_1'). Hence, all σ-chains (except possibly the one containing s_1 and s_1') are colorful. Finally, we color the edge $e_i = v_iv_{i+1}$, and then determine the admissible colors for s_i or s_i' if they have become halfedges. When $i = n$, the color of e_n should be distinct from $\lambda(e_1)$. After all these steps, we also color s_1 and s_1' if they are halfedges. Otherwise, we change them into halfedges as in general steps. We prove that no critical (open) σ-chains arise, and this enables us to color the rest of the graph by applying induction.

There are several things that we have to take care of during the coloring procedure. First of all, we have to avoid critical open σ-chains. This is achieved by an appropriate choice of v_1 and C at the beginning, and by a careful coloring at a general step. Additionally, we make sure by an appropriate selection of the orientation of C that, at the general step, we never encounter a vertex that is incident with two proper edges that are contained in an open σ-chain P such that one has 4 and the other has 5 admissible colors left. If this happened, in general it would not be possible to achieve that the new σ-chain is not critical. Exceptions to this rule are the cases when we know for sure that there is another proper edge in P that still has 5 admissible colors. Another important situation is when a (critical) closed σ-chain R turns into an open σ-chain P. Suppose that this happens at v_{i+1}. Again, we have to assure that P is not critical. Usually, this is achieved by assuring that there are two possible colors for the edge e_i. In such a case, by Lemma 2.3(d) R is critical for at most one of the colors. Situations when this approach does not apply are treated separately. That there are two admissible colors to color e_i is also needed when $i = n$. In that case, the color of e_n must be distinct from the color of e_1, and, if there are two colors available, this is an easy task. Additionally, two colors are also needed in case (6) below. The details for how to achieve that there are two available colors (and the exceptions to this) are explained at the particular steps of the coloring procedure. Let us remark that two colors are not needed for e_1.

B. How to Start

The selection of the vertex v_1, the orientation of C, and the choice of the color $x = \lambda(e_1)$ need special care. They are chosen according to the following cases, where each new case assumes that the assumptions of previous cases cannot be met at any of the vertices of G. If not stated otherwise, the orientation of C is arbitrary. A proper edge will be called a *chord* (of H) if it does not belong to $E(H)$. We denote by R the σ-chain containing s_1 and s_1'. Let us recall that, after choosing a partial L-edge-coloring and changing some proper edges into halfedges, R and other σ-chains may change. Now, in order to avoid that R becomes critical, we distinguish the following cases:

(1) *There is a vertex v_1 such that $L(e_1)\backslash(L(s_1) \cup L(s_1')) \neq \emptyset$.* Then we choose $x \in L(e_1)\backslash(L(s_1) \cup L(s_1'))$. This choice does not restrict admissible colors on s_1 and s_1' and does not introduce critical σ-chains.

(2) *There is a vertex v_1 such that s_1 and s_1' are chords with their other ends out of C.* If R is a closed chain, we select $x \in L(e_1)$ arbitrarily. By Lemma 2.3, R does not become critical, since either s_1 or s_1' has 5 admissible colors, or exactly these two edges on R have precisely 4 admissible colors. If R is open, then R may become critical. However, by Lemma 2.2(d), x can be selected such that this is not the case.

(3) *There is a vertex v_1 such that s_1' is a halfedge and s_1 is a chord with the other end not in C.* We distinguish two subcases:

  (3.1) If s_1' is σ-constrained and $\sigma(s_1')$ is incident with s_1, we select $x \in L(e_1)$ so that $L(s_1')\backslash\{x\} \cup L(\sigma(s_1')) \not\subseteq L(s_1)\backslash\{x\}$ and set $\lambda(e_1) = x$. It is easy to see that excluding (1) such an x always exists. Note that it may happen that s_1 retains only 2 and s_1 only 4 admissible colors, but, since R consists only of s_1, s_1', and $\sigma(s_1')$, this does not introduce any troubles in the general step.

(3.2) Otherwise, we choose $x \in L(e_1)\backslash L(s'_1)$. Observe that there are at least two such choices. If R is open, Lemma 2.1(a) implies that at least one choice is such that R does not become critical after removing x from $L(s_1)$.

(4) *There is a vertex v_1 such that s_1 is a chord of C and $s'_1 = v_1u$ is a chord with $u \notin V(C)$.* Notice that the edge s of R, incident with u and distinct from s'_1 is a chord (since (3) is excluded) that is not incident with C (since (2) is excluded). Therefore, we choose $x \in L(e_1)$ arbitrarily and observe that R will not be critical, since s with $|L(s)| = 5$ remains unchanged when coloring the edges incident with C.

(5) *There is a vertex v_1 such that s_1 is a chord of C and s'_1 is a halfedge.* Let $v_i \in V(C)$ be the other end of $s_1 = s'_i$. If s_i is a chord, then its other end is also a vertex of C, say v_j. In this case, by choosing the orientation of C, we can assure that $j < i$. Now, we take $x \in L(e_1)\backslash L(s'_1)$. If R is open, one of the choices for x is such that R does not become critical. The chosen orientation of C assures that we do not encounter a vertex incident with two proper edges having 4 and 5 admissible colors left, respectively.

(6) *There is a vertex v_1 such that s_1 and s'_1 are halfedges.* Let $R = f_1f_2\cdots f_k$ (possibly closed). Since $|L(s_1) \cup L(s'_1)| \geq 5$ and σ-free halfedges have exactly two admissible colors, we have $k \geq 4$ and we may assume that s'_1 is σ-constrained. Moreover, $f_1, \dots, f_k$ are halfedges, because, after excluding (3) and (5), no vertex of G is incident with both a halfedge and a proper edge. Additionally, exclusion of (1) gives $|L(f_i) \cup L(f_{i+1})| \geq 5, i = 1, 3, 5, \dots$. The last property implies that no σ-chain emerging from R during the coloring procedure is critical. On the other hand, after coloring e_1 by a color $x \in (L(s'_1)\backslash L(s_1)) \cap L(e_1)$, the edge s'_1 will be left with only two admissible colors. We have to assure that, when coloring $\sigma(s'_1)$, these two colors still remain admissible, i.e., $\sigma(s'_1)$ should be colored by a color from $L(\sigma(s'_1))\backslash(L(s'_1)\backslash\{x\})$. There is nothing to take care of if $\sigma(s'_1)$ is not incident with $V(C)$. So, suppose that $\sigma(s'_1) = s_i$, where $i > 2$. Then $L(s_i) \cap L(s'_i)$ is either empty or contains one color, say z. In the latter case, x can be selected such that $L(s'_1)\backslash\{x\} \neq L(s_i)\backslash\{z\}$. This assures that we will be able to color s_i by a color distinct from z. Additionally, in the general step, this case has to be treated separately when coloring s_i and s'_i. The remaining possibility is when $i > 2$ cannot be achieved. In that case, R is closed and $\sigma(s'_j) = s_{j+1}, j = 1, 2, \dots, n$. The set $(L(s'_1)\backslash L(s_1)) \cap L(e_1)$ contains at least two colors, say a and b. Put $\lambda(e_1) = a, \lambda(s'_1) = b$. This selection does not restrict available colors for s_1. We continue with the general step until reaching the edge e_n. Note that s_1 has become a σ-free halfedge and that we have two colors available for e_n. Any coloring of e_n leaves an available color for s_1. If we cannot color e_n, the available colors for e_n are precisely a and b. If $\lambda(s_2) \notin L(s'_1)$, then we can recolor s'_1 by a color distinct from a, b, and this partial coloring can be extended to a coloring of e_n and s_1. When coloring the edges incident with v_2, we can select $\lambda(s_2) \in L(s_2)\backslash L(s'_1)$

unless $L(s_2) = L(s'_1)$, which we assume henceforth. If $\lambda(s_2) \in L(e_1)$, we can swap the colors of e_1 and s_2 and extend the resulting partial coloring to a coloring of e_n and s_1. Otherwise, e_1 can be recolored by the color from $L(e_1)\backslash\{\lambda(e_2), \lambda(s'_2), a, b\}$, and again the resulting coloring has an extension to e_n and s_1:

(7) *In the remaining possibility, all σ-chains are closed and composed of chords of C only*. In particular, C is a Hamilton cycle of a connected component of G. We distinguish three subcases:

(7.1) *There exist an orientation of C and a vertex $v_1 \in V(C)$, where $s_1 = s_i, s'_i = s_k$, and $s'_1 = s_j, s'_j = s_l$ such that $k < i$ and $l < j$*. Then we select $x \in L(e_1)$ arbitrarily. The assumptions guarantee that at the general step we never encounter a vertex incident with two proper edges having 4 and 5 admissible colors left, respectively.

(7.2) *There is a vertex $v_1 \in V(C)$ such that $L(s_1) \neq L(s'_1)$*. Let $s_1 = s_i$ and $s'_1 = s_j$. We may assume that $L(s'_1) \neq L(e_1)$. Then we choose $x \in L(e_1)\backslash L(s'_1)$ arbitrarily. This selection introduces only one edge, namely s_1, with 4 admissible colors. If $i < j$, then, when coloring the edges incident with v_i, R will not be critical, since it will contain the proper edge s'_1 with 5 admissible colors. Suppose now that $i > j$. Let $s'_i = s_k$ and $s'_j = s_l$. If $k > i$ and $l < j$, then R will not be critical when reaching v_i, since it will contain the proper edge s'_k with 5 admissible colors (s'_k will still be a proper edge, since we have excluded (7.1)). Since we have already checked for (7.1), the only remaining possibility is that $k < i$ and $l > j$. In that case, when considering v_i, s_k is already a halfedge.

(7.3) *For every pair of incident chords s, s' of C, we have $L(s) = L(s')$*. Excluding (1), all chords have the same list of colors. Therefore, the chords of C can be colored by using only three colors altogether. Moreover, we may achieve that there is an edge $e \in E(C)$ such that only two distinct colors are used on the chords incident with e. Hence, each edge of C contains at least 2 admissible colors that are not used on the adjacent chords and e contains at least 3 such colors. This guarantees that $E(C)$ can also be colored.

C. General Coloring Step

Let $R_1 = R$ be the σ-chain that contains s_1 and s'_1. During the coloring procedure, this σ-chain changes. Let R_i be the σ-chain containing s_1, s'_1 after coloring $e_{i-1}, i = 2, \ldots, n+1$. By Lemma 2.3, every critical closed σ-chain contains a proper edge with 4 admissible colors. In our case, the only proper edges with 4 colors may be s_1 and s'_1. Hence, the only possible critical closed σ-chain is R_1 (and R_i later in the general step).

Let us now explain how the general step proceeds. Having colored edges up to e_{i-1} $(2 \leq i \leq n)$, we select colors for s_i, s'_i, and e_i as follows. Suppose first that s_i and s_i' are both chords. We shall color e_i by a color from $L(e_i)\backslash\{\lambda(e_{i-1})\}$, and then consider s_i, s_i' as σ-constrained halfedges with a triple of colors from $L_1 = L(s_i)\backslash\{\lambda(e_{i-1}), \lambda(e_i)\}$, and $L_2 = L(s_i')\backslash\{\lambda(e_{i-1}), \lambda(e_i)\}$, respectively. Suppose that $|L(s_i)| = |L(s_i')| = 5$. If s_i and s_i' are contained in a closed σ-chain, we color e_i arbitrarily. Let us observe that there are 4 candidates for $\lambda(e_i)$. Otherwise, suppose that after coloring e_i by y, the new chain becomes critical. Then, by (O1), there is a color x such that both subchains P_1 and P_2 (as defined in (O1)) are also critical. By Lemma 2.1(a), the colors $L_1\backslash\{x\}$ and $L_2\backslash\{x\}$ are uniquely determined. Therefore, by taking $\lambda(e_i)$ to be a color distinct from $x, y, \lambda(e_{i-1})$, the new chain is not critical. Note that, in each case, there are at least two appropriate colors for $\lambda(e_i)$. Suppose now that one of s_i, s_i', say s_i, has only 4 admissible colors. The initial choice of v_1 and of the orientation of C was made in such a way that we meet the pair of edges with 5 and 4 admissible colors only in cases (4) or (7.2). In those cases, $3 \leq i < n$. Moreover, R_{i+1} will not be critical, irrespective of the choice of $\lambda(e_i)$. Now, if $\lambda(e_{i-1}) \notin L(s_i)$, then we choose a color for e_i as above. Otherwise, there is a color $y \neq \lambda(e_{i-1})$ contained in $L(e_i)\backslash L(s_i)$. By choosing y for $\lambda(e_i)$, we get new colorful lists of admissible colors. Let us remark that, in this case, we do not need two candidates for $\lambda(e_i)$.

Suppose now that s_i and s_i' are both halfedges. If s_i and s_i' are contained in an open σ-chain, say $Q = f_1 \cdots f_k$, where $s_i = f_r, s_i' = f_{r+1}$, then we choose colors $d_1 \in L(s_i)\backslash\{\lambda(e_{i-1})\}$ and $d_2 \in L(s_i')\backslash\{\lambda(e_{i-1})\}$ by using Lemma 2.2(a). We change σ so that $\sigma(s_i) = s_i$ and $\sigma(s_i') = s_i', \sigma(f_{r-1}) = f_{r-1}$ and $\sigma(f_{r+2}) = f_{r+2}$ (if $r \geq 2$ and $r \leq k-2$, respectively), and remove d_1 and d_2 from $L(f_{r-1})$ and $L(f_{r+2})$, respectively. Lemma 2.2(a) guarantees that, after these changes, no critical σ-chains arise. Finally, we color e_i with a color from $L(e_i)\backslash\{\lambda(e_{i-1}), d_1, d_2\}$. Let us observe that there are at least two candidates for $\lambda(e_i)$. The same procedure is used if s_i and s_i' are contained in a noncritical closed σ-chain Q, except that (C1) is used instead of Lemma 2.2. Suppose now that Q is a closed σ-chain, which is critical at v_i. By Lemma 2.3(b), $L(s_i) = L(s_i')$. Since $Q = R_i$ contains an edge with four admissible colors, the vertex v_1 was not chosen according to (1). Therefore, $i \geq 3$ and, hence, there are two candidates for $\lambda(e_{i-1})$. By Lemma 2.3(d), Q may be critical for each of them only when the admissible colors on Q depend on the choice of $\lambda(e_{i-1})$. In that case, s_{i-1} and s_{i-1}' were chords with 5 admissible colors, and there were 4 candidates that may have been used for $\lambda(e_{i-1})$. One of them is not contained in $L(s_i)$. By Lemma 2.3(a), its selection gives the chain Q, which is not critical for $\lambda(e_{i-1})$.

A special treatment is needed if s_i and s_i' are the halfedges from (6). Recall that $i > 2$ and $s_i = \sigma(s_1')$. In that case, s_1' has only two admissible colors, say a and b, and we do not want to use them to color s_i. The choice of x in (6) guarantees that there is a color $c \in L(s_i)\backslash(L(s_i') \cup \{a, b\})$. Since $i > 1$, there are two choices for $\lambda(e_{i-1})$. Hence, we may assume that $\lambda(e_{i-1}) \neq c$. Now, we set $\lambda(s_i) = c$ and choose $\lambda(s_i')$ from $L(s_i')\backslash\{\lambda(e_{i-1})\}$. Finally, we color e_i by a color from $L(e_i)\backslash\{\lambda(e_{i-1}), \lambda(s_i), \lambda(s_i')\}$.

The last case is when s_i is a halfedge and s_i' is a proper edge. If s_i and $s_i' = v_i u$ are contained in an open σ-chain $Q = f_1 \cdots f_k$, where $s_i = f_r$ and (without loss of generality) $s_i' = f_{r+1}$, then we can choose a color $d \in L(s_i)\backslash\{\lambda(e_{i-1})\}$ by applying Lemma 2.2(b), where c equals $\lambda(e_{i-1})$. Now we set $\lambda(s_i) = d$ and change s_i' into a σ-free halfedge incident with u and with admissible colors $L' = L(s_i')\backslash\{\lambda(e_{i-1}), d\}$. If $r > 1$ (i.e., s_i is σ-constrained), then we also replace $L(f_{r-1})$ by $L(f_{r-1})\backslash\{d\}$ and change σ so that $\sigma(f_{r-1}) = f_{r-1}$. Lemma 2.2(b) guarantees that, after these changes, no critical σ-chains arise. Next, we select a color $b \in L(e_i)\backslash\{\lambda(e_{i-1}), d\}$ such that the σ-chain containing s_i' is not critical with respect to the remaining admissible colors (i.e., the colors of s_i' are $L'\backslash\{b\}$). If $|L'| > 2$, then Lemma 2.1(a) shows that there are at least two candidates for b. In particular, this is the case when $i = n$. If $|L'| = 2$, then $L(s_i')$ contains only 4 colors. Therefore, $s_i' \in \{s_1, s_1'\}$, and, hence, $i < n$. Moreover, the σ-chain R_{i+1} is open and we are not in the case of the previous paragraph. Therefore, we do not need two distinct colors for e_i.

Let us now consider the case when the σ-chain Q containing s_i and s_i' is closed. If Q is not critical at v_i for the color $\lambda(e_{i-1})$, then we color s_i and transform s_i' into a halfedge by applying (C2) or (C3). The color $\lambda(e_i)$ is then determined as above. If Q is critical, then $|L(s_i')| = 4$. In particular, $i > 2$. The previous steps of the coloring procedure assure that there are two candidates for $\lambda(e_{i-1})$. By Lemma 2.3(d), Q is not critical at v_i for both of them. So we may assume the above case. However, it is possible that the admissible colors on Q depend on the choice of $\lambda(e_{i-1})$. This can happen only when s_{i-1} and s_{i-1}' were chords. The choice of v_1 and the orientation of C guarantee that s_{i-1} and s_{i-1}' contained 5 colors each. In that case, however, one of the four possible colors in $L(e_{i-1})\backslash\{\lambda(e_{i-2})\}$ for $\lambda(e_{i-1})$ is not contained in $L(s_i)$, and we choose that color as $\lambda(e_{i-1})$. By Lemma 2.3(a), this guarantees that Q is not critical for $\lambda(e_{i-1})$.

The coloring procedure starts by coloring e_1 and then proceeds for $i = 2, 3, \ldots, n$ as described above. In the case when $i = n$, there are two available admissible colors for $e_i = e_n$. One of them is distinct from $\lambda(e_1)$, and it can be used to color e_n.

It remains to explain how to color s_1 and s_1'. Let $Q = R_{n+1}$ be the σ-chain containing s_1, s_1' after all edges of C have been colored. If Q is open, then it is not critical, and we apply Lemma 2.2(a)–(c) depending on whether s_1, s_1' are halfedges or proper edges.

If Q is closed, we distinguish two cases. If Q is not critical at v_1 for $\lambda(e_n)$, then we apply one of (C1), (C2), or (C3). Otherwise, the coloring procedure started in case (3), one of s_1, s_1', say s_1, is a chord with 4 admissible colors, and the other is a σ-constrained halfedge with 3 colors. The selection of $\lambda(e_1)$ in (3.1) also guarantees that $\sigma(s_1')$ is not incident with s_1. Let s be the edge of Q incident with $\sigma(s_1')$. If s was a halfedge at the beginning, then Q cannot be critical, since we have excluded (1). On the other hand, if s was a chord, then it is not incident with C, since we have excluded (2) and Q is still closed. Therefore, s still has 5 admissible colors, a contradiction. This shows that Q cannot be critical at v_1 for $\lambda(e_n)$ and, thus, completes the proof. ∎

The result mentioned in the title of this article immediately follows from Theorem 3.1.

Corollary 3.1. *Every graph G with $\Delta(G) \leq 4$ is 5-edge-choosable.*

Proof. We may assume that G has no halfedges. Then G is a subgraph of a 4-regular graph without halfedges. Therefore, we may assume that G is 4-regular. By Petersen's Theorem, G has a 2-factor H. Let L be a list assignment of G with 5 admissible colors for each edge. By Theorem 3.1, G has an L-edge-coloring. This completes the proof. ∎

Corollary 3.1 in particular verifies Vizing's List Chromatic Index Conjecture for graphs of class 2 of maximum degree 4. The proof of Theorem 3.1 also shows that there is a polynomial time algorithm for list edge-coloring graphs with maximum degree 4, if the lists contain at least 5 colors each.

4. APPLICATION TO SIMULTANEOUS EDGE-FACE COLORING OF PLANE GRAPHS

Corollary 3.1 implies that edges and faces of a 2-edge-connected plane graph of maximum degree $\Delta \leq 4$ can be simultaneously list-colored (so that incident or adjacent elements receive distinct colors), if each list contains at least $\Delta + 3$ colors. To see this, we first list-color the faces (by [13], lists of size 5 suffice). This leaves at least $\Delta + 1$ colors on each of the edges. In case $\Delta = 4$, Corollary 3.1 applies, while for $\Delta \in \{2, 3\}$, the proof is straightforward.

Proposition 4.1. *Let G be a 2-edge-connected plane graph of maximum degree $\Delta \leq 4$. Then the choice number for simultaneous edge and face coloring of G is at most $\Delta + 3$.*

A general result of this kind was recently proved (for usual colorings only) by Sanders and Zhao [11]. See also Waller [16].

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