Face covers and the genus problem for apex graphs[1]

Bojan Mohar*

Department of Mathematics,
University of Ljubljana,
1111 Ljubljana, Slovenia
E-mail: bojan.mohar@uni-lj.si

A graph G is an *apex graph* if it contains a vertex w such that $G - w$ is a planar graph. It is easy to see that the genus $\mathbf{g}(G)$ of the apex graph G is bounded above by $\tau - 1$, where τ is the minimum face cover of the neighbors of w, taken over all planar embeddings of $G - w$. The main result of this paper is the linear lower bound $\mathbf{g}(G) \geq \tau/160$ (if $G - w$ is 3-connected and $\tau > 1$). It is also proved that the minimum face cover problem is **NP**-hard for planar triangulations and that the minimum vertex cover is **NP**-hard for 2-connected cubic planar graphs. Finally, it is shown that computing the genus of apex graphs is **NP**-hard.

Key Words: planar graph, apex graph, genus, NP-complete, face cover, vertex cover

1. INTRODUCTION

The *genus* of a graph G, denoted by $\mathbf{g}(G)$, is the minimum genus of an orientable surface in which the graph can be embedded. This parameter has been extensively studied in the literature (cf., e.g. [6, 13]). The original motivation to study the genus of graphs was the Heawood problem which concerns the maximum chromatic number of graphs embeddable in a fixed surface. The solution of the Heawood problem turned out to be equivalent to determining the genus of complete graphs (cf. [10]). Apart from many results concerning rather specific families of graphs, there are no general results available that would enable us to efficiently determine (or lower bound) genera of general graphs. This mysterious lack of available tools was explained when Thomassen [11] proved that the genus problem is **NP**-

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complete. (The *genus problem* asks if for a given graph G and an integer g, $\mathbf{g}(G) \leq g$.)

The main motivation for this paper is the following question: "How does the addition of a new vertex to the given graph G influence the genus." Such influence is most transparent when G is a planar graph. The second motivation is the study of the genus parameter in minor closed families of graphs. By the Robertson-Seymour theory of graph minors, generic cases of minor closed families are: (a) graphs for which the deletion of a bounded number of vertices yields a graph embeddable in a fixed surface, (b) graphs embeddable in a fixed nonorientable surface, and (c) graphs of bounded tree-width. Some recent studies show that the genus problem may be polynomially solvable for (b) and (c). On the other hand, the results in Section 5 show, rather surprisingly, that (a) is **NP**-hard even in the simplest case of apex graphs.

A graph G is an *apex graph* if it contains a vertex w (called an *apex* of G) such that $G - w$ is a planar graph. Although apex graphs seem to be close to planar graphs, it is easy to see that their genus can be arbitrarily large. A general result about the genus of apex graphs, presented here as Theorem 3.3, was known to the author for several years. However, the general characterization of obstructions to small genus of apex graphs (by means of nonexistence of small face covers as presented in Section 3) became apparent only recently. It is easy to see that the genus $\mathbf{g}(G)$ of the apex graph G is bounded above by $\tau - 1$, where τ is the minimum face cover of the neighbors of w, taken over all planar embeddings of $G - w$. The main result of this paper is Theorem 3.1 which yields linear lower bound $\mathbf{g}(G) \geq \tau/160$ (proved for the case when $G - w$ is 3-connected and $\tau > 1$). Observe that $\mathbf{g}(G) = 0$ if $\tau = 1$.

From the computational complexity point of view, the genus of graphs was one of the toughest open cases from the list of Garey and Johnson [3]. Thomassen proved in 1989 [11] (and has later found a simpler proof [12]) that the genus problem is **NP**-complete. The main result of Section 5 shows that the genus problem remains **NP**-complete even if we restrict ourselves to apex graphs. This solves a problem raised by Neil Robertson in 1988 (private communication). As a side result we also prove that vertex cover and the (maximum) independent set problem are **NP**-complete for planar cubic graphs. Garey, Johnson, and Stockmeyer [4] proved that these problems are **NP**-complete for planar graphs of maximum degree 6. Our results resolve their question [4] how much the degree condition can be narrowed so that the problem remains **NP**-complete (if **P**$\neq$**NP**).

Our treatment of graph embeddings follows essentially [9]. An *embedding* of a connected graph G is a pair $\Pi = (\pi, \lambda)$ where $\pi = \{\pi_v \mid v \in V(G)\}$ is a collection of *local clockwise rotations*, i.e., π_v is a cyclic permutation of the edges incident with v ($v \in V(G)$), and $\lambda : E(G) \to \{+1, -1\}$ is a *signature*.

The local rotation π_v describes the cyclic clockwise order of edges incident with v on the surface, and the signature $\lambda(uv)$ of the edge uv is positive if and only if the local rotations π_u and π_v both correspond to the clockwise (or both to anticlockwise) rotations when traversing the edge uv on the surface. An embedding of the graph G is *orientable* if every cycle of G has an even number of edges with negative signature.

The embedding Π determines a set of Π-*facial walks*. If a Π-facial walk is a cycle, it is also called a Π-*facial cycle*. Suppose that Π is an orientable embedding. If f is the number of Π-facial walks, then the number $\mathbf{g}(G,\Pi) = 1 - \frac{1}{2}(|V(G)| - |E(G)| + f)$ is called the *genus* of Π. (The underlying surface of the embedding Π is obtained by pasting discs along the Π-facial walks in G. Then $\mathbf{g}(G,\Pi)$ is the genus of that surface, by Euler's formula.) The minimum of $\mathbf{g}(G,\Pi)$ taken over all orientable embeddings of G is the *genus* of the graph G and is denoted by $\mathbf{g}(G)$.

If G is a Π-embedded graph and H is a subgraph of G, then Π *induces* an embedding Π' of H which we call the *induced embedding* of H, or the *restriction* of Π to H. Let us observe that $\mathbf{g}(H,\Pi') \le \mathbf{g}(G,\Pi)$, possibly with strict inequality. We refer to [9] for further definitions and basic properties of embeddings which are used in the sequel.

2. BOUQUETS OF CYCLES AND EMBEDDINGS

In this section we prove two auxiliary results (which may be of independent interest) that are used in the proof of Theorem 3.1.

Let $\mathcal{F}$ be a collection of cycles of a graph G. Suppose that there is a vertex $x \in V(G)$ such that the intersection of any two distinct cycles in $\mathcal{F}$ is either x or an edge incident with x. Then we say that $\mathcal{F}$ is a *bouquet*. The union $\bigcup\{C \cap C' \mid C, C' \in \mathcal{F}, C \ne C'\}$ is called the *center* of $\mathcal{F}$. More generally, a collection of cycles $\mathcal{F}$ is called a *collection of bouquets* if it is the union of bouquets such that any two cycles in distinct bouquets are disjoint.

Lemma 2.1. *Let G be a graph embedded in an orientable surface of genus g and let $w \in V(G)$. Let $\mathcal{F}$ be a collection of bouquets in $G - w$ such that each $C \in \mathcal{F}$ is noncontractible and contains a vertex adjacent to w which is not in the center of the bouquet containing C. Then $|\mathcal{F}| \le 4g$.*

Proof. Let $C_1, \dots, C_k$ ($k = |\mathcal{F}|$) be the cycles in $\mathcal{F}$, and let $\mathcal{F}_1, \dots, \mathcal{F}_p$ be the bouquets in $\mathcal{F}$. We may assume that the center of each bouquet $\mathcal{F}_j$ is either empty (if $|\mathcal{F}_j| = 1$) or a single vertex, $1 \le j \le p$. (If not, we contract the edges in the center of $\mathcal{F}_j$.) Let $k_j = |\mathcal{F}_j|$, $1 \le j \le p$. Cut the surface of the embedded graph along the cycles $C_1, \dots, C_k$, and let $\Sigma_1, \dots, \Sigma_r$ be the connected components (surfaces with boundary) resulting in this way. It is easy to see that the sum of the Euler characteristics of these components is equal to

$$\sum_{i=1}^{r} \chi(\Sigma_i) = \chi(\Sigma) + k - p = 2 - 2g + k - p. \tag{1}$$

The component (say Σ_1) which contains w has at least p boundary components since w is adjacent to a vertex in each bouquet. Each cycle in $\mathcal{F}$ gives rise to two arcs on the boundaries of $\Sigma_1, \ldots, \Sigma_r$. At least k of these arcs are in Σ_1 since every cycle in $\mathcal{F}$ contains a neighbor of w distinct from the centre of the corresponding bouquet. Suppose that some component Σ_i $(2 \le i \le r)$ is a disk. Since none of the cycles in $\mathcal{F}$ is contractible, the boundary of such a disk contains at least two of the arcs. Therefore, there are at most $k/2$ disk components. They have Euler characteristic 1. All other components have nonpositive Euler characteristic. Since Σ_1 has p or more boundary components, $\chi(\Sigma_1) \le 2 - p$. These properties and (1) imply that

$$2 + k - p - 2g = \sum_{i=1}^{r} \chi(\Sigma_i) \le \chi(\Sigma_1) + \frac{k}{2} \le 2 - p + \frac{k}{2}$$

and hence $k \le 4g$. ∎

It is not hard to embed a bouquet of 4 cycles in the torus so that the conditions of Lemma 2.1 are satisfied. By taking g such bouquets on the surface of genus g, we see that the bound of Lemma 2.1 is best possible.

Lemma 2.2. *Let G be a graph embedded in an orientable surface of genus g and let $w \in V(G)$. Let $\mathcal{F}$ be a collection of bouquets in $G - w$ such that each $C \in \mathcal{F}$ is noncontractible and contains a vertex adjacent to w which is not in the center of the bouquet containing C. Suppose, moreover, that any two cycles in $\mathcal{F}$ are disjoint. Then $|\mathcal{F}| \le 2g$.*

Proof. A proof similar to the proof of Lemma 2.1 gives a better bound in this case since there are no disk components Σ_i $(1 \le i \le p)$. ∎

Next we prove that any collection of facial cycles of a 3-connected planar graph contains a large collection of bouquets.

Lemma 2.3. *Let G be a 3-connected planar graph and let $\mathcal{F}$ be a collection of facial cycles of G. Then $\mathcal{F}$ contains a subset $\mathcal{F}_0$ which is a collection of bouquets in which no two cycles intersect more than in a vertex and such that $|\mathcal{F}_0| \ge \frac{1}{40}|\mathcal{F}|$.*

Proof. Let us observe that any two distinct facial cycles of G are either disjoint, or they intersect in a vertex or an edge. By applying the 4-color theorem it is easy to see that there is a subset $\mathcal{F}'$ of $\mathcal{F}$ such that $|\mathcal{F}'| \geq |\mathcal{F}|/4$ and such that no two cycles in $\mathcal{F}'$ have an edge in common. Then, it suffices to prove that there is a vertex $x \in V(G)$ and there is a bouquet $\mathcal{F}_0 = \{C_1, \ldots, C_r\} \subseteq \mathcal{F}'$ containing r cycles ($r \geq 1$) such that $x \in V(C_1 \cap \cdots \cap C_r)$ and such that the number of cycles in $\mathcal{F}'$ which intersect $C_1 \cup \cdots \cup C_r$ (not counting $C_1, \ldots, C_r$) is at most $9r$. To prove this claim we may assume that $\cup\mathcal{F}'$ is connected. We may also assume that $|\mathcal{F}'| \geq 10$.

Let H be a bipartite graph obtained as follows. Its vertex set is $\mathcal{F}' \cup U$ where U is the set of all vertices of G in which two or more of the cycles from $\mathcal{F}'$ intersect. There is an edge $Cu \in V(H)$ if and only if $C \in \mathcal{F}'$ and $u \in U$ are incident. Then H is connected and has a natural embedding in the plane obtained from the embedding of G by putting each vertex in $\mathcal{F}'$ in the corresponding face of G. Since no two cycles in $\mathcal{F}'$ have more than a vertex in common, the girth of H is at least 6. Let $f = |\mathcal{F}'|$, $w = |U|$, $e = |E(H)|$, and let q be the number of facial walks of H. By Euler's formula, $f + w - e + q = 2$. The girth condition and the assumption that $f \geq 3$ imply that $q \leq e/3$, and so

$$3f + 3w - 2e \geq 6. \tag{2}$$

If $u \in U$ has degree j in H, then u is called a *j-vertex*. If $C \in \mathcal{F}'$ has degree j in H, then it is called a *j-face*. For $j \geq 0$, let n_j be the number of j-vertices and let f_j be the number of j-faces. Then $n = \sum_j n_j$ and $f = \sum_j f_j$. Since U and $\mathcal{F}'$ form a bipartition of H, we also have $e = \sum_j jn_j = \sum_j jf_j$. By putting these relations into (2), the following inequality results:

$$\sum_j \Big(3 - \frac{3}{2}\,j\Big) n_j + \sum_j \Big(3 - \frac{1}{2}\,j\Big) f_j \geq 6. \tag{3}$$

To prove the existence of $\mathcal{F}_0$ we will apply the discharging method on (3). First, we define the charge of each j-vertex to be $3 - \frac{3}{2}\,j$ and the charge of each j-face to be $3 - \frac{1}{2}\,j$. By (3), the sum of charges of all vertices of H is positive. Now, we redistribute the charges in two steps according to the following rules:

Step 1. If $1 \leq j \leq 5$ and $C \in \mathcal{F}'$ is a j-face, then send charge $1/2$ from C to each adjacent 3-vertex, send $3/4$ to each adjacent 4-vertex or 5-vertex, and send charge 1 to each 6-vertex adjacent to C.

Step 2. Suppose that $1 \leq j \leq 5$ and that C is a j-face which still has positive charge $c > 0$ after step 1. If $j = 1$ and the neighbor of C is an i-vertex u where $i \geq 11$, then send the charge c from C to u. If $2 \leq j \leq 5$ and C has $t \geq 1$ neighbors of degree at least 7 in H, then send equal charges c/t to each such neighbor.

After the charge redistribution, the total charge remains the same as before. Therefore, there is a vertex of H with positive charge.

Suppose first that a j-face C has positive charge. Recall that the initial charge of C was $3 - j/2$. Since in steps 1 and 2 the charge is always sent from $\mathcal{F}'$ to U, the current charge of C cannot be larger than initially. Hence $j \leq 5$. If $j = 5$, then C has sent no charge to its neighbors. In particular, its neighbors are all 2-vertices and hence C intersects only 5 other cycles in $\mathcal{F}'$. If $j = 4$, then C is adjacent to at most one vertex of degree at least 3 in H, and if there is one, its degree is at most 5. Hence, C intersects at most 7 other faces in $\mathcal{F}'$. Similarly we see that in the case when $j = 3$, C intersects at most 7 other faces in $\mathcal{F}'$. If $j = 2$, C is not adjacent to an i-vertex where $i \geq 7$, and it is not adjacent to two 6-vertices. Therefore, it intersects at most 9 other faces in $\mathcal{F}'$. Similarly if $j = 1$. In each of these cases we may take $\mathcal{F}_0 = \{C\}$.

Suppose now that a j-vertex u has positive charge. Since there are no 1-vertices, the initial charge $3 - 3j/2$ of u was not positive. No charge is ever sent to a 2-vertex. Therefore, $j \geq 3$. Any i-vertex with $3 \leq i \leq 6$ receives additional charge only in step 1. Since they have precisely i neighbors in H, it is easy to see that their charge cannot become positive. Hence $j \geq 7$. Let us recall that charge $c > 1$ may be sent from an i-face C to u only in step 2 and precisely in the following five cases:

(i) C is a 3-face which is adjacent to two 2-vertices and to u. In this case $c = 1.5$.

(ii) C is a 2-face which is adjacent to a 2-vertex and to u. In this case $c = 2$.

(iii) C is a 2-face which is adjacent to a 3-vertex and to u. In this case $c = 1.5$.

(iv) C is a 2-face which is adjacent to a 4-vertex or 5-vertex and to u. In this case $c = 1.25$.

(v) C is a 1-face and $j \geq 11$. In this case $c = 2.5$.

If a charge greater than one is sent to u only once, then it is easy to see that $j = 7$ and that case (ii) was applied. Then the corresponding 2-face C intersects only 7 other faces in $\mathcal{F}'$. Thus we may assume that (at least) two of the cases (i)–(v) have been applied to u. Consider the bouquet consisting of the corresponding two cycles in $\mathcal{F}'$. This bouquet intersects at most $(j - 2) + 8 = j + 6$ other faces in $\mathcal{F}'$. Hence we are done if $j \leq 12$.

Suppose now that $j \geq 13$. The initial charge at u was $3-3j/2 \leq -j-\frac{3}{2}\frac{j}{6}$. This implies that cases (i)–(v) apply to u more than $j/6$ times. Let $\mathcal{F}_0$ be the bouquet consisting of the corresponding $r = \lceil j/6 \rceil$ cycles in $\mathcal{F}'$. Each of the cycles in $\mathcal{F}_0$ intersects at most 4 cycles of $\mathcal{F}'$ in addition to those which contain u. Therefore $\mathcal{F}_0$ intersects at most $j-r+4r = j+3r \leq 9r$ other cycles in $\mathcal{F}'$. ∎

3. EMBEDDINGS OF APEX GRAPHS AND FACE COVERS

Let Π be an embedding of the apex graph G, and let Π_0 be the induced embedding of $G_0 = G - w$, where w is the apex of G. Denote by W the set of all neighbors of w in G. A Π_0-*face cover* (or simply a *face cover*) of W is a set of Π_0-facial walks such that each vertex in W is contained in at least one of them. Denote by $\tau(W, G_0, \Pi_0)$ the smallest cardinality of a face cover of W. Minimum face covers in plane graphs have been studied in [1, 2].

Minimum genus embeddings of graphs and minimum face covers are related as shown below.

LEMMA 3.1. *Let G be a graph and $w \in V(G)$ such that the graph $G' = G - w$ is connected. Let W be the set of neighbors of w in G, and let Π' be an orientable embedding of G'. The minimum genus of all orientable embeddings of G, whose restriction to G' is Π', is equal to $\mathbf{g}(G', \Pi') + \tau(W, G', \Pi') - 1$.*

Proof. Let $\tau = \tau(W, G', \Pi')$. Let $F_0, \ldots, F_{\tau-1}$ be a minimum Π'-face cover of W. Add a vertex v_i in F_i and join it to all vertices of W in F_i $(0 \leq i < \tau)$. By adding the edges $v_0v_1, \ldots, v_0v_{\tau-1}$ we get a graph $\tilde{G}$ which contains G as a minor. Since adding an edge increases the genus by at most one, $\tilde{G}$ (and hence also G) has an embedding whose restriction to G' is Π' and whose genus is at most $\mathbf{g}(G', \Pi') + \tau - 1$.

Conversely, let Π be an embedding of G such that the induced embedding of G' is Π'. Let $F_1, \ldots, F_r$ be the Π'-facial walks that are not Π-facial. Then $F_1, \ldots, F_r$ is a Π'-face cover of W, and hence $r \geq \tau$. For each F_i, let $e_i = v_iw$ be an edge of G incident with w such that, in the local clockwise rotation around v_i, the edge e_i is placed between the edges of G' which are consecutive on F_i (i.e., F_i is not facial in the induced embedding of $G' + e_i$). Let $G_1 = G' + e_1 + \cdots + e_r \subseteq G$. It is easy to prove by induction on r that the genus of the induced embedding of G_1 is equal to $\mathbf{g}(G', \Pi') + r - 1$. This completes the proof. ∎

Lemma 3.1 can also be formulated for the case when $G - w$ is not connected. This implies:

PROPOSITION 3.1. *Let G be an apex graph with apex w, and let $G_1, \dots, G_k$ be the connected components of $G - w$. Then*

$$\mathbf{g}(G) = \sum_{i=1}^{k} \min\Big\{\mathbf{g}(G_i, \Pi_i) + \tau(W_i, G_i, \Pi_i) - 1\Big\}$$

where W_i is the set of neighbors of w in G_i, and the minimum runs over all orientable embeddings Π_i of G_i, $i = 1, \dots, k$.

Now we are prepared for our main result.

THEOREM 3.1. *Let G be an apex graph with apex w. Suppose that $G_0 = G - w$ is 3-connected. Denote by W the set of neighbors of w in G. Let Π_0 be the plane embedding of G_0, and let $\tau = \tau(W, G_0, \Pi_0)$. If $\tau \geq 2$, then*

$$\frac{1}{160}\tau \leq \mathbf{g}(G) \leq \tau - 1. \tag{4}$$

Proof. The upper bound $\mathbf{g}(G) \leq \tau - 1$ is clear by Lemma 3.1.

To prove the lower bound, consider an arbitrary orientable embedding Π of G. Let $\mathcal{F}'$ be the set of Π_0-facial cycles which are not Π-facial. For every $v \in W$, $vw \in E(G)\backslash E(G_0)$. This implies that $\mathcal{F}'$ is a Π_0-face cover of W. Let $\mathcal{F} \subseteq \mathcal{F}'$ be a minimal Π_0-face cover of W contained in $\mathcal{F}'$. Then $|\mathcal{F}| \geq \tau$. Facial cycles of 3-connected graphs in the plane are induced and nonseparating (cf., e.g., [9]). Therefore, every $C \in \mathcal{F}$ is also induced in G. Since $\tau > 1$, w has a neighbor outside C. This implies that C is also nonseparating in G. Consequently, C is Π-noncontractible.

By Lemma 2.3, $\mathcal{F}$ contains a collection of bouquets $\mathcal{F}_0$ which has at least $\tau/40$ members. Since no proper subset of $\mathcal{F}$ is a face cover of W, each cycle $C \in \mathcal{F}_0$ contains a neighbor of w which is not in the centre of the bouquet containing C. By Lemma 2.1, $\mathbf{g}(G, \Pi) \geq |\mathcal{F}_0|/4 \geq \tau/160$. This completes the proof. ∎

The lower bound in (4) can be improved (with a more complicated proof) but the resulting bound is still far from the worst case examples that we can construct.

At the end of Section 5 we prove that there are apex graphs G with 3-connected planar subgraph $G_0 = G - w$ such that $\mathbf{g}(G) = \frac{1}{2}\tau(W, G_0, \Pi_0)$, where Π_0 is the plane embedding of G_0.

If the vertices in W are "far apart", the bound of Theorem 3.1 can be greatly improved (possibly even to an exact result) as shown below. The distance that will be used is the following. Let G_0 be a plane graph and u, v distinct vertices of G_0. We say that u, v are at *face distance* at least k if there are no facial walks $F_1, \ldots, F_{k-1}$ such that $u \in V(F_1)$, $v \in V(F_{k-1})$, and $F_i \cap F_{i+1} \neq \emptyset$ for $i = 1, \ldots, k-2$. For example, $\tau(W, G_0, \Pi_0) = |W|$ if and only if any two vertices in W are at face distance at least 2.

THEOREM 3.2. *Let G be an apex graph with apex w. Suppose that $G_0 = G - w$ is 3-connected. Denote by W the neighbors of w in G. Let Π_0 be the plane embedding of G_0, and suppose that any two vertices in W are at face distance at least* 3. *Then*

$$\frac{1}{2}|W| \leq \mathbf{g}(G) \leq |W| - 1. \tag{5}$$

Proof. Face distance at least 3 implies that any two Π_0-facial cycles which contain distinct vertices of W are disjoint. Now, we follow the proof of Theorem 3.1 and observe that $|\mathcal{F}| = |W|$ and that $\mathcal{F}_0 = \mathcal{F}$. This saves a factor of 40. Moreover, applying Lemma 2.2 instead of Lemma 2.1 saves another factor of 2. This implies (5). ▮

THEOREM 3.3. *Let G be an apex graph with apex w. Suppose that $G_0 = G - w$ is 3-connected. Denote by W the neighbors of w in G. Let Π_0 be the plane embedding of G_0, and suppose that any two vertices in W are at face distance at least* 4. *Then*

$$\mathbf{g}(G) = |W| - 1. \tag{6}$$

Proof. The claim is obvious if $|W| = 1$, so we may assume that $|W| \geq 2$. Let Π be a minimum genus embedding of G, and let Π' be the induced embedding of G_0. For each $v \in W$ there is a Π_0-facial cycle C_v which is not Π-facial. Since C_v is an induced nonseparating cycle of G_0 and since $C_v \cap W = \{v\}$ and $|W| \geq 2$, C_v is also induced and nonseparating cycle of G. Therefore, C_v is not surface separating in the embedding Π. Similarly we see that for any proper subset $W' \subset W$, the collection of disjoint cycles $\{C_v \mid v \in W'\}$ is an induced and nonseparating subgraph of G. (However, the union of all cycles C_v $(v \in W)$ separates w from the rest of the graph.) This implies (cf. [8, Lemma 2.4]) that $\mathbf{g}(G) = \mathbf{g}(G, \Pi) \geq |W| - 1$. Clearly, $\mathbf{g}(G, \Pi) \leq |W| - 1$. This completes the proof. ▮

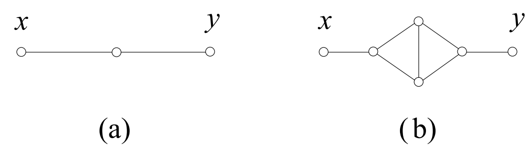

FIG. 1. Replacing vertices of degree 2 in G_1

4. COMPUTING MINIMUM FACE COVERS

Bienstock and Monma [2] proved that finding minimum face covers of planar graphs is **NP**-hard. Their reduction is from planar vertex cover and implies that the problem remains **NP**-hard also for instances whose maximum face size is at most 6. (The authors of [2] claim that their proof also works for triangulations. However, the premises used for such a case are based on a wrong interpretation of results of Garey, Johnson, and Stockmeyer [4].) In this section we prove that finding a minimum face cover is **NP**-hard also for 3-connected instances in which each face is of size 3.

THEOREM 4.1. (a) *The minimum vertex cover and the maximum independent set problem are* **NP***-hard also when restricted to 2-connected cubic planar graphs.*

(b) *The minimum face cover problem is* **NP***-hard also when restricted to planar triangulations.*

Proof. Kratochvil [7] proved that the following problem (called *planar 3-satisfiability*) is **NP**-complete. Let X be a set of logical variables and $\mathcal{C}$ a set of clauses, each clause containing exactly 3 distinct variables of X. Let H be the graph whose vertices are the elements of X and $\mathcal{C}$, and whose edges are xc for every $x \in X$ and $c \in \mathcal{C}$ such that x or $\neg x$ is in c. With the additional requirement that H is planar and 3-connected, the problem of deciding whether such a set $\mathcal{C}$ of clauses is satisfiable is NP-complete.

Suppose that we are given an instance $X, \mathcal{C}$ of planar 3-satisfiability. Let H be the corresponding planar graph. Define the graph G_1 which is obtained from H by replacing each clause vertex $c \in \mathcal{C}$ by a triangle T_c and replacing each vertex $x \in X$ by a cycle C_x of length $2k$ where k is the number of clauses that contain x or $\neg x$ (i.e., k is the degree of x in H). Label the vertices of T_c by the three variables occurring in c, and label the vertices of C_x respectively by $c_1^1, c_1^2, c_2^1, c_2^2, \ldots, c_k^1, c_k^2$ where $c_1, c_2, \ldots, c_k$ are the clauses in which x or $\neg x$ appears, enumerated in the cyclic order determined by the local clockwise rotation around x in the plane embedding of H. Finally, if x (resp. $\neg x$) occurs in the clause c_i, add the edge joining the vertex of T_{c_i} corresponding to x with the vertex c_i^1 (resp. c_i^2) of C_x. If $\mathcal{C}$ contains p clauses, then H has $3p$ edges, and the constructed graph G_1 has $12p$ edges. Clearly, G_1 is a 2-connected planar graph with $3p$ vertices of degrees 2 and $6p$ vertices of degree 3.

Every vertex cover of G_1 contains at least 2 vertices of each T_c and at least k vertices of each cycle C_x of length $2k$. Therefore, it has at least $2p + 3p = 5p$ vertices. It is easy to see that $\mathcal{C}$ is satisfiable if and only if G_1 has a vertex cover of cardinality precisely $5p$, or, equivalently, has an independent set of size $4p$. Replace each vertex v of degree 2 in G_1 by the 4-vertex graph as shown in Figure 1. Then the resulting cubic graph G_2 on $18p$ vertices has a vertex cover of size $11p$ if and only if G_1 has a vertex cover of size $5p$. This completes the proof of (a).

Note that G_1 and G_2 are planar graphs. Let G_3 be the planar dual of G_2 (with respect to some embedding of G_2 in the plane; however, since H is 3-connected and G_1 is a subdivision of a 3-connected graph, any two such duals G_3 are plane isomorphic). Then G_3 is a triangulation (with several parallel edges). Subdivide each edge of G_3 by inserting a vertex of degree 2. Denote by W the set of all vertices of degree 2 obtained in this way. Finally, for each face of the resulting graph add a 3-cycle joining the three vertices of W in that face. The resulting graph G is a triangulation (without parallel edges and hence 3-connected), and there is a face cover of W of cardinality r if and only if G_2 has a vertex cover of cardinality r. In particular, G has face cover of W of cardinality $18p$ if and only if $\mathcal{C}$ is satisfiable. This completes the proof of (b). ▮

Garey, Johnson, and Stockmeyer proved [4] that the vertex cover problem is **NP**-complete for planar graphs of maximum degree 6. Theorem 4.1(a) answers their question what is the strongest degree restriction so that the problem remains **NP**-complete for planar graphs.

5. COMPUTING THE GENUS OF APEX GRAPHS

The main result of this section is the following:

Theorem 5.1. *It is* **NP**-*complete to decide if the genus of the given apex graph G is smaller or equal to the given integer g.*

The proof of Theorem 5.1 occupies the rest of the section. In Section 3 we established a close connection of the genus problem for apex graphs with the (**NP**-hard) problem of a minimum face cover. However, the reduction that we shall use to prove Theorem 5.1 is from an entirely different problem, proved to be **NP**-complete by Garey, Johnson, and Tarjan [5].

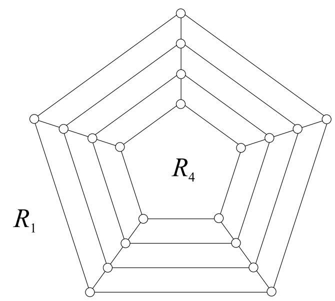

FIG. 2. The graph $H_{5,4}$

Theorem 5.2 (Garey, Johnson, and Tarjan [5]). *The decision problem whether a given cubic planar graph contains a Hamilton cycle is* **NP**-*complete.*

We need some preparation. Let $H_{p,r}$ be the Cartesian product of the p-cycle with the path on r vertices. It is shown in Figure 2 for $p = 5$ and $r = 4$. Denote by $R_1, \ldots, R_r$ the nested p-cycles of $H_{p,r}$, where R_1 is the outer cycle and R_r is the innermost facial cycle.

Lemma 5.1. *Let G be a graph and let H be a subgraph of G isomorphic to $H_{p,r}$ ($r \geq 2$, $p \geq 3$) such that only the vertices of H on the cycle corresponding to R_1 may be incident with an edge in $E(G) \backslash E(H)$. If Π is an orientable embedding of G and $r \geq \mathbf{g}(G, \Pi) + 2$, then there is an orientable embedding Π_1 of G such that:*

(a) *The induced embeddings of Π and Π_1 on $G - V(H)$ are the same.*

(b) *The induced embedding of Π_1 on H is of genus 0. Moreover, H contains no Π_1-noncontractible cycles.*

(c) $\mathbf{g}(G, \Pi_1) \leq \mathbf{g}(G, \Pi)$.

Proof. By induction on r. If $r = 2$, then $\mathbf{g}(G, \Pi) = 0$, so we may take $\Pi_1 = \Pi$. Suppose now that $r > 2$. Let C_i be the cycle of H corresponding to R_i, $i = 1, \ldots, r$. If C_r is Π-contractible, then we consider the induced embedding Π' of $G' = G - \cup_{i=1}^{r-1} E(C_i)$. Since C_r is an induced and nonseparating cycle of G, it is Π-facial and hence also Π'-facial. Each vertex of C_r is incident with precisely one edge of $G' - E(C_r)$. Since all such edges are Π'-embedded in the Π'-exterior of C_r, it is easy to see how one can extend Π' to an embedding Π_1 of G in the same surface satisfying (a)–(c).

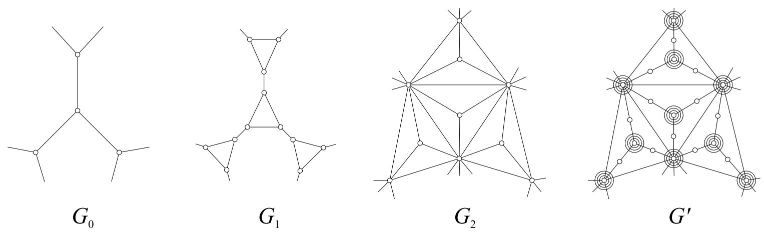

FIG. 3. From G_0 to G'

Suppose now that C_r is Π-noncontractible. Then the induced embedding of $G - V(C_r)$ has genus less than $\mathbf{g}(G, \Pi)$, and $H - V(C_r)$ is a subgraph isomorphic to $H_{p,r-1}$ satisfying the premises of the lemma for $r-1$. By the induction hypothesis, there is an embedding of $G-V(C_r)$ under which C_{r-1} is contractible. Since C_{r-1} is an induced and nonseparating cycle of $G - V(C_r)$, it is facial. Therefore, we may add C_r and the edges joining C_r with C_{r-1} into the face bounded by C_{r-1} to get the desired embedding Π_1 of G. ▌

Let G_0 be a (2-connected) cubic planar graph. We shall now introduce some related graphs and fix some notation that will be used in the sequel to prove Theorem 5.1. Let $n_0 = |V(G_0)|$. Denote by G_1 the cubic graph which is the truncation of G_0, i.e., the graph obtained from G_0 by first subdividing each edge of G_0 by inserting two vertices of degree 2, and then $Y\Delta$ each vertex v of G_0 into a triangle T_v. (Each $Y\Delta$ operation deletes the vertex v and adds the triangle T_v on the vertices adjacent to v in the subdivided graph.) Denote by $\mathcal{T} = \{T_v \mid v \in V(G_0)\}$ the set of triangles of G_1. Let G_2 be the plane dual of G_1 (with respect to some embedding of G_0 and G_1 in the plane). Replace each vertex u of G_2 by a distinct copy of the graph $H_r(u)$ which is isomorphic to $H_{d,r}$, where d is the degree of u in G_2 and $r = 2n_0 + 1$. Now, replace each edge uv of G_2 by a new edge e_{uv} joining the outer cycles of $H_r(u)$ and $H_r(v)$ so that no two such edges share an end and such that the resulting graph is planar. Finally, subdivide each edge e_{uv}, where uv is an edge dual to an edge of some 3-cycle in $\mathcal{T}$, by inserting a vertex w_{uv} of degree 2. Denote by G' the resulting graph, and let W be the set of all vertices w_{uv} (i.e., vertices of degree 2 in G'). The construction of G' is locally represented in Figure 3.

Let G be the graph obtained from G' by adding a new vertex w whose neighbors are precisely the vertices in W. Let Π_0' be the plane embedding of G'. It is easy to see that $\tau(W, G', \Pi_0') = 2n_0$ (since two faces are necessary and sufficient to cover the vertices in W corresponding to any triangle in $\mathcal{T}$). Therefore, $\mathbf{g}(G) \le 2n_0 - 1$ by Proposition 3.1. Since $r = 2n_0 + 1 \ge \mathbf{g}(G) + 2$, Lemma 5.1 implies that the genus of G is attained at an embedding Π whose induced embedding to G' satisfies condition (b) of Lemma 5.1 for each of the subgraphs $H_r(u)$, $u \in V(G_2)$. The set of all such embeddings of G' will be denoted by $\mathcal{E}$.

Two embeddings which have the same set of facial walks are said to be *equivalent*. A *local change* of the embedding $\Pi = (\pi, \lambda)$ at the vertex v changes π_v to its inverse π_v^{-1} and $\lambda(e)$ is replaced by $-\lambda(e)$ for edges e that are incident with v. It is easy to see that two embeddings are equivalent if and only if one can be obtained from the other by a sequence of local changes.

Claim 5.1. The set of equivalence classes of embeddings in $\mathcal{E}$ is in a bijective correspondence with the families $\mathcal{Q} = \{Q_1, \ldots, Q_p\}$ $(p \geq 0)$ of pairwise disjoint cycles of G_1.

Proof. For each $w_{uv} \in W$, let f_{uv} and f'_{uv} be the edges incident with w_{uv} in G'. The embeddings in $\mathcal{E}$ have fixed local clockwise rotation at all vertices and have positive signature on all edges of $H_r(u)$, $u \in V(G_2)$. The only freedom is that the signature of edges e_{uv} or their subdivision edges f_{uv} and f'_{uv} may be negative or positive. We may also assume that the signature of f'_{uv} is positive for each $w_{uv} \in W$. After these restrictions, the equivalence classes of embeddings in $\mathcal{E}$ are in a bijective correspondence with selections of positive or negative signatures for the edges e_{uv} and f_{uv} in G'.

Since G_2 is a triangulation (possibly with parallel edges), each Π'_0-facial cycle C contains at most 3 edges with negative signature in the embedding $\Pi' \in \mathcal{E}$. Since we only consider orientable embeddings, the number of such edges on C is even, so it is either 0 or 2. This implies that the edges e_{uv} and f_{uv} with negative signature determine a collection of pairwise disjoint cycles of G_1 whose edges are precisely the edges dual to an edge uv such that e_{uv} or f_{uv} has negative signature. Conversely, each such family $\mathcal{Q}$ of cycles determines an orientable embedding of G' with the same local clockwise rotations as the plane embedding Π'_0 of G' whose negative edges e_{uv} or f_{uv} are precisely those which are dual to the edges of the cycles in $\mathcal{Q}$. It is easy to see that this correspondence is bijective. ∎

If $\mathcal{Q} = \{Q_1, \ldots, Q_p\}$, $p \geq 0$, is a collection of pairwise disjoint cycles of G_1, let $\Pi'(\mathcal{Q}) \in \mathcal{E}$ denote the corresponding embedding of G'.

A cycle C of an embedded (cubic) graph is a *zig-zag cycle* (also known as a *Petrie cycle*) if no three consecutive edges of C are consecutive edges of some facial walk.

Claim 5.2. Let $\mathcal{Q} = \{Q_1, \dots, Q_p\}$, $p \geq 0$, be a collection of pairwise disjoint cycles of G_1. Denote by p_1 and p_2 the number of odd and even cycles in $\mathcal{Q}$, respectively, and let z_2 be the number of even zig-zag cycles in Q. Let N be the number of vertices in $V(\mathcal{Q}) = V(Q_1) \cup \cdots \cup V(Q_p)$, and let N_3 be the number of triangles $T \in \mathcal{T}$ of G_1 such that all three vertices of T are contained in $V(\mathcal{Q})$. If $\Pi' = \Pi'(\mathcal{Q})$, then

$$\mathbf{g}(G', \Pi') = \frac{1}{2}N - p + \frac{1}{2}\,p_1 \tag{7}$$

and

$$\tau(W, G', \Pi') = 2n_0 - N + N_3 + p_1 + 2p_2 - z_2. \tag{8}$$

Proof. Observe that Π' is obtained from the plane embedding Π'_0 of G' by changing the signatures along the edges dual to the cycles in $\mathcal{Q}$. If Q_i $(1 \leq i \leq p)$ is an even cycle of length $2l$ (say), then $2l$ Π'_0-facial cycles are replaced by precisely two facial cycles F_i, F'_i (we fix their notation now for later reference). Thus, the Euler characteristic drops by $2l - 2$, and hence the genus increases by $l - 1 = \frac{1}{2}|V(Q_i)| - 1$. Similarly, if Q_i is an odd cycle of length $2l - 1$, then $2l - 1$ Π'_0-facial cycles are replaced by a single Π'-facial cycle F_i. Hence, the Euler characteristic drops by $2l - 2$, and the genus increases by $l - 1 = \frac{1}{2}|V(Q_i)| - 1 + \frac{1}{2}$. This implies (7).

For each $T \in \mathcal{T}$ which is disjoint from $V(\mathcal{Q})$, two Π'-facial cycles are necessary and sufficient to cover the corresponding three vertices of W. Consider now an odd cycle $Q_i \in \mathcal{Q}$. Then F_i covers all vertices in W corresponding to the triangles in $\mathcal{T}$ intersected by Q_i. If Q_i is an even zig-zag cycle, then one of F_i or F'_i covers all vertices in W corresponding to the triangles in $\mathcal{T}$ intersected by Q_i. If Q_i is an even cycle which is not zig-zag, then F_i and F'_i do the same. Observe that the number of triangles $T \in \mathcal{T}$ which intersect some cycle Q_i is equal to $n_T = \frac{1}{2}(N - N_3)$. The above conclusions show that

$$\begin{aligned}\tau(W, G', \Pi') &\leq 2(n_0 - n_T) + p_1 + 2p_2 - z_2 \\ &= 2n_0 - N + N_3 + p_1 + 2p_2 - z_2.\end{aligned} \tag{9}$$

To prove (8), we have to show that equality holds in (9). It suffices to see that no single Π'-facial cycle covers all vertices of W corresponding to Q_i if Q_i is even and not zig-zag. If Q_i intersects some $T \in \mathcal{T}$ in precisely two vertices, then it is easy to see that neither F_i nor F'_i (nor any other Π'-facial walk) contains all three vertices of W corresponding to T. If such T does not exist, then there are adjacent triangles

$T, T' \in \mathcal{T}$ used by Q_i such that the edge connecting T and T' and the two adjacent edges used by Q_i are consecutive on a facial cycle of G_1. In this case, each of F_i, F_i' contains precisely 5 of the 6 vertices of W corresponding to T and T', hence the claim. This completes the proof. ▮

Claim 5.3. The genus of the graph G is at least n_0, and $\mathbf{g}(G) = n_0$ if and only if G_0 has a Hamilton cycle.

Proof. By Lemma 3.1, Claims 5.1 and 5.2, and the remark preceding Claim 5.1, the genus of G is equal to

$$\min_{\mathcal{Q}} \Big(\frac{1}{2}N - p + \frac{1}{2}p_1 + 2n_0 - N + N_3 + p_1 + 2p_2 - z_2 - 1\Big) \qquad (10)$$

where the minimum runs over all collections $\mathcal{Q} = \{Q_1, \dots, Q_p\}$ of disjoint cycles of G_1. Since $p_1 + p_2 = p$, (10) is equal to

$$\min_{\mathcal{Q}} \Big(2n_0 - 1 - \frac{1}{2}N + N_3 + \frac{1}{2}p_1 + p_2 - z_2\Big). \qquad (11)$$

If $\mathcal{Q}$ contains a 3-cycle Q_i, then $\mathcal{Q}\backslash\{Q_i\}$ gives the same value in (11) as $\mathcal{Q}$. Similarly, if some $Q_i \in \mathcal{Q}$ contains 3 vertices of the same triangle $T \in \mathcal{T}$, then replacing Q_i by the cycle, which is the same except that it intersects T in only two vertices, does not increase the value in (11). Therefore, the minimum in (11) may be taken only over collections of cycles $\mathcal{Q}$ such that each $Q_i \in \mathcal{Q}$ intersects each $T \in \mathcal{T}$ in 0 or 2 vertices. Then, clearly, $N_3 = 0$, $p_1 = 0$, and $z_2 = 0$. Therefore, (11) becomes

$$2n_0 - 1 + \min\Big(p_2 - \frac{1}{2}N\Big). \qquad (12)$$

Clearly, for such collections $\mathcal{Q}$, $N/2 - p_2 \le n_0 - 1$, where the equality holds if and only if $p = p_2 = 1$ and $N = 2n_0$, i.e., $\mathcal{Q} = \{Q_1\}$ where Q_1 visits all n_0 3-cycles of G_1. Clearly, existence of Q_1 is equivalent to the existence of a Hamilton cycle in G_0. This implies that $\mathbf{g}(G) \ge n_0$, and the equality holds if and only if G_0 has a Hamilton cycle. ▮

Starting with an arbitrary (2-connected) cubic planar graph G_0 we constructed in polynomial time the apex graph G whose genus is equal to $|V(G_0)|$ if and only if G_0 contains a Hamilton cycle. Theorem 5.2 then implies Theorem 5.1.

To show that the genus problem remains **NP**-complete for apex graphs G for which the corresponding planar subgraph $G - w$ is a triangulation (and hence 3-connected), we apply the following construction. First, subdivide also the remaining edges e_{uv} of G' (while keeping W unchanged). Then triangulate each 9-face of the resulting subdivision of G' by joining the three vertices of degree 2 and adding diagonals in the resulting 4-gons. The obtained graph G'' is a triangulation. It is easy to see that the genus of $G''+w$ is the same as the genus of G. The details are left to the reader.

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