Flexibility of polyhedral embeddings of graphs in surfaces[1]

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Whitney's theorem states that 3-connected planar graphs admit essentially unique embeddings in the plane. We generalize this result to embeddings of graphs in arbitrary surfaces by showing that there is a function $\xi : \mathbb{N}_0 \to \mathbb{N}_0$ such that every 3-connected graph admits at most $\xi(g)$ combinatorially distinct embeddings of face-width ≥ 3 into surfaces whose Euler genus is at most g.

Key Words: Graph theory, surface, polyhedral embedding, face-width

1. INTRODUCTION

Whitney proved [15] that every 3-connected planar graph has an essentially unique embedding in the plane. This means that face boundaries and local rotations are uniquely determined. This result was obtained as a corollary of a stronger statement that any two embeddings of a 2-connected planar graph are Whitney equivalent, i.e., one can be obtained from the other by a sequence of simple local re-embeddings. (See, e.g., [9] for more details on Whitney equivalence.) Robertson and Vitray [13] extended that

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result to an arbitrary surface of genus g by assuming that the face-width of the embedding is at least $2g+3$. Seymour and Thomas [14] and Mohar [8] improved the bound on the face-width to $O(\log g/\log\log g)$. Archdeacon [1] proved that an assumption on large face-width is necessary by showing that, for each integer k, there are graphs which admit distinct embeddings of face-width at least k. On the other hand, it has been noted in [5] that the finiteness of the number of irreducible triangulations for each fixed surface S implies that there is a bound $b = b(S)$ such that every graph admits at most b triangular embeddings in S. Thomassen [12] extended Whitney's uniqueness theorem under a hypothesis of large edge-width. Let us observe that the edge-width and the face-width are the same in the special case of triangulations.

In this paper we show that for each surface S, there is a constant $\xi = \xi(S)$ such that every 3-connected graph admits at most ξ embeddings of face-width ≥ 3 in S. The assumption on 3-connectivity is clearly necessary for such a result, and the following example shows that also the bound on the face-width cannot be weakened.

Let H_0 be a 4-connected plane graph whose outer face is a 4-cycle $v_1v_2v_3v_4$. For $n \geq 3$, let G_n be the graph obtained by taking n copies $H_1, \ldots, H_n$ of H_0 and, for $i = 1, \ldots, n$, identifying the edge v_1v_2 of H_i with the edge v_4v_3 of H_{i+1} (indices modulo n). The graph G_n is 4-connected and planar and has $2^{n-1} - 1$ embeddings of face-width 2 in the torus obtained by "flipping" one or more copies H_i "up or down" as shown by an example in Figure 1. Each such embedding is determined by a subset $A \subseteq \{1, \ldots, n\}$, and the embedding corresponding to the complementary subset $\overline{A}$ has the same set of facial walks as A. If $A = \emptyset$ or $A = \{1, \ldots, n\}$, the face-width is zero.

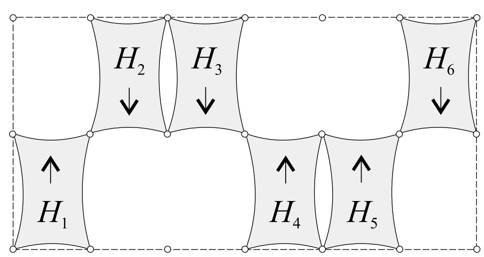

FIG. 1. An embedding of G_6 in the torus

This example can be easily transformed into a similar one where the graph G_n is nonplanar.

The following example shows that also increasing connectivity to 6 does not help to get bounded flexibility. Let us observe that increasing con-

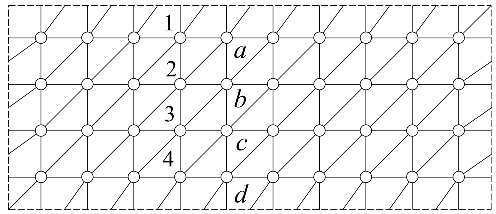

FIG. 2. A 6-connected triangulation of the torus

nectivity to 7 or more does not make sense since for each surface S, there is only a finite number of 7-connected graphs that can be embedded in S. (This can be easily seen by bounding the average degree of the graph by using Euler's formula.) Let T_n be the 6-connected triangulation of the torus represented in Figure 2. If we replace the 8 triangular faces between the 4-cycles 1234 and $abcd$ with the following four facial cycles: $12b34d$, $23c41a$, $ab3cd1$, and $bc4da2$, we get an embedding of face-width 2 in the orientable surface S_3 of genus 3. Since such a change can be performed between any two consecutive vertical 4-cycles, this example gives rise to 6-connected graphs which admit arbitrarily many embeddings of face-width 2 in S_3.

2. PRELIMINARIES

All graphs in this paper are undirected, finite and simple. We follow standard terminology as used, for example, in [2]. A subgraph C of a graph G is *induced* if every pair of non-adjacent vertices in C is also non-adjacent in G. It is *non-separating* if $G - V(C)$ is connected.

Let H be a subgraph of G. An *H-bridge* in G is a subgraph of G which is either an edge not in H but with both ends in H, or a connected component of $G - V(H)$ together with all edges which have one end in this component and the other end in H. Let B be an H-bridge. The vertices of $B \cap H$ are *vertices of attachment* of B, and each edge of B incident with a vertex of attachment is a *foot* of B.

Our treatment of graph embeddings follows essentially [9]. An *embedding* of a connected graph G is a pair $\Pi = (\pi, \lambda)$ where $\pi = \{\pi_v \mid v \in V(G)\}$ is a collection of *local rotations*, i.e., π_v is a cyclic permutation of the edges incident with v ($v \in V(G)$), and $\lambda : E(G) \to \{+1, -1\}$ is a *signature*. The local rotation π_v describes the cyclic clockwise order of edges incident with v on the surface, and the signature $\lambda(uv)$ of the edge uv is positive if and only if the local rotations π_u and π_v both correspond to the clockwise (or both to anticlockwise) rotations when traversing the edge uv on the surface. If we consider the graph G together with its embedding Π, we say that G is Π-*embedded*. The embedding Π determines a set of Π-*facial walks*. Facial walks are closed and are not distinguished if they differ only by choice of the initial vertex or by reversal of order of traversal. Each edge is either contained in two Π-facial walks or it appears twice in the same facial walk. If a Π-facial walk is a cycle, it is also called a Π-*facial cycle*. Two embeddings of G are *equivalent* if they have the same set of facial walks. A (contiguous, possibly closed) subwalk with at least one edge of a facial walk is called a *facial segment*.

The surface S of an embedding Π is given by attaching open discs (the faces) to the graph along the Π-facial walks.

The *Euler genus* of Π (or the Π-*genus* of G) is the integer $g = \mathbf{eg}(G, \Pi)$ defined by Euler's formula, $g = 2 - |V(G)| + |E(G)| - f$ where f denotes the number of Π-facial walks of G. The *Euler genus* $\mathbf{eg}(G)$ of the graph G is the minimum of Euler genera over all embeddings of G.

If G is a Π-embedded graph and H is a connected subgraph of G, then Π *induces* an embedding of H which is also denoted by Π and called the Π-*embedding* of H. Note that $\mathbf{eg}(H, \Pi) \le \mathbf{eg}(G, \Pi)$ and that strict inequality may occur.

Let G be a Π-embedded graph, and let $C = x_1 \dots x_k$ be a k-cycle of G. C is said to be Π-*onesided* if the number of edges of C with negative signature is odd. Otherwise, C is Π-*twosided*. By passing to an equivalent embedding (for example, by reversing the clockwise local rotations to anticlockwise at some of the vertices x_i and changing the signature of all edges incident with x_i), we may assume that all edges of C except possibly x_1x_k have positive signature. Let $e = x_ix_{i+1}$ $(1 \le i \le k)$ be an edge of C, and let $d = deg_G(x_i)$. Then there is an l such that $1 \le l < d$ and $\pi^l_{x_i}(e) \in E(C)$. We say that the edges $\pi_{x_i}(e), \dots, \pi^{l-1}_{x_i}(e)$ are incident with C on its *right side* and the edges $\pi^{l+1}_{x_i}(e), \dots, \pi^{d-1}_{x_i}(e)$ are incident with C on its *left side*. By *cutting G along C*, a new Π'-embedded graph G' is obtained from G as follows: If C is Π-twosided, we delete C and add instead two disjoint cycles $C' = x'_1 \dots x'_k$ and $C'' = x''_1 \dots x''_k$. If C is Π-onesided, we replace C by a single $2k$-cycle $C' = x'_1 \dots x'_k x''_1 \dots x''_k$. In each case, the vertex x'_i (resp., x''_i) is adjacent to a vertex $y \in V(G) \setminus V(C)$ if and only if $x_iy \in E(G)$ and the edge x_iy is incident with C on its left side (resp., right side). We let Π' be the same as Π except that G' has no edges incident with C' (resp., C'') on its right side (resp., left side), $i = 1, \dots, k$. We say that the cycles C' and C'' *correspond* to C. Clearly, C' and C'' are Π'-facial cycles of G'. Let W' be a Π'-facial walk of G' different from C' and C''. If W' contains no point in $V(C') \cup V(C'')$, then W' is a Π-facial walk of G. Otherwise, the walk W obtained from W' by replacing x'_i and x''_i by x_i is a Π-facial walk of G. Conversely, if W is a Π-facial walk of G, then there is a Π'-facial walk W' of G' such that W is obtained from W' by replacing x'_i

and x_i'' by x_i. If G' is connected, then $\mathbf{eg}(G', \Pi) < \mathbf{eg}(G, \Pi)$. Otherwise, $G' = G_1' \cup G_2'$ where $G_1' \cap G_2' = \emptyset$ and $C' \subseteq G_1'$, $C'' \subseteq G_2'$. In this case, $\mathbf{eg}(G, \Pi) = \mathbf{eg}(G_1', \Pi') + \mathbf{eg}(G_2', \Pi')$. If $\min\{\mathbf{eg}(G_1', \Pi'), \mathbf{eg}(G_2', \Pi')\} = 0$, then C is said to be Π-*contractible*. If $\mathbf{eg}(G_1', \Pi') = 0$, then we write $G_1' = \mathrm{Int}(C, \Pi)$ and $G_2' = \mathrm{Ext}(C, \Pi)$.

Disjoint cycles C, C' of G are (*freely*) Π-*homotopic* if either C and C' are both Π-contractible, or C and C' are Π-twosided and cutting along C and C' results in a graph which has a component D which contains precisely one copy of C and one copy of C' and whose Π-genus is zero. In the latter case we write $D = \mathrm{Int}(C, C', \Pi)$, and we denote by $\mathrm{Ext}(C, C', \Pi)$ the other component(s) containing copies of C and C'.

If a, b are distinct vertices of G and P_1, P_2 are internally disjoint paths from a to b in G, then P_1 and P_2 are said to be Π-*homotopic* if the cycle $C = P_1 \cup P_2$ is Π-contractible. This definition extends to the case when P_1, P_2 are cycles with the common vertex $a = b$ or even cycles with an edge or a path in common (cf. [9]). In all these cases, we define $\mathrm{Int}(P_1, P_2, \Pi)$ to be the disk bounded by C.

LEMMA 2.1. *Let G be a Π-embedded graph and let $\mathcal{C}$ be a set of noncontractible Π-homotopic cycles. Suppose that there is a path P (possibly $P = \emptyset$) such that $C \cap C' = P$ for any distinct cycles $C, C' \in \mathcal{C}$. Then the cycles in $\mathcal{C}$ can be enumerated, $\mathcal{C} = \{C_1, \ldots, C_r\}$, such that for each i and j $(1 \le i < j \le r)$, $\mathrm{Int}(C_i, C_j, \Pi) = \cup_{t=i}^{j-1} \mathrm{Int}(C_t, C_{t+1}, \Pi)$.*

Proof. We assume that $P \neq \emptyset$; the case of pairwise disjoint cycles has similar proof and we leave the details to the reader. By contracting P to a point, we may assume that $P = \{v\}$ is just a vertex.

The proof is by induction on $r = |\mathcal{C}|$. There is nothing to prove if $r \le 2$, so assume $r \ge 3$. By removing an arbitrary cycle $C \in \mathcal{C}$, the remaining cycles can be enumerated, by the induction hypothesis, as $C_1', \ldots, C_{r-1}'$ to satisfy the conclusion of the lemma. If $C \subseteq \mathrm{Int}(C_t', C_{t+1}', \Pi)$ for some t, $1 \le t < r-1$, then we insert C between C_t' and C_{t+1}' in the ordering for $\mathcal{C}$, and use the fact that $\mathrm{Int}(C_t', C_{t+1}', \Pi) = \mathrm{Int}(C_t', C, \Pi) \cup \mathrm{Int}(C, C_{t+1}', \Pi)$ to complete the proof. Otherwise, $C \subseteq \mathrm{Ext}(C_1', C_{r-1}', \Pi)$. If $\mathrm{Int}(C, C_{r-1}', \Pi)$ does not contain C_1', then $\mathrm{Int}(C_1', C, \Pi) = \mathrm{Int}(C_1', C_{r-1}', \Pi) \cup \mathrm{Int}(C_{r-1}', C, \Pi)$ and we set $C_t = C_t'$ for $1 \le t < r$, and $C_r = C$. If $C_1' \subseteq \mathrm{Int}(C, C_{r-1}', \Pi)$, then $\mathrm{Int}(C_1', C_{r-1}', \Pi) \subseteq \mathrm{Int}(C, C_{r-1}', \Pi)$. Now, we set $C_1 = C$ and $C_t = C_{t-1}'$ for $1 < t \le r$. ∎

We will refer to the natural ordering of $\mathcal{C}$ as in Lemma 2.1 or in Corollary 2.1 below as the *linear nesting* of homotopic cycles.

COROLLARY 2.1. *Let G be a Π-embedded graph and let $\mathcal{C}$ be a set of noncontractible Π-homotopic cycles such that the intersection of any two* *of them is either empty or a path. If the cycles in* $\mathcal{C}$ *are* Π*-twosided, then they can be enumerated,* $\mathcal{C} = \{C_1, \ldots, C_r\}$*, such that for each* i *and* j $(1 \le i < j \le r)$*,* $\mathrm{Int}(C_i, C_j, \Pi) = \cup_{t=i}^{j-1} \mathrm{Int}(C_t, C_{t+1}, \Pi)$*.*

Proof. Since the cycles in $\mathcal{C}$ are Π-twosided, they can be separated by splitting vertices of their intersection. It is easy to see that by appropriate splitting of vertices, we can get a graph H that is Π'-embedded in the same surface and such that $\mathcal{C}$ gives rise to a set of pairwise disjoint homotopic cycles. Now we apply Lemma 2.1 and observe that the Π-interior is obtained from the Π'-interior by contractions of edges in $E(H) \backslash E(G)$. ▮

We will make use of the following lemma which is a special case of the main theorem in [6].

Lemma 2.2 (Juvan, Malnič, Mohar [6]). *For each Euler genus* g *there is a positive integer* $c_1 = c_1(g)$ *such that for every* Π*-embedded graph* G *where* $\mathbf{eg}(G, \Pi) = g$*, and for every family of* r *cycles (paths) of* G *which pairwise intersect in at most two common segments, there is a subset of at least* $\lceil r/c_1 \rceil$ *pairwise* Π*-homotopic cycles (paths).*

In the proof of Theorem 5.1 we shall use the following lemma of Fisk and Mohar [4]. We add its proof since in [4] it is formulated only for paths while here we allow walks.

Lemma 2.3. *Let* $k \ge 1$ *and* $r \ge 1$ *be integers. There exists an integer* $\varphi(k, r)$ *such that the following holds: If a multigraph* H *contains* $\varphi(k, r)$ *walks of length at most* k *joining vertices* v_1 *and* v_2*, such that the initial edges of the walks incident with* v_1 *are all distinct, then there is a vertex* $v \ne v_1$ *of* H *such that* v_1 *and* v *are joined by* r *internally disjoint subwalks of the given walks.*

Proof. It suffices to give the proof for simple graphs since H has a subdivision which is simple and such that each walk has length at most $3k$.

For simple graphs we prove the lemma by induction on $k + r$ and with $\varphi(k, r) = r^{k-1}(k-1)!$. For $r = 1$, there is nothing to prove. For $k \le 2$, all the walks are disjoint paths, so we proceed to the induction step. Now, let H be a graph that has $\varphi(k+1, r)$ walks from v_1 to v_2 of length at most $k+1$ and with distinct initial edges. Pick a walk P. If there are at least $k\varphi(k, r)$ walks intersecting $P - \{v_1, v_2\}$, then some $\varphi(k, r)$ of these walks meet the same vertex of $P - \{v_1, v_2\}$, and so we obtain the desired walks by induction. Otherwise, let $P_1, \ldots, P_q$ be a maximal collection of internally disjoint walks from v_1 to v_2 (taken from the $\varphi(k+1, r)$ walks). As each of the $\varphi(k+1,r)$ walks (which is not the edge v_1v_2) has an intermediate vertex in $P_1 \cup \cdots \cup P_q$, we have $\varphi(k+1,r) \le qk\varphi(k,r)$, and hence $q \ge r$. ▮

Let us observe that Lemma 2.3 holds also in the case when $v_1 = v_2$ (in which case the walks or subwalks may be closed).

3. POLYHEDRAL EMBEDDINGS

Let G be a Π-embedded graph. If $\mathbf{eg}(G,\Pi) \ge 1$, the *face-width* of Π (also called the *representativity*), $\mathrm{fw}(G,\Pi)$, is the smallest integer r such that G has a Π-noncontractible cycle which is the union of r segments, each of which is contained in a Π-facial walk. If $\mathbf{eg}(G,\Pi) = 0$, we let $\mathrm{fw}(G,\Pi) = \infty$.

Let C_1 and C_2 be distinct Π-facial walks. We say that C_1 and C_2 *meet properly* if the intersection of C_1 and C_2 is either empty, a single vertex or an edge. Π is said to be a *polyhedral embedding* if every Π-facial walk is a cycle and any two Π-facial walks meet properly. The following results are due to Robertson and Vitray [13].

Proposition 3.1. *Let G be a connected Π-embedded graph. Then Π is a polyhedral embedding if and only if $\mathrm{fw}(G,\Pi) \ge 3$ and G is 3-connected.*

Proposition 3.2. *Let G be a 3-connected Π-embedded graph. If $\mathrm{fw}(G,\Pi) \ge 3$, then every facial cycle is an induced nonseparating cycle.*

4. COMPARING DISTINCT EMBEDDINGS

Lemma 4.1. *Let Π and Π' be embeddings of a 3-connected graph G such that $\mathrm{fw}(\Pi) \ge 3$. Suppose that $C_1, \ldots, C_r$ are distinct Π'-facial cycles such that any two of them meet properly. If $C_1, \ldots, C_r$ are all Π-noncontractible and Π-homotopic to each other, then $\mathbf{eg}(\Pi') \ge \frac{r}{13} - 1$.*

Proof. Suppose first that $C_1, \ldots, C_r$ are Π-twosided. Then we may assume that $C_1, \ldots, C_r$ is a linear nesting, by Corollary 2.1. If C_i intersects C_1 in $\mathrm{Ext}(C_1, C_i, \Pi)$, then C_i and C_r do not intersect in $\mathrm{Ext}(C_i, C_r, \Pi)$ $(1 \le i \le r)$. Therefore, we may assume that for $i = 1, \ldots, t = \lceil r/2 \rceil$, the cycles C_1 and C_i do not intersect in $\mathrm{Ext}(C_1, C_t, \Pi)$. Let $H = \mathrm{Int}(C_1, C_t, \Pi)$. By inserting a new vertex into each of the faces C_1, C_t of H and joining each of them to all vertices of C_1 and C_t, respectively, we get a plane graph H' without vertices of degree 2 whose facial cycles meet properly. By

Proposition 3.1, H' is 3-connected. By Menger's Theorem there are three disjoint (C_1, C_t)-paths P'_1, P'_2, P'_3 in H. These paths determine disjoint paths P_1, P_2, P_3 in G since C_1 and C_t do not intersect in $\mathrm{Ext}(C_1, C_t, \Pi)$. Each of P_1, P_2, P_3 intersects all cycles C_i, $i = 1, \ldots, t$. Let v_{ik} be a vertex of $P_k \cap C_i$ ($k = 1, 2, 3$; $i = 1, \ldots, t$). If $v_{ik} = v_{i'k'}$, then $k = k'$ but i and i' may be distinct. Let G'' be the graph obtained from G by adding new vertices $v_1, \ldots, v_t$ and joining each v_i with the vertices v_{i1}, v_{i2}, v_{i3}, $i = 1, \ldots, t$. Since $C_1, \ldots, C_t$ are Π'-facial, Π' can be extended to an embedding Π'' of G'' in the same surface as Π'. By contracting P_1, P_2, P_3 to single vertices in the subgraph of G'' consisting of P_1, P_2, P_3 and the stars of vertices v_i ($i = 1, \ldots, t$), we obtain $K_{3,t}$ as a minor in G''. Therefore,

$$\mathbf{eg}(\Pi') = \mathbf{eg}(G'', \Pi'') \geq \mathbf{eg}(K_{3,t}) \geq \left\lceil \frac{t-2}{2} \right\rceil \geq \frac{r}{4} - 1. \qquad (1)$$

(The second inequality in (1) is an easy corollary of Euler's formula and biparticity of G. Ringel [11] proved that this is indeed an equality.)

Suppose now that $C_1, \ldots, C_r$ are Π-onesided. Then any two cycles intersect (and cross each other locally in Π). Let $p = \lceil 2r/13 \rceil$. If p of the cycles intersect in the same point, then those cycles can be enumerated as concluded in Lemma 2.1. (The details are left to the reader.) As in (1) we get the inequality:

$$\mathbf{eg}(\Pi') \geq \mathbf{eg}(K_{3,p}) \geq \frac{r}{13} - 1. \qquad (2)$$

So, we may assume that no p of the cycles intersect in the same point. Let $v_{ij} \in V(C_i) \cap V(C_j)$. For each vertex $v \in \{v_{ij} \mid 1 \leq i < j \leq r\}$, select a pair (i, j) such that $v = v_{ij}$. Now, we define a graph G'' obtained from G by adding r new vertices $u_1, \ldots, u_r$, and joining u_l ($1 \leq l \leq r$) to all vertices v_{il} and v_{lj} whose selected pair contains l. Clearly, Π' can be extended to an embedding of G'' in the same surface. The new vertices and edges form a subgraph of G'' which is a subdivision of a simple graph H with r vertices and at least $\binom{r}{2}/(p-1)$ edges. Euler's formula implies that

$$\begin{aligned}\mathbf{eg}(\Pi') &\geq \mathbf{eg}(H) \geq 2 - |V(H)| + \frac{1}{3}|E(H)| \\ &\geq 2 - r + \frac{13r(r-1)}{6(2r-1)} \geq \frac{r+11}{12} > \frac{r}{13}.\end{aligned}$$

∎

Let Π and Π' be embeddings of a graph G. A closed walk in G is said to be (Π, Π')-*unstable* if it is Π-facial and is not Π'-facial.

Lemma 4.2. *Let G be a 3-connected graph with embeddings Π and Π' such that* $\mathrm{fw}(\Pi) \geq 3$ *and* $\mathrm{fw}(\Pi') \geq 3$. *If r is the number of (Π, Π')-unstable* *cycles, then*

$$r \leq 13(g+1)c_1$$

where $g = \mathbf{eg}(G, \Pi)$ *and* $c_1 = c_1(g)$ *is the constant from Lemma 2.2.*

Proof. By Proposition 3.1, the unstable cycles meet properly. By Lemma 2.2, $\lceil r/c_1 \rceil$ of them are pairwise Π'-homotopic. Proposition 3.2 implies that they are Π'-noncontractible. By Lemma 4.1, $\mathbf{eg}(G, \Pi) \geq r/(13c_1) - 1$. This proves the lemma. ∎

Lemma 4.3. *Let* G *be a 3-connected graph with distinct embeddings* Π *and* Π' *such that* $\mathrm{fw}(\Pi) \geq 3$ *and* $\mathrm{fw}(\Pi') \geq 3$. *Suppose that* C *is a* Π-*facial cycle and that* C' *is a* Π'-*facial cycle. Let* p *denote the number of connected components of* $C \cap C'$. *Then* p *is smaller than the number of* (Π, Π')-*unstable cycles.*

Proof. We may assume that C' is not Π-facial since otherwise $p \leq 1$ (and there are at least two unstable cycles). For each edge $e \in E(C')$ there is a (Π, Π')-unstable cycle $C(e)$ which contains e. Therefore each connected component P of $C \cap C'$ intersects the Π-facial cycle $C(e) \neq C$ where e is the edge of C' following P. Since C and $C(e)$ meet properly, these cycles $C(e)$ are distinct. Since C is also unstable, p is smaller than the number of (Π, Π')-unstable cycles. ∎

Let H be a graph with k connected components. The number $\beta(H) = |E(H)| - |V(H)| + k$ is called the *Betti number* (or the *cyclomatic number*) of H.

Lemma 4.4. *Let* G *be a 3-connected graph with distinct embeddings* Π *and* Π' *such that* $\mathrm{fw}(\Pi) \geq 3$ *and* $\mathrm{fw}(\Pi') \geq 3$. *Let* H *be the union of all* (Π, Π')-*unstable cycles. Then*

$$\beta(H) < 85(g+1)^2\, c_1^2$$

where $g = \mathbf{eg}(G, \Pi)$ *and* $c_1 = c_1(g)$ *is from Lemma 2.2.*

Proof. Let $C_1, \ldots, C_r$ be the (Π, Π')-unstable cycles. We prove by induction on t that $\beta_t = \beta(C_1 \cup \cdots \cup C_t) \leq 1 + t(t-1)/2$, $t = 1, \ldots, r$. Clearly, $\beta_1 = 1$. So assume that $t > 1$. Let $S_1, \ldots, S_q$ be the maximal segments of C_t which are edge-disjoint from $C_1 \cup \cdots \cup C_{t-1}$. Since the Π-facial cycles meet properly, $q \leq t - 1$. By the induction hypothesis, $\beta_t = \beta_{t-1} + q \leq 1 + (t-1)(t-2)/2 + t - 1 = 1 + t(t-1)/2$.

Finally, $r \leq 13c_1\,(g+1)$ by Lemma 4.2. This shows that $\beta(H) < 85c_1^2(g+1)^2$. ∎

Let C and C' be cycles of a Π'-embedded graph G. Suppose that one of the following holds:

(a) $C \cap C' = \{u\}$ where $u \in V(G)$ and the edges of C and C' incident with u interlace in the Π'-clockwise ordering around u (cf. Fig. 3(a)).

(b) $C \cap C'$ is the edge uv and the edges of C and C' incident with u and v interlace in the Π'-clockwise ordering around u and v as shown in Fig. 3(b).

Then we say that C and C' *interlace* in Π'.

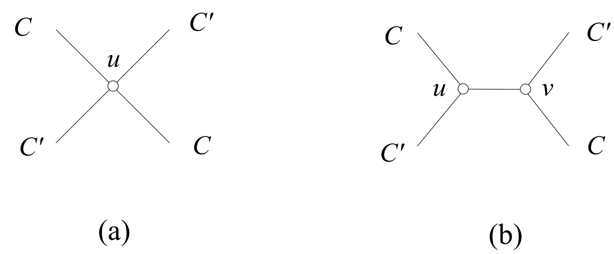

FIG. 3. The cycles C and C' interlace in Π'

LEMMA 4.5. *Let G be a graph and let Π and Π' be polyhedral embeddings of G. Let C be a Π-facial cycle and let S be a segment of C. Suppose that in the embedding Π', there is an interior vertex of S which has an edge on the right and there is an interior vertex of S with an edge on the left side of C. Then there are interior vertices u, v of S and a Π-facial cycle C' such that $C \cap C' = \{u\}$ or $C \cap C' = \{uv\}$ and C and C' interlace in Π'.*

Proof. S contains an interior vertex u which has an edge on the right and contains an interior vertex v with an edge on the left side of C. Since each vertex of C has either an edge on the left or on the right side of C, we may assume that either $u = v$, or that u and v are adjacent on C. We assume that $u = v$ since the proof of the case when $uv \in E(C)$ proceeds in the same way. In the Π-clockwise ordering around u, there are consecutive edges $e, f \notin E(C)$ such that e is on the left side of C and f is on the right side in Π'. Now, we let C' be the Π-facial walk containing e and f. ∎

5. FLEXIBILITY OF EMBEDDINGS OF FACE-WIDTH 3

Suppose that G is a 3-connected graph and that $\Pi_0, \ldots, \Pi_N$ are distinct embeddings of G each of which has face-width at least 3. For distinct integers $i, j \in \{0, \ldots, N\}$ we introduce the following notation. We let $\mathcal{C}_{ij}$ be the set of all (Π_i, Π_j)-unstable cycles, and let $c_{ij} = |\mathcal{C}_{ij}|$. The subgraph $U_{ij} = \cup \mathcal{C}_{ij}$ of G is called the (Π_i, Π_j)-*unstable part* of G. Let us observe that the complements of U_{ij} and U_{ji} in G are the same since they represent the union of the facial cycles that are common to the two embeddings. Therefore, $U_{ij} = U_{ji}$. By a face count in the two embeddings, $c_{ij} - \mathbf{eg}(G, \Pi_i) = c_{ji} - \mathbf{eg}(G, \Pi_j)$. In particular, if $\mathbf{eg}(G, \Pi_i) = \mathbf{eg}(G, \Pi_j)$, then $c_{ij} = c_{ji}$.

LEMMA 5.1. *Suppose that Π_0, Π_1, Π_2 are embeddings of G in the same surface and that $c_{01} = c_{02} = c_{12}$. Suppose, moreover, that $\mathcal{C}_{01} \cap \mathcal{C}_{02} = \emptyset$. Then $U_{01} = U_{02} = U_{12}$.*

Proof. Since $\mathcal{C}_{01} \cap \mathcal{C}_{02} = \emptyset$, we have $\mathcal{C}_{02} \subseteq \mathcal{C}_{12}$. Now, $c_{02} = c_{12}$ implies that $\mathcal{C}_{02} = \mathcal{C}_{12}$. Similarly, $\mathcal{C}_{01} = \mathcal{C}_{21}$. Hence, $U_{02} = U_{12} = U_{21} = U_{01}$. ▮

Now we turn to the main theorem of this paper.

THEOREM 5.1. *There is a function $\xi : \mathbb{N}_0 \to \mathbb{N}_0$ such that every 3-connected graph admits at most $\xi(g)$ embeddings of face-width ≥ 3 into surfaces whose Euler genus is at most g.*

The rest of this section is devoted to the proof of Theorem 5.1. The proof is by induction on g. Clearly, $\xi(0) = 1$ by Whitney's Theorem. So, we let $g \geq 1$. We now assume (reductio ad absurdum) that there is no upper bound on the number of distinct embeddings of face-width ≥ 3 of 3-connected graphs G in a surface S of Euler genus g. Let $\Pi_0, \ldots, \Pi_N$ be such embeddings, where $\mathbf{eg}(G, \Pi_i) = g$, $i = 0, 1, \ldots, N$. We assume that G can be chosen so that N is as large as we want. During the proof, we will occasionally select and continue working with a subset of $\Pi_0, \ldots, \Pi_N$ but we will always be able to argue that the new set of embeddings is still as large as we want. Our main concern will be a smooth flow of the proof, and we have no intention to derive good bounds on $\xi(g)$.

Claim 5.1. There is an integer function $r(N, g)$ such that, for each fixed $g > 0$, $\lim_{N \to \infty} r(N, g) = \infty$, and there is an integer c and a subset I of $\{0, \ldots, N\}$ of cardinality $r(N, g)$ such that $c_{ij} = c$ for any distinct elements $i, j \in I$.

Proof. By Lemma 4.2, $c_{ij} = c_{ji}$ are bounded by a constant depending only on g. Now, the existence of $r(N, g)$ follows by Ramsey's Theorem (see, e.g., [10, Theorem 1.1]). ∎

By using Claim 5.1 and by passing to the subset of embeddings $\Pi_i, i \in I$, we may assume henceforth that $c_{ij} = c_{ji} = c$ for $0 \le i < j \le N$, and that N is still as large as we want.

Claim 5.2. Suppose that $\log_2 N \ge \log_2^2(2c)$. Then there is a number $\alpha > 0$ which depends only on g, and there is a subset $I \subseteq \{0, \ldots, N\}$ such that $|I| \ge \alpha\sqrt{\log_2 N}$ and such that for each $i \in I$, there is a Π_i-facial cycle C_i which is Π_j-nonfacial for every $j \in I\backslash\{i\}$.

Proof. Suppose that each (Π_0, Π_1)-unstable cycle is Π_i-facial where $i \ge 2$. Then $\mathcal{C}_{01} \cap \mathcal{C}_{0i} = \emptyset$. By the proof of Lemma 5.1, $U_{01} = U_{0i} = U_{1i}$. By Lemma 4.4, the Betti number of U_{01} is bounded by $c^2/2$. Since $\beta(U_{01})$ is the dimension of the cycle space of U_{01} over $GF(2)$, U_{01} contains less than $2^{\beta(U_{01})}$ cycles. Hence less than $\alpha_1 = 2^{\beta(U_{01})c}$ embeddings Π_i $(i \ge 1)$ have their unstable part U_{0i} contained in U_{01}. We remove all such embeddings Π_i. In each of the remaining $N - \alpha_1$ embeddings Π_i $(i \ge 2)$, one of the cycles in $\mathcal{C}_{01}$ is Π_i-nonfacial. Since $|\mathcal{C}_{01}| = c$, there is a Π_0-facial cycle C_0 which is nonfacial in at least $N_1 = (N - \alpha_1)/c$ embeddings. Clearly, $N_1 \ge N/2c$ if $N \ge 2\alpha_1$ (which we may assume).

By passing to the subset of the remaining embeddings and continuing the process, let us assume that $1 \le i \le \alpha\sqrt{\log_2 N}$ where α will be determined below. Suppose that we have cycles $C_0, \ldots, C_{i-1}$ as claimed, and now we want to find C_i. We are left with $N_i \ge N/(2c)^i$ embeddings $\Pi_i, \ldots, \Pi_{N_i+i-1}$.

Let $U_i = U_{01} \cup \cdots \cup U_{0,i-1}$. As before, the Betti number of U_i is bounded above by $r^2/2$ where $r = ic$ is an upper bound on the number of cycles in $\mathcal{C}_{01} \cup \cdots \cup \mathcal{C}_{0,i-1}$. Then U_i contains less than $2^{\beta(U_i)}$ cycles, and hence less than $\alpha_i = 2^{\beta(U_i)c}$ embeddings Π_j $(j \ge i)$ have their unstable part U_{0j} contained in U_i. We will prove below that $N_i - \alpha_i \ge N_i/2$. Hence, we may assume that $U_{0i} \not\subseteq U_i$. Denote by $Q_1, \ldots, Q_p$ $(1 \le p \le c)$ the (Π_i, Π_0)-unstable cycles which are not contained in U_i. Each cycle Q_s $(1 \le s \le p)$ contains an edge which is not in U_i. This implies that Q_s is (Π_i, Π_j)-unstable for $j = 0, \ldots, i-1$. Let $U' = U_i \cup Q_1 \cup \cdots \cup Q_p$. Since r is also an upper bound on the number of Π_0-facial cycles forming U', $\beta(U') \le r^2/2$ and hence at least $N_i - 2^{cr^2/2}$ embeddings Π_j $(j > i)$ satisfy $U_{0j} \not\subseteq U'$. Assuming $2^{cr^2/2} < N_i/2$, we get at least $N_i/2p \ge N_i/2c$ remaining embeddings and a Π_i-facial cycle C_i (where $C_i = Q_s$ for some $1 \le s \le p$) which is nonfacial in all other embeddings. By retaining only those embeddings, we can continue the process. The reader can verify that the choice $\alpha = \frac{1}{2}c^{-3/2}$ guarantees that $2^{cr^2/2} < N_i/2$ for $i \le \alpha\sqrt{\log_2 N}$. ∎

By Claim 5.2 we may assume that for each $i \in \{0, \ldots, N\}$, there is a Π_i-facial cycle C_i which is Π_j-nonfacial for every $j \in \{0, \ldots, N\}\backslash\{i\}$. In particular, the cycles $C_0, \ldots, C_N$ are distinct.

The cycle $C_i \in \mathcal{C}_{i0}$ is contained in $U_{0i} = \cup\mathcal{C}_{0i}$. Since $\mathcal{C}_{0i}$ contains c cycles, any two of which meet properly, C_i can be written as the union of no more than c^2 Π_0-facial segments (by Lemma 4.3). Let $S_{i1}, \ldots, S_{i,\kappa_i}$ ($\kappa_i \le c^2$) be these Π_0-facial segments.

Claim 5.3. Suppose that there are pairwise disjoint Π_0-facial segments $A_i \subseteq S_{i1}$ ($i = 1, \ldots, N$), and suppose that there are distinct vertices v_{ij} of $A_i \cap C_j$ such that an edge $e_{i,j}$ of C_j incident with v_{ij} is not in the same Π_0-facial cycle as A_i ($1 \le i < j \le N$). Then $N < \kappa$ where κ is an integer which depends only on g.

Proof. Let $1 \le p(i,j) \le c^2$ be the index of the segment of C_j such that $v_{ij} \in S_{i1} \cap S_{j,p(i,j)}$. If N is large enough, then Ramsey's Theorem (cf. [10, Theorem 1.1]) implies that there is a set $I \subseteq \{1, \ldots, N\}$ with $k_0 = 2\lceil 9\sqrt{g} + 18\rceil + 2c$ elements and there exists an integer p such that $p(i,j) = p$ for all $i, j \in I$, $i < j$. We may assume that $I = \{1, \ldots, k_0\}$. Let D_i be the Π_0-facial cycle containing A_i, and let D'_i be the Π_0-facial cycle containing S_{ip}, $i = 1, \ldots, k_0$. Suppose that for $1 \le a < b < d < k_0 - 3$, $D_a = D_b = D_d$. Then v_{a,k_0}, v_{b,k_0}, and v_{d,k_0} all belong to $D_a \cap S_{k_0,p}$. Since the Π_0-facial cycles meet properly, $D_a = D'_{k_0}$. Similarly we prove that $D_a = D'_{k_0-1} = D'_{k_0-2}$. The vertices v_{1,k_0-2}, v_{1,k_0-1}, and v_{1,k_0} all belong to $D_1 \cap D'_{k_0}$, and so $D_1 = D'_{k_0}$. Similarly, $D_1 = D_2 = \cdots = D_{2d+1} = D'_{k_0}$. Now, the edges $e_{1,k_0}, e_{2,k_0}, \ldots, e_{2d+1,k_0}$ show that $C_{k_0} \cap D_1$ consists of more than d components, a contradiction to Lemma 4.3. This proves that there is a subset of $k = \lceil 9\sqrt{g} + 18\rceil$ cycles, say $C_1, \ldots, C_k$, such that $D_1, \ldots, D_k$ are all distinct. Now we distinguish two cases.

Case 1: $p = 1$. We can extend the embedding Π_0 to an embedding in the same surface of a graph $\tilde{G} \supseteq G$ which contains a subdivision of the complete graph K_k as follows. We insert a new vertex x_i into each Π_0-face D_i and add edges inside D_i from x_i to v_{ij} and inside D_j from v_{ij} to x_j, $1 \le i < j \le k$. Since $\mathbf{eg}(K_k) \ge (k-3)(k-4)/6$, we get a contradiction to the fact that $k \ge 9\sqrt{g} + 18$.

Case 2: $p \ne 1$, say $p = 2$. Suppose that $2 < i < j < k$ and $D'_i = D'_j = D'_k = D'$. Then D' intersects D_1 in three distinct vertices. Since the Π_0-facial cycles meet properly, $D' = D_1$. Similarly we see that $D' = D_2$, a contradiction. This implies that we may assume that $D'_3, \ldots, D'_{k_1}$ are all distinct, where $k_1 = \lceil k/2\rceil$. Let $k_2 = \lfloor k_1/3\rfloor > 1.5\sqrt{g} + 2$. Then we may assume that $D'_{k_2+1}, \dots, D'_{2k_2}$ are distinct Π_0-facial cycles which are distinct from each of $D_1, \dots, D_{k_2}$. We now extend the embedding Π_0 to an embedding in the same surface of a graph $\tilde{G} \supseteq G$ which contains a subdivision of the complete bipartite graph K_{k_2,k_2} in the same way as in Case 1 by using vertices in the Π_0-faces $D_1, \dots, D_{k_2}$ and $D'_{k_2+1}, \dots, D'_{2k_2}$, respectively, and joining them through the vertices v_{ij}, $i = 1, \dots, k_2$, $j = k_2 + 1, \dots, 2k_2$. Since $\mathbf{eg}(K_{k_2,k_2}) \geq (k_2 - 2)^2/2$, we get a contradiction to the fact that $k_2 > 1.5\sqrt{g} + 2$. ▌

Claim 5.4. Let A, B be segments of distinct Π_0-facial cycles or disjoint segments of the same Π_0-facial cycle. Suppose that for $i = 1, \dots, N$, the cycle C_i contains a segment S_i joining A and B. If $S_1, \dots, S_N$ are pairwise internally disjoint, then $N < 6480c_1(c_1 + 1)^4$.

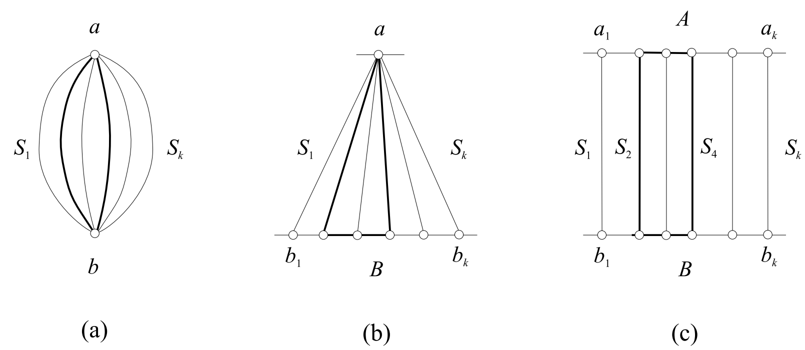

FIG. 4. The Π_0-homotopic segments $S_1, \dots, S_k$

Proof. Suppose that $N \geq 6480c_1(c_1 + 1)^4$. Since Π_0-facial cycles meet properly, $B - V(A)$ consists of at most two facial subsegments of B. Hence, if $2N/5$ or more of the segments end up in $B - V(A)$, then B has a subsegment disjoint from A such that at least $N/5$ of the segments S_i end up in that subsegment. A similar conclusion holds if at least $2N/5$ of the segments end up in $A - V(B)$. Otherwise, at least $N/5$ of the segments start and end in $A \cap B$. Therefore, we may assume that either $A = B$ or $A \cap B = \emptyset$, and that $N \geq 6^4 c_1(c_1 + 1)^4$. We may also assume that the interior vertices of each segment S_i are not in $A \cup B$. For $i = 1, \dots, N$, let $a_i \in A$ and $b_i \in B$ be the ends of S_i, and let $S'_i = S_i - \{a_i, b_i\}$.

By imagining that A and B are contracted to point(s), we may speak of homotopy of the segments S_i. By Lemma 2.2, there is a set I_0 of $6^4(c_1+1)^4$ segments which are Π_0-homotopic. Clearly, there is a subset I_1 of I_0, where

$|I_1| \geq |I_0|^{1/2}$, such that the ends a_i $(i \in I_1)$ are either all distinct or all the same. Similarly, there is a subset I_2 of I_1, where $|I_2| \geq |I_1|^{1/2}$, such that the vertices b_i $(i \in I_2)$ are either all distinct or all the same. Since $|I_2| \geq 6(c_1 + 1)$, we may assume that $S_1, \ldots, S_k$ $(k = 6c_1 + 5)$ are Π_0-homotopic segments, their ends a_i (resp. b_i), $i = 1, \ldots, k$, are either all distinct or all the same, and they are enumerated in the same way as concluded in Lemma 2.1. For $1 \leq i < j \leq k$, let $A_{i,j}$ (resp. $B_{i,j}$) be the segment of A (resp. B) from a_i to a_j (resp. b_i to b_j).

Since S_i and S_j are Π_0-homotopic, $D_{ij} = S_i \cup A_{i,j} \cup S_j \cup B_{i,j}$ is Π_0-contractible. We will denote $\mathrm{Int}(D_{ij}, \Pi_0)$ by $\overline{D}_{ij}$. If $1 < i < j < k$, then D_{ij} is a cycle unless $A_{1,k} = B_{1,k}$ is a single vertex. We have one of the cases shown in Figure 4 where D_{24} is drawn by thicker lines. In the case of Figure 4(a), it is possible that $a = b$.

Suppose that $1 < i < j < k$ and $j \neq i + 1$. Suppose first that $A_{1,k} = \{a\}$ and $B_{1,k} = \{b\}$ are just vertices. Since S_i is a Π_i-facial segment, Proposition 3.2 implies that $G - S_i$ is connected. The same holds for S_j. In particular, this implies that no Π_0-facial walk in $\overline{D}_{1k}$ contains both a and b. Therefore, there is a path $P \subseteq \overline{D}_{1k} - \{a, b\}$ which joins S'_1 and S'_k. No edge connects S'_i and S'_j; such an edge would be either in $\overline{D}_{ij}$ (in which case it would cross S_{i+1}) or not (in which case it would cross S_1), yielding a contradiction in each case. Since S_i and S_j are induced subgraphs of G, no $(S_i \cup S_j)$-bridge in G is just an edge, except possibly the edge ab. Suppose now that Q is an $(S_i \cup S_j)$-bridge in $\overline{D}_{ij}$, and $v \in V(Q) \backslash (S_i \cup S_j)$. If $v \in V(\overline{D}_{i,i+1})$, then there is a path in $G - S_i$ from v to S'_j since $G - S_i$ is connected. Such a path intersects S_{i+1} before it reaches S'_j. Therefore $Q \supseteq S_{i+1}$. A similar argument shows that $Q \supseteq S_{i+1}$ if Q contains a vertex in $\overline{D}_{i+1,j}$. This shows that Q is the only $(S_i \cup S_j)$-bridge in $\overline{D}_{ij}$. If Q is an $(S_i \cup S_j)$-bridge which is not in $\overline{D}_{ij}$, then we similarly see that this is the only such bridge. This shows that there are precisely two or three $(S_i \cup S_j)$-bridges, and if there are three, one of them is just the edge ab, which is not in $\overline{D}_{ij}$.

Suppose now that $B_{1,k}$ is not just a vertex. Then we can use similar arguments as above to prove that there are precisely two $(S_i \cup S_j)$-bridges which are not edges. Also, there are no edges joining $S_i \backslash S_j$ with $S_j \backslash S_i$. Since C_i is an induced cycle, if $a_i b_i \in E(G)$, then $C_i = S_i + a_i b_i$ must lie in $\overline{D}_{1k}$, so C_i would be Π_0-contractible, a contradiction. Similarly for S_j. This shows that there are precisely two $(S_i \cup S_j)$-bridges in G.

Let $i = 3c_1 + 3$. By Lemma 2.2, there are indices $2 \leq p < q < r \leq 3c_1 + 1$ such that S_p, S_q, S_r are Π_i-homotopic. We claim that $\overline{D}'_{pr} := \mathrm{Int}(D_{pr}, \Pi_i) = \overline{D}_{pr}$. Since there are at least two D_{pr}-bridges in G, D_{pr} is not a Π_i-facial cycle. Denote by Q and R the $(S_p \cup S_r)$-bridges in G (distinct from ab) where $Q \subseteq \overline{D}_{pr}$.

If $A_{1,k} = \{a\}$ and $B_{1,k} = \{b\}$ are just vertices and $ab \in E(G)$, then $C_p = S_p + ab$. We already argued above that $ab \notin E(\overline{D}_{pr})$ and, similarly, $ab \notin E(\overline{D}'_{pr})$. Thus, to prove the above claim, it suffices to show that $Q \subseteq \overline{D}'_{pr}$ and that $R \not\subseteq \overline{D}'_{pr}$.

Suppose that $R \subseteq \overline{D}'_{pr}$. Since R contains S_i and since C_i is Π_i-facial, $C_i \subseteq \overline{D}'_{pr}$. If D_{pr} is a cycle, then by using the embeddings of $\overline{D}_{pr}$ and $\overline{D}'_{pr}$, we easily construct an embedding of G of genus 0. This gives a contradiction since planar graphs have no nonplanar embeddings of face-width 3 or more (cf. [13]). This shows that D_{pr} is not a cycle, i.e., we have $A_{1,k} = B_{1,k} = \{a\}$ where $a \in V(G)$. Then the Π_i-noncontractible cycles $C_{i-1} = S_{i-1}$ and $C_{i+1} = S_{i+1}$ are both in R and hence they are Π_i-homotopic. It is easy to see that $C_i \subseteq \overline{D}'_{i-1,i+1} = \overline{D}_{i-1,i+1}$. By Lemma 4.5 (applied on $S = S_i$, $\Pi = \Pi_i$, $\Pi' = \Pi_0$) we see that there is a Π_i-facial cycle C' which intersects C_i in a vertex or an edge disjoint from a and which interlaces with C_i in Π_0. Since $C' \subseteq \overline{D}_{1k}$, C' and C_i have another point of intersection. This contradiction to Proposition 3.1 proves that $R \not\subseteq \overline{D}'_{pr}$.

Suppose now that $Q \not\subseteq \overline{D}'_{pr}$. Then there is a D_{pr}-bridge $Q' \subseteq Q$ which is not in $\overline{D}'_{pr}$. As we proved above, $\overline{D}'_{pr} \subseteq D_{pr} \cup Q$. This implies that $\overline{D}'_{pr}$ contains a Π_i-facial walk D' which contains all vertices of attachment of Q'. This, in particular, implies that Q' is not just an edge. Moreover, each Π_i-facial walk that contains a foot of Q', contains precisely one other foot of Q'. Hence, if f is the number of feet of Q', then there are precisely f Π_i-facial walks $\overline{Q}_1, \ldots, \overline{Q}_f$ containing feet of Q'. Let $Q_j = \overline{Q}_j \cap Q'$, $j = 1, \ldots, f$. Each Q_j intersects D' in its end(s) x, y. If $x \neq y$, then (because $\overline{Q}_j$ and D' meet properly) $xy \in E(D')$ and $\overline{Q}_j = Q_j + xy$. Thus $\overline{Q}_j$ is an induced nonseparating cycle which is contained in the disk $\overline{D}_{pr}$. Therefore, it is also Π_0-facial. It is easy to see that this is not possible for $j = 1, \ldots, f$. This proves that Q' does not exist and hence $\overline{D}'_{pr} = \overline{D}_{pr}$.

Similarly we see that there are Π_i-homotopic segments S_s, S_t, S_u where $i + 1 < s < t < u < 6c_1 + 5$.

Suppose that $A_{1,k} = \{a\}$ and $B_{1,k} = \{b\}$ are just vertices (possibly $a = b$). As proved above, no Π_0-facial walk in $\overline{D}_{1k}$ contains a and b. Therefore S_i has an intermediate vertex in which there are edges on the left side of S_i, and an intermediate vertex with an edge on the right side of S_i in the embedding Π_0. We say that S_i has the *left-right property*.

Suppose now that $B_{1,k}$ is not just a vertex. Then $\overline{D}_{1k}$ is a disk and hence $C_i \not\subseteq \overline{D}_{1k}$. We claim that the segment $\overline{S}_i$ of C_i from a_i through S_i until $S_r \cup S_s$ has the left-right property. If not, then one of the Π_0-facial walks containing an edge e of S_i contains the entire $\overline{S}_i$. In particular, it contains

e and also a vertex of $(S_r \cup S_s)\backslash\{a_i\}$. Clearly, this is not possible, hence the claim.

Since S_i (or $\overline{S}_i$) has the left-right property, Lemma 4.5 (applied on this segment as S, $\Pi = \Pi_i$, $\Pi' = \Pi_0$) shows that there is a Π_i-facial cycle C' which intersects S_i (or $\overline{S}_i$) in an internal vertex or edge x and interlaces with C_i in Π_0. Therefore C' is not Π_0-facial. In particular, C' and C_i are Π_0-noncontractible. This implies that $C' \not\subseteq \overline{D}_{1k}$. (In the case when $A_{1,k} = B_{1,k} = \{a\}$ is the same vertex, $\overline{D}_{1k}$ contains Π_0-noncontractible cycles. Then we use the fact that $C' \cap C_i = x$, so $a \notin V(C')$.) Since $\overline{D}_{pr} = \overline{D}'_{pr}$ and $\overline{D}_{su} = \overline{D}'_{su}$, we have $C' \cap \overline{D}_{pr} \subseteq A \cup B$ and $C' \cap \overline{D}_{su} \subseteq A \cup B$. Since C_i and C' interlace around x in Π_0, we may assume that $C' \cap \overline{D}_{pr} = A_{p,r}$ and $C' \cap \overline{D}_{su} = B_{s,u}$. In particular, neither $A_{1,k}$ nor $B_{1,k}$ is just a vertex. Similarly, $C_i \cap \overline{D}_{pr} = B_{p,r}$ and $C_i \cap \overline{D}_{su} = A_{s,u}$. Now, let S''_i be the segment of C_i from the edge of S_i incident with a_i to a_s (so that $S'_i \cap S''_i = e$). Then S''_i has the left-right property, and Lemma 4.5 shows that there is a Π_i-facial walk C'' which intersects C_i in the interior of S''_i and which interlaces with C_i in Π_0. As above, we conclude that $C'' \cap \overline{D}_{pr} = A_{p,r}$ and $C'' \cap \overline{D}_{su} = B_{s,u}$. Thus, the Π_i-facial cycles C' and C'' do not meet properly. This contradiction completes the proof. ∎

Now, we will apply Lemma 2.3 to prove

Claim 5.5. Let $A_0, \dots, A_{p-1}$ ($p \geq 1$) be pairwise disjoint Π_0-facial segments. Suppose that each of the cycles C_i ($1 \leq i \leq N$) intersects A_0 and leaves A_0 by an edge e_i such that $e_1, \dots, e_N$ are all distinct. Suppose also that $N \geq \varphi(k, 6480c_1(c_1+1)^4)$ where φ is the function of Lemma 2.3. Then there is a cycle C_i and a Π_0-facial segment A_p contained in C_i and disjoint from $A_0, \dots, A_{p-1}$ such that at least $k/(4pc^3) - 1$ other cycles C_j ($1 \leq j \leq N$) intersect and leave C_i in distinct vertices of the segment A_p.

Proof. Let H be the graph obtained from $C_1 \cup \cdots \cup C_N \cup A_0 \cup \cdots \cup A_{p-1}$ by contracting each segment $A_0, \dots, A_{p-1}$ into a vertex and splitting the vertex v_0 which is obtained from A_0 into two vertices v_1, v_2 so that v_1 is incident with $e_1, \dots, e_N$, and v_2 is incident with all other edges incident with v_0 before the splitting. Finally, we suppress all vertices of degree 2. Let $\overline{C}_i$ be the walk in H from v_1 to v_2 corresponding to C_i. Suppose that no $\overline{C}_i$ has more than k vertices of intersection with other walks $\overline{C}_j$. Then we apply Lemma 2.3 to get a set of $6480c_1(c_1+1)^4$ internally disjoint subwalks of the walks $\overline{C}_i$ between v_1 and some vertex v of H. These subwalks determine internally disjoint paths (or cycles) in G joining v_1 and v (or the corresponding Π_0-facial segments). Now, we get a contradiction by applying Claim 5.4.

Therefore, some $\overline{C}_i$ intersects other walks in at least k distinct vertices, and so other cycles intersect and leave C_i in at least $k-p$ distinct vertices disjoint from $A_0,\ldots,A_{p-1}$. The cycle C_i is the union of at most c^2 Π_0-facial segments. Hence, there is a Π_0-facial segment A contained in C_i such that other cycles intersect and leave A in at least $(k-p)/c^2$ distinct vertices. Therefore, at least $(k-p)/c^3$ distinct cycles intersect and leave A in distinct vertices. Since A intersects each Π_0-facial segment A_j $(0 \le j < p)$ at most once, there is a subsegment A_p of A disjoint from $A_0,\ldots,A_{p-1}$ such that at least $(k-p)/(c^3(p+1)) \ge k/(2pc^3)-1$ other cycles C_j intersect and leave A_p in distinct vertices. ▮

Now we have all the main assumptions and main ingredients to conclude the proof of Theorem 5.1. Define the function $\Phi : \mathbb{N} \to \mathbb{N}$ inductively as follows. Set $\Phi(0) = 1$, and for $k \ge 1$, $\Phi(k) = \varphi(2\kappa c^3(\Phi(k-1)+1), 6480c_1(c_1+1)^4)$, where κ is the integer from Claim 5.3, and φ is the function of Lemma 2.3.

Now, assume that $N > 6c_1(c_1+1)\Psi$, where $\Psi = (c^2\Phi(\kappa))^{c^2\Phi(\kappa)}$. Let us first assume that each cycle C_i intersects less than Ψ other cycles. Then there is a subset of $6c_1(c_1+1)$ disjoint cycles. By Lemma 2.2, there is a subset of $6(c_1+1)$ disjoint Π_0-homotopic cycles, say $C_1,\ldots,C_{6c_1+6}$. Similarly to the proof of Claim 5.4, we set $i = 3c_1+3$ and take Π_i-homotopic cycles C_p, C_q, C_r and C_s, C_t, C_u where $1 < p < q < r < i < s < t < u < 6c_1+5$. Clearly, $\mathrm{Int}(C_p, C_r, \Pi_i) = \mathrm{Int}(C_p, C_r, \Pi_0)$ and $\mathrm{Int}(C_s, C_u, \Pi_i) = \mathrm{Int}(C_s, C_u, \Pi_0)$. This shows that every Π_i-facial walk which intersects C_i is contained in $\mathrm{Int}(C_r, C_s, \Pi_0)$. In particular, this holds for the Π_i-facial walk C' obtained by Lemma 4.5 which interlaces with C_i in Π_0, a contradiction.

Suppose now that there is a Π_0-facial segment A_0 in which at least $\Phi(\kappa)$ cycles intersect and leave A_0 using distinct edges. In such a case, let $\mathcal{C}$ be the set of those cycles. We may assume that $\mathcal{C} = \{C_1,\ldots,C_{\Phi(\kappa)}\}$. Now we successively apply Claim 5.5 as follows. By Claim 5.5, one of the cycles, say C_1, contains a Π_0-facial segment A_1 which is disjoint from A_0 and such that at least $\Phi(\kappa-1)$ other cycles $C_j \in \mathcal{C}$ intersect and leave A_1 in distinct vertices. Inductively, we find cycles $C_1,\ldots,C_\kappa$ such that each C_i contains a Π_0-facial segment A_i disjoint from $A_0,\ldots,A_{i-1}$ in which all cycles C_j, $i < j \le \kappa$, intersect in distinct vertices. This is a contradiction to Claim 5.3.

In what it remains, we may assume that C_1 intersects Ψ other cycles and that, for $i = 1,\ldots,N$, the cycles intersecting C_i leave C_i in at most $c^2(\Phi(\kappa)-1)$ distinct edges (since C_i is composed of at most c^2 Π_0-facial segments). Starting with C_1, there is an edge e_1 such that a set $\mathcal{C}_1$ of at least $(\Psi-1)/(c^2(\Phi(\kappa)-1)) \ge \Psi/(c^2\Phi(\kappa))$ cycles leave C_1 through e_1. Let $C_2 \in \mathcal{C}_1$. For each $C_i \in \mathcal{C}_1\backslash\{C_2\}$, let f_i be the first edge after e_1 at which C_i leaves C_2. Then there is an edge e_2 such that $f_i = e_2$ for at least $\Psi/(c^2\Phi(\kappa))^2$ cycles $C_i \in \mathcal{C}_1$. Now we define $\mathcal{C}_2$ as the set of all these cycles C_i. We select $C_3 \in \mathcal{C}_2$, find the next edge e_3, etc. Eventually, we end up with a sequence of cycles $C_1, C_2, \ldots, C_{c\Phi(\kappa)}$. The construction shows that the cycles $C_1, C_2, \ldots, C_{c^2\Phi(\kappa)-1}$ leave $C_{c^2\Phi(\kappa)}$ using distinct edges. This contradiction to the above assumption completes the proof of Theorem 5.1.

6. SOME EXAMPLES

Two embeddings of G are *isomorphic* if there is a homeomorphism of the surface taking the first embedded $G$to the other embedded G, which does not necessarily respect the labeling of the vertices.

If K_n ($n \geq 7$) triangulates a surface S, then every embedding Π of K_n in S is triangular and hence of face-width 3. It is easy to see that precisely $2n(n-1)$ automorphisms of K_n preserve the embedding Π. Therefore, by taking all $n!$ automorphisms of K_n we obtain $\frac{1}{2}(n-2)!$ nonequivalent embeddings of K_n isomorphic to Π. Bonnington et al. [3] constructed, for all values of n congruent to 7 or 19 modulo 36, at least $2^{n^2/54-O(n)}$ nonisomorphic triangular embeddings of K_n in orientable surfaces. This shows that K_n (for these restricted values of n) admits at least $(n-2)!\,2^{n^2/54-O(n)}$ nonequivalent embeddings of face-width 3 in the orientable surface of Euler genus $g = (n-3)(n-4)/6$.

However, unless the number of nonisomorphic triangular embeddings of K_n can be proved to be much larger, there are even better candidates for maximum flexibility of embeddings of face-width 3 in the same surface. Let G_0 be a triangulation of the 2-sphere with at least k facial triangles $T_1, \ldots, T_k, \ldots$. For each T_i ($1 \leq i \leq k$), add a new copy of the complete graph K_7 and identify three of its vertices with the three vertices of T_i. Denote the resulting graph by G_k. Since K_7 has 48 nonequivalent embeddings into the torus such that a fixed triangle is a face, these embeddings used on each of the added graphs result in 48^k distinct embeddings of G_k in the orientable surface S_k of Euler genus $2k$. This shows that $\xi(2g) \geq 48^g$. Similarly we see that $\xi(2g+1) \geq 6 \cdot 48^g$, by using 6 embeddings of K_6 with a fixed facial triangle in the projective plane to get embeddings of odd Euler genus. This is better than the aforementioned bound for K_n if k is large enough.

Although the bounds for $\xi(g)$ in the proof of Theorem 5.1 are enormous, we conjecture that there is a constant C such that $\xi(g) \leq C^g$.

Another interesting aspect of flexibility of embeddings of face-width 3 is the following. For a fixed surface S, there are finitely many graphs without vertices of degree 2 which are embedded with face-width 3 but the removal of every edge gives an embedding of face-width 2 [7] (cf. also [9]). Such

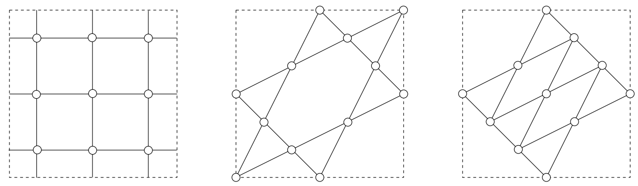

FIG. 5. Three embeddings of the line graph of $K_{3,3}$

graphs (and their embeddings) are said to be *minimal of face-width 3*. If G is embedded with face-width 3 or more, then it contains a subdivision of an embedded graph H which is minimal of face-width 3. It is easy to see that the embedding of H uniquely extends to an embedding of G (if G is 3-connected). Such an observation was used in [5] to show that triangulations of a fixed surface have bounded flexibility. Unfortunately, this does not yield a simple proof of Theorem 5.1 since the subgraph H may have embeddings of smaller face-width or even smaller genus. Figure 5 shows three embeddings of the line graph of $K_{3,3}$ in the torus having face-width 3, 2, and 1, respectively (the first one being minimal of face-width 3).

REFERENCES

1. D. Archdeacon, Densely embedded graphs, J. Combin. Theory Ser. B 54 (1992) 13–36.
2. J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, North-Holland, New York, 1981.
3. P. Bonnington, M. Grannell, T. Griggs, and J. Širáň, Exponential families of non-isomorphic triangulations of complete graphs, J. Combin. Theory Ser. B 78 (2000) 169–184.
4. S. Fisk, B. Mohar, Surface triangulations with isometric boundary, Discrete Math. 134 (1994) 49–62.
5. Z. Gao, L. B. Richmond, C. Thomassen, Irreducible triangulations and triangular embeddings on a surface, Research Report CORR 91–07, University of Waterloo, 1991.
6. M. Juvan, A. Malnič, B. Mohar, Systems of curves on surfaces, J. Combin. Theory Ser. B 68 (1996) 7–22.
7. A. Malnič, R. Nedela, k-Minimal triangulations of surfaces, Acta Math. Univ. Comenian. 64 (1995) 57–76.
8. B. Mohar, Uniqueness and minimality of large face-width embeddings of graphs, Combinatorica 15 (1995) 541–556.
9. B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, MD, 2001.

10. J. Nešetřil, Ramsey theory, Chapter 25 in "Handbook of Combinatorics," Ed. R. Graham, M. Grötschel, and L. Lovász, Elsevier, 1995, pp. 1331–1403.

11. G. Ringel, Der vollständige paare Graph auf nichtorientierbaren Flächen, J. reine angew. Math. 220 (1965) 88–93.

12. C. Thomassen, Embeddings of graphs with no short noncontractible cycles, J. Combin. Theory Ser. B 48 (1990) 155–177.

13. N. Robertson, R. P. Vitray, Representativity of surface embeddings, in: Paths, Flows, and VLSI-Layout (B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver Eds.), Springer-Verlag, Berlin, 1990, pp. 293–328.

14. P. D. Seymour, R. Thomas, Uniqueness of highly representative surface embeddings, J. Graph Theory 23 (1996) 337–349.

15. H. Whitney, 2-isomorphic graphs, Amer. Math. J. 55 (1933) 245–254.