Some Topological Methods in Graph Coloring Theory

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Abstract

Attempts to solve the famous Four Color Problem led to fruitful discoveries and rich coloring theories. In this talk, some old and some recent applications of tools from topology to graph coloring problems will be presented. In particular, the following subjects will be treated: The use of Euler's formula and local planarity conditions, Kempe equivalence, homotopy, winding number and its higher dimensional analogues.

Key words: Coloring, Euler's formula, Winding number, Homotopy, Surface, Kempe equivalence, Critical graph, Local planarity.

1 Introduction

The following propositions are well-known:

(a) If G is a plane graph such that all faces of G are of even size, then G is bipartite, and hence 2-colorable.
(b) If G is a triangulation of the disk such that every vertex of G in the interior of the disk is of even degree, then G is 3-colorable.

A common feature of these results is that the local structure (being locally 2- or 3-colorable, respectively) implies a global structure. It turns out that these results are true because the plane (or the disk) is *simply connected*, i.e., every simple closed curve is contractible to a point. This suggests to look on

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the problem of colorability through the eyes of homotopy, and yields a deeper insight into possible global obstructions to colorability.

The above example is an illustration of a topological tool applied to graph coloring theory. We shall overview some old and some recent results on graph coloring whose proofs use tools from topology. The following sections give excerpts of results related to particular topics covered in the talk.

2 Euler's formula, discharging, and local planarity

For $g \geq 0$, let $H(g) = \left\lfloor \frac{1}{2}(7 + \sqrt{1 + 24g}) \right\rfloor$. Heawood proved the following analogue of the Four Color Theorem for general surfaces: Let S be a surface with Euler genus $g > 0$ and let G be a graph embedded in S. Then $\chi(G) \leq H(g)$. Dirac (and Albertson and Hutchinson) proved: If G is a graph embedded in the surface of Euler genus $g \geq 1$, then $\chi(G) < H(g)$ unless G contains $K_{H(g)}$ as a subgraph. Recent extensions of Dirac's theorem have been obtained by Škrekovski [13] and by Böhme, Mohar, and Stiebitz [2].

Thomassen [14] proved that for each surface S, there are only finitely many 6-critical graphs that can be embedded in S. The complete list of 6-critical graphs is known only for the sphere, the projective plane, and the torus. It can be shown that there are infinitely many k-critical graphs on a surface S if and only if $k \in \{3, 4, 5\}$. This implies that the problem of k-colorability of graphs on a fixed surface is polynomially decidable if $k \geq 5$. The problem of 3-coloring graphs on any surface is **NP**-complete. It is still open if 4-colorability on a fixed surface is polynomially decidable.

We shall present a neat topological proof, using homotopy, that shows that 4-colorability is polynomially decidable in the class of Eulerian triangulations of the projective plane [10]. Although this is a very simple result, no direct proofs are known.

3 Winding number and Kempe equivalence

The *edge-width* $\mathbf{ew}(G)$ of a graph embedded in a nonsimply connected surface is defined as the length of a shortest noncontractible cycle in G. Hutchinson [6] proved that if G is embedded in an orientable surface with large edge-width such that all facial walks have even length, then G is 3-colorable. This extends proposition (a) from the introduction. The condition on large width (depending on the genus) is necessary since there are quadrangulations of surfaces whose underlying graph has arbitrarily large edge-width and arbitrarily large chromatic number. The result of Hutchinson does not extend to nonorientable surfaces (Youngs [15], Klavžar and Mohar [7]). Mohar and Seymour [11] recently obtained a complete characterization of those locally bipartite graphs on surfaces whose chromatic number is more than 3:

Theorem 3.1 *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that the following holds. Let G be a graph embedded in a surface of Euler genus g with edge-width $\geq f(g)$ and all faces of even size. Then G is 4-colorable and:*

(a) *If every 4-cycle is facial and there is face of size > 4, then G is 3-colorable.*
(b) *If G is a quadrangulation, then G is not 3-colorable if and only if there exist disjoint surface separating cycles $C_1, \ldots, C_g$ such that, after cutting along $C_1, \ldots, C_g$, we obtain a sphere with g holes and g Möbius strips, an odd number of which is nonbipartite.*

Let c be a fixed 3-coloring of the graph G. If $W = v_1 v_2 \ldots v_k v_1$ is a closed walk in G, then the coloring of $V(W)$ can be viewed as a mapping onto the 3-cycle C_3 and we may speak of the *winding number* $w_c(W)$. The main tools used in the proof of Theorem 3.1 are some results about graph minors, combined with the follwing property of the winding number. Let G be a quadrangulation of some surface and let c be a 3-coloring of G. If W and W' are homotopic closed walks of G, then $w_c(W) = w_c(W')$.

There exist generalizations of the winding number to 4-colorings, and even to graph homomorphisms to "spheres" and graphs of spherical complexes, e.g., homomorphisms to the graph of the 3-cube.

The simplest extension of the winding number from 3-colorings to 4-colorings, leads to homology. The modulo 2 homology is related to the notion of the *parity* of a 4-coloring which was investigated in the fifties by Tutte. The parity is the only known invariant of colorings which is preserved under the Kempe equivalence. A simple but far from straightforward is the following result whose proof again refers to simple connectivity of the plane.

Theorem 3.2 (Mohar, 2000) *Let G be a 3-colorable planar graph. Then all 4-colorings of G are Kempe equivalent.*

4 Other

Fisk [4] proved that there is a nice hierarchy between the following four variations of 4-colorings of triangulated surfaces: 4-coloring, the dual edge-coloring (nowhere-zero 4-flow), the Heawood coloring, and the local 4-coloring. They coincide in the sphere. However, on a general surface S, one type of a coloring implies another only in the order of this hierarchy. Obstructions for the converse can be described by means of certain easily described homomorphism, $\varphi : \pi_1(S) \to S_3$, of the fundamental group $\pi_1(S)$ of S into the symmetric group S_3. A series of papers by Fisk [3–5] contains many further results of the same flavor.

Finally, one cannot speak about topological tools in graph theory without mentioning Lovász' proof where he determines the chromatic numbers of Kneser graphs [9,1].

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