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GRAPH MINORS AND GRAPHS ON SURFACES

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Graph minors and graphs on surfaces[1]

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Abstract

Graph minors and the theory of graphs embedded in surfaces are fundamentally interconnected. Robertson and Seymour used graph minors to prove a generalization of the Kuratowski Theorem to arbitrary surfaces [37], while they also need surface embeddings in their Excluded Minor Theorem [45]. Various recent results related to graph minors and graphs on surfaces are presented.

1 Introduction

A graph H is a *minor* of another graph G if H can be obtained from a subgraph of G by contracting edges.

Graph minors and the theory of graphs embedded in surfaces are fundamentally interconnected. The family of all graphs which are embeddable in a fixed surface $\mathbb{S}$ is closed under taking minors. Therefore the graphs embeddable in $\mathbb{S}$ can be characterized by specifying the list $\mathbf{Forb}_0(\mathbb{S})$ of *minimal forbidden minors*, i.e., minor minimal graphs which do not embed in $\mathbb{S}$. (Similarly, they can be characterized by excluding, as subgraphs, all subdivisions of graphs in the set $\mathbf{Forb}(\mathbb{S})$ which is defined as the set of graphs of minimum degree ≥ 3 which cannot be embedded in $\mathbb{S}$ but all of whose proper subgraphs have embeddings in $\mathbb{S}$.) Robertson and Seymour used graph minors to prove that $\mathbf{Forb}_0(\mathbb{S})$ (and hence also $\mathbf{Forb}(\mathbb{S})$) is finite for every surface $\mathbb{S}$ [37]. This result is a generalization of the Kuratowski Theorem to arbitrary surfaces. On the other hand, Robertson and Seymour needed surface embeddings in their Excluded Minor Theorem [45] where they determine a general structure of graphs which do not have a fixed graph H as a minor. This interplay between the two theories is visible in many other results, some of which are presented here.

The main purpose of this survey is to present up-to-date information on some of the most appealing results about graph minors and their relation to the study of graphs on surfaces.

Besides a stimulating survey article on minors and embeddings by Thomassen [59], there are numerous existing texts that cover this subject. A good introduction to graph minors is Diestel [14, Chapter 12], while excluded minor theorems are treated in Thomas [57]. Graph minors and tree-width are studied in Reed [28], for tree-width and algorithms we refer to [5] and [6]. Embeddings of graphs in surfaces are treated in Mohar and Thomassen [26]; minors and embeddings are also covered in Robertson and Vitray [54]. The proof of the

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Graph Minor Theorem is sketched in Robertson and Seymour [29], and a more recent survey with focus on the related disjoint paths problem is [52].

2 Basic definitions

It is convenient to view minors as substructures. Then, a subgraph $\bar{H}$ of G is said to be an H*-minor* in G if $\bar{H}$ can be written as the union of $r = |V(H)|$ pairwise disjoint trees $T_1, \ldots, T_r$ and $m = |E(H)|$ edges $e_1, \ldots, e_m$ such that for $i = 1, \ldots, m$, the edge e_i joins T_j and T_l if the ith edge of H connects the jth and lth vertex of H. In Figure 1, a graph G with subtrees $T_1, \ldots, T_5$ (represented by thick lines) is exhibited to show that the graph K_5 minus an edge is a minor of G.

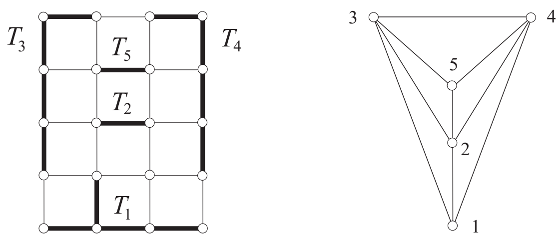

Figure 1: K_5 minus an edge as a minor

A family $\mathcal{F}$ of graphs is *minor closed* if for every graph in $\mathcal{F}$, all its minors are also in $\mathcal{F}$. There are two basic classes of examples of minor closed families. The first class are families related to embeddings in various topological spaces. Such examples include graphs embeddable in a fixed surface, graphs embeddable in $\mathbb{R}^3$ in some specific way, for instance, *linklessly embeddable graphs* [53, 57] (i.e., graphs which admit an embedding in $\mathbb{R}^3$ such that no two disjoint cycles of the graph are linked in $\mathbb{R}^3$), *knotlessly embeddable graphs* (every cycle of the graph is embedded as an unknot), etc. The second important class of minor closed families is related to the tree-width. Classes of both types are discussed below.

Every closed surface is either homeomorphic to the orientable surface $\mathbb{S}_g$ of *genus* $g \geq 0$, or to the nonorientable surface $\mathbb{N}_g$ of *nonorientable genus* $g \geq 1$. Surfaces of the same orientability type can be distinguished by their Euler characteristic, and to unify the genus parameters for the surfaces $\mathbb{S}_g$ and $\mathbb{N}_{2g}$, which have the same Euler characteristic, it is convenient to introduce the *Euler genus* which is defined by $\mathbf{eg}(\mathbb{S}_g) = 2g$ and $\mathbf{eg}(\mathbb{N}_g) = g$. An embedding of a graph G in a surface $\mathbb{S}$ is a *2-cell embedding* if every face is homeomorphic to an open disk in the plane. In that case, the number of faces is equal to $|E(G)| - |V(G)| + 2 - \mathbf{eg}(\mathbb{S})$. This relation is known as Euler's formula.

Embeddings of graphs in surfaces, in particular the 2-cell embeddings, can be represented combinatorially. One such combinatorial description, known as the *Heffter-Edmonds-Ringel representation*, can be taken as a definition, and then one can work with combinatorial embeddings without any reference to topology. We refer to Mohar and Thomassen [26] for a thorough combinatorial treatment of surface embeddings. Following [26], we **define** an *embedding* of a connected graph G as a pair $\Pi = (\pi, \lambda)$ where $\pi = \{\pi_v \mid v \in V(G)\}$ is a collection of *local clockwise rotations*, i.e., π_v is a cyclic permutation of the edges incident with v ($v \in V(G)$), and $\lambda : E(G) \to \{+1, -1\}$ is a *signature*. The local rotation π_v describes the cyclic clockwise order of edges incident with v on the surface, and the signature $\lambda(uv)$ of the edge uv is positive if and only if the cyclic permutations π_u and π_v both correspond to the clockwise (or both to anticlockwise) cyclic order of edges incident with u and v as seen on the surface when traversing the edge uv. An embedding of the graph G is *orientable* if every cycle of G has an even number of edges with negative signature.

The embedding $\Pi = (\pi, \lambda)$ determines a set of Π-*facial walks*. They are determined by the following process, called the *face traversal procedure*. We start with an arbitrary vertex v and an edge $e = vu$ incident with v. Traverse the edge e from v to u. We continue the walk along the edge $e' = \pi_u(e)$ which follows e in the π-clockwise ordering around u. If $\lambda(e) = -1$, the π-anticlockwise rotation is used instead, i.e., $e' = \pi_u^{-1}(e)$. We continue using the π-anticlockwise ordering until the next edge with signature -1 is traversed, and so forth. The walk is completed when the initial edge e is encountered in the same direction from v to u and we are in the same mode (the π-clockwise ordering) which we started with. The other Π-facial walks are determined in the same way by starting with other edges. Two facial walks are considered the same if a cyclic shift of the first one gives rise to the second one or to the reverse of the second walk.

If f is the number of Π-facial walks, then the number

$$\mathbf{eg}(G, \Pi) = 2 - |V(G)| + |E(G)| - f$$

is called the *Euler genus* of the embedding Π. The underlying surface of the embedding Π which is obtained by pasting discs along the facial walks in G has the same Euler genus.

A *tree decomposition* of a graph G is a pair (T, Y), where T is a tree and Y is a family $\{Y_t \mid t \in V(T)\}$ of vertex sets $Y_t \subseteq V(G)$ (called *parts* of the tree decomposition) such that the following two properties hold:

(T1) $\bigcup_{t \in V(T)} Y_t = V(G)$, and every edge of G has both ends in some Y_t.

(T2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The pair (T, Y) is a *path decomposition* if T is a path. The *width* of the tree decomposition (T, Y) is $\max_{t \in V(T)}(|Y_t| - 1)$.

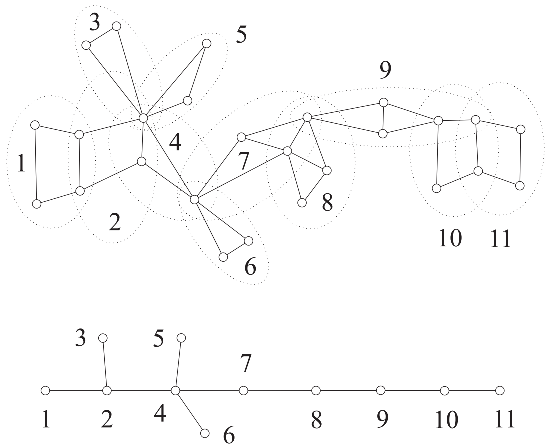

Figure 2: A graph and its tree decomposition of width 3

Figure 2 shows a graph G, a tree decomposition of width 3, and the underlying tree T. Let us observe that the graph G is outerplanar and hence it also has a tree decomposition of width 2.

It was shown in [27] that if a graph G has a tree decomposition of width at most w, then G has a tree decomposition of width at most w that further satisfies:

(T3) For every two vertices t, t' of T and every positive integer k, either there are k disjoint paths in G between Y_t and $Y_{t'}$, or there is a vertex t'' of T on the path between t and t' such that $|Y_{t''}| < k$.

(T4) If t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$.

(T5) If $t_0 \in V(T)$ and B is a component of $T - t_0$, then $\bigcup_{t \in V(B)} Y_t \setminus Y_{t_0} \neq \emptyset$.

The *tree-width* $\mathbf{tw}(G)$ (*path-width*) of a graph G is the smallest width of a tree decomposition (path decomposition) of G.

Let G_1 and G_2 be vertex disjoint graphs and let k be an integer. Suppose that V_i is a k-clique in G_i, and let G_i' be a subgraph of G_i obtained by deleting some (possibly none) of the edges joining pairs of vertices in V_i, $i = 1, 2$. If a graph G is obtained from $G_1' \cup G_2'$ by pairwise identifying the vertices of V_1 with the vertices of V_2, then we say that G is a *k-sum* of G_1 and G_2, or that G is a *clique sum* of G_1 and G_2 of *order* k.

3 The Excluded Minor Theorem

Robertson and Seymour proved that in any infinite sequence $G_1, G_2, G_3, \ldots$ of graphs there are indices $i < j$ such that G_i is a minor of G_j [30]–[51].

This seminal result, which establishes the well-quasi-ordering[2] of graphs with respect to the minor relation, is known as the **Graph Minor Theorem**. In the proof, one may assume (reductio ad absurdum) that none of the graphs $G_2, G_3, \ldots$ contains G_1 as a minor. Robertson and Seymour then prove that these graphs have a special structure. In particular, if G_1 is a forest, then the graphs have bounded path-width [30]. If G_1 is a planar graph, then the graphs have bounded tree-width [34]. It takes a lot of work to reach the **Excluded Minor Theorem** 3.1 [45] which describes the structure of the sequence when a more general graph is an excluded minor. To express this result, an additional definition is needed.

Let G be a graph, $\mathbb{S}$ a surface, and k an integer. We say that G can be *k-nearly embedded* in $\mathbb{S}$ if G has a set A of at most k vertices such that $G - A$ can be written as $G_0 \cup G_1 \cup \cdots \cup G_k$ where the graphs $G_0, G_1, \ldots, G_k$ satisfy the following conditions:

(i) G_0 is embedded in $\mathbb{S}$.

(ii) The graphs $G_1, \ldots, G_k$ are pairwise disjoint.

(iii) For $i = 1, \ldots, k$, let $U_i = \{u_1^{(i)}, u_2^{(i)}, \ldots, u_{r_i}^{(i)}\} := V(G_0) \cap V(G_i)$. Then G_i has a path decomposition $(P_{r_i}, Y^{(i)})$ of width $\leq k$ such that for $t = 1, \ldots, r_i$, $Y_t^{(i)} \cap U_i = \{u_t^{(i)}\}$.

(iv) There are (not necessarily distinct) faces $F_1, \ldots, F_k$ of G_0 in $\mathbb{S}$, and there are pairwise disjoint disks $D_1, \ldots, D_k$ in $\mathbb{S}$, such that for $i = 1, \ldots, k$, $D_i \subset F_i$, $D_i \cap G_0 = U_i$, and the cyclic order of vertices in U_i on the boundary of D_i is $u_1^{(i)}, u_2^{(i)}, \ldots, u_{r_i}^{(i)}$.

Theorem 3.1 (Robertson and Seymour [45]) *For every graph H there exists an integer $k \geq 0$ such that every graph which does not contain H as a minor can be obtained by clique sums of order $\leq k$ from graphs that can be k-nearly embedded in some surface, in which H cannot be embedded.*

The main application of this impressive result is the proof of the Graph Minor Theorem by Robertson and Seymour. As Theorem 3.1 is very general and has not appeared in print till very recently, not many other applications are known. Two such examples, Theorems 3.2 and 3.7 below, have been obtained recently.

Theorem 3.2 (Böhme, Maharry, and Mohar [8, 9]) *For every positive integer k there exists an integer $N = N(k)$ such that every 7-connected graph of order at least N contains $K_{3,k}$ as a minor.*

[2] A *well-quasi-ordering* of a set X is a reflexive and transitive relation $\preceq$ such that, for every infinite sequence $x_1, x_2, x_3, \ldots$ of elements of X, there are indices i and j such that $i < j$ and $x_i \preceq x_j$.

Theorem 3.2 is sharp in the sense that the 7-connectivity condition cannot be relaxed. There are arbitrarily large 6-connected graphs which can be embedded on the torus. Since $K_{3,7}$ cannot be embedded in the torus, none of these graphs contains $K_{3,7}$ as a minor. The following construction [8] gives arbitrarily large graphs of tree-width $3a - 1$ none of which contain a $K_{a,2a+1}$-minor. Let $m \geq 4$ and $a \geq 3$ be integers, and let $N_{m,a}$ be the graph with vertices $v_{x,y}$ where $1 \leq x \leq m$ and $1 \leq y \leq a$, in which the vertex $v_{x,y}$ is adjacent to another vertex $v_{w,z}$ if and only if $w \in \{x-1, x, x+1\}$ where $x \pm 1$ is considered modulo m.

Theorem 3.3 (Böhme, Maharry, and Mohar [8]) *There is a function $c : \mathbb{N} \to \mathbb{N}$ such that for any $a \geq 3$ the following holds. For any positive integers k and w there exists a constant $N = N(k, w)$ such that every $c(a)$-connected graph of tree-width less than w and of order at least N contains $K_{a,k}$ as a minor.*

Böhme, Maharry, and Mohar [8] conjectured the following extensions:

Conjecture 3.4 *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that any 9-connected graph on at least $f(k)$ vertices contains a $K_{4,k}$-minor.*

Conjecture 3.5 *There are functions $f : \mathbb{N} \to \mathbb{N}$ and $c : \mathbb{N} \to \mathbb{N}$ such that any $c(a)$-connected graph on at least $f(k)$ vertices contains a $K_{a,k}$-minor.*

In [8] it is remarked that the sequence of graphs $K_{a,k}$, where a is fixed and k tends to infinity, is essentially the only family of graphs for which a result like Theorem 3.2 or 3.3 holds. More precisely:

Proposition 3.6 *Let c and $w \geq c$ be positive integers, and let H_k $(k \geq 1)$ be a sequence of graphs such that $\lim_{k\to\infty} |V(H_k)| = \infty$. Suppose that for any positive integer k there exists an integer $N(k)$ such that every c-connected graph of tree-width $\leq w$ and of order at least $N(k)$ contains H_k as a minor. Then H_k is a minor of $K_{c,N(k)}$ for $k \geq 1$.*

Proof Clearly, the graph $K_{c,N(k)}$ is c-connected and has tree-width $c \leq w$. By the assumption on the family H_k, $K_{c,N(k)}$ contains H_k as a minor. □

Böhme, Mohar, and Reed [10] showed that Theorem 3.2 can be strengthened by modifying the connectivity assumptions. Recall that a connected graph G is *t-tough* if for every separating vertex set S, the subgraph $G - S$ of G has at most $|S|/t$ connected components.

If d and k are positive integers, then P_k^d denotes the dth power of the path on k vertices, i.e., distinct vertices v_i and v_j of P_k^d are adjacent if and only if $|j - i| \leq d$.

Theorem 3.7 (Böhme, Mohar, and Reed [10]) *For any positive integers d and k there exist numbers $t = t(d)$ and $N = N(k, d)$ such that every t-tough graph of order at least N contains P_k^d as a minor.*

4 Excluded minors for a fixed surface

One of the highlights in the Robertson-Seymour theory on graph minors is the proof of the finiteness (for each fixed surface $\mathbb{S}$) of the set $\mathbf{Forb}_0(\mathbb{S})$ of the minimal forbidden minors for $\mathbb{S}$.

Theorem 4.1 (Robertson and Seymour [37]) *For each surface $\mathbb{S}$, the set $\mathbf{Forb}_0(\mathbb{S})$ of minimal forbidden minors is finite.*

Unfortunately, the complete list of graphs in $\mathbf{Forb}_0(\mathbb{S})$ is known only for the 2-sphere, where $\mathbf{Forb}_0(\mathbb{S}_0) = \{K_5, K_{3,3}\}$, and for the projective plane $\mathbb{N}_1$, where there are precisely 35 minimal forbidden minors [18, 1].

The original proof of Theorem 4.1 by Robertson and Seymour is nonconstructive in the sense that it does not provide a bound on the number or the size of graphs in $\mathbf{Forb}_0(\mathbb{S})$. A constructive proof for the case of nonorientable surfaces was obtained by Archdeacon and Huneke [4], while the first constructive proof for orientable surfaces appeared just recently (Mohar [25]). An independent constructive proof based on graph minors was also obtained by Seymour [55]. Seymour's bound on the size of graphs in $\mathbf{Forb}_0(\mathbb{S})$ is $2^{2^{(3g+9)^9}}$, where g is the Euler genus of $\mathbb{S}$. This number is enormous already for the torus and the Klein bottle ($g = 2$). Even today, it remains a challenge to verify the following

Conjecture 4.2 *Every minimal forbidden minor for the torus has less than 30 vertices.*

In the late 90's, Thomassen observed the possibility of obtaining a short proof of Theorem 4.1. He found a very short proof of the following result.

Theorem 4.3 (Thomassen [60]) *Let $G \in \mathbf{Forb}(\mathbb{S}_g)$. Then G contains no $k \times k$ grid as a minor, where $k = \lceil 3300g^{3/2} \rceil$.*

Theorem 4.3 implies Theorem 4.1 when combined with two other important results in the Robertson-Seymour theory, that graphs of large tree-width contain large grid minors [34], and that graphs of bounded tree-width are well-quasi-ordered [33]. For the former of these two results, a short proof with constructive bounds was obtained by Diestel, Gorbunov, Jensen, and Thomassen.

Theorem 4.4 (Diestel, Gorbunov, Jensen, Thomassen [15]) *Let r, m be positive integers, and let G be a graph of tree-width at least $r^{4m^2(r+2)}$. Then G contains either K_m or the $r \times r$ grid as a minor.*

The second result, the well-quasi-ordering of graphs of bounded tree-width, was proved by Robertson and Seymour in [33]. The proof is lengthy and technical as it provides general machinery for the graph minor theory. A shorter direct proof of this result was recently obtained by Geelen, Gerards and Whittle [17]. In the sequel we give a new, much simpler proof of this result restricted to graphs in $\mathbf{Forb}_0(\mathbb{S})$.

Theorem 4.5 *Let g and w be positive integers and let $\mathbb{S}$ be a surface of Euler genus g. Then there is an integer N such that every graph in $\mathbf{Forb}_0(\mathbb{S})$ with tree-width $< w$ has at most N vertices.*

Theorem 4.5 combined with Theorems 4.3 and 4.4 clearly implies Theorem 4.1. Theorem 4.3 is stated for orientable surfaces only but it is not difficult to extend its proof to include the nonorientable case as well.

Proof Suppose that $S \subseteq V(G)$. An S-bridge in G is a subgraph of G which is either an edge with both ends in S or a connected component C of $G - S$ together with all edges joining C with S. We start the proof by establishing some facts about bridges of embedded graphs.

Suppose that x, y is a separating pair of vertices of a graph G. An $\{x, y\}$-bridge B is said to be *nonplanar* if $B + xy$ is a nonplanar graph.

(1) If $G \in \mathbf{Forb}_0(\mathbb{S})$, then every $\{x, y\}$-bridge containing at least two edges is nonplanar.

This is easy to argue since the replacement of a nontrivial planar $\{x, y\}$-bridge by the edge xy would give a proper minor of G but would not decrease the genus of the graph.

Suppose that S is a vertex separating set of a connected graph G which is Π-embedded in $\mathbb{S}$. Let $W = v_1e_1v_2e_2 \dots v_ke_kv_1$ be a Π-facial walk. A triple $e_{i-1}v_ie_i$ in W (including the triple $e_kv_1e_1$) is called a *mixed angle* if the edges e_{i-1} and e_i belong to distinct S-bridges in G. Let R be the multigraph embedded in $\mathbb{S}$ obtained by joining vertices of consecutive mixed angles in the Π-facial walks. Then $G \cup R$ has an embedding $\tilde{\Pi}$ in $\mathbb{S}$ which extends the embedding Π. Consider the induced embedding Π^R of R in $\mathbb{S}$. Let us observe that this embedding is not always 2-cell.

(2) The faces of Π^R in $\mathbb{S}$ can be partitioned into two classes, $\mathcal{F}_A$ and $\mathcal{F}_B$, such that every edge of R is incident with a face in $\mathcal{F}_A$ and a face in $\mathcal{F}_B$. The faces in $\mathcal{F}_A$ are 2-cells and correspond to the faces of G with mixed angles. The faces in $\mathcal{F}_B$ and the S-bridges in G which are $\tilde{\Pi}$-embedded in these faces are in bijective correspondence.

The existence of the partition $\mathcal{F}_A \cup \mathcal{F}_B$ is obvious. Let $F \in \mathcal{F}_B$. The boundary of F in $\mathbb{S}$ is composed of one or more closed walks in R. Let e be an edge on one of them, joining vertices v_i and v_j $(i < j)$ of the Π-facial walk W. Since $e_{i-1}v_ie_i$ and $e_{j-1}v_je_j$ are consecutive mixed angles on W, all edges $e_i, e_{i+1}, \dots, e_{j-1}$ belong to the same S-bridge B. Consider the local clockwise rotation of $\tilde{\Pi}$ around v_j. We may assume that e is followed by e_{j-1}. Then e_{j-1} is followed by some other edges of B (possibly none) until a mixed angle in some face is reached, in which case an edge e' of R would follow the edges of B. Clearly, e' follows e on the boundary of F. By using the same argument at e', etc., we see that the edges of G entering the face F at the considered component of the boundary of F all belong to the same S-bridge B. If the face F has another boundary component, it must be incident with the same bridge; otherwise the embedding of G would not be 2-cell. Clearly, every S-bridge lies in a single face of R. This completes the proof of (2).

(3) Let G be a connected graph and $S \subseteq V(G)$ a separating set such that no vertex of S is a cutvertex and for any two vertices $x, y \in S$, every $\{x, y\}$-bridge containing at least two edges is nonplanar. If G is embedded in $\mathbb{S}$, and $s = |S|$ then

$$|E(R)| \leq 6g + s^2 + 5s - 12. \tag{4.1}$$

Let $q = |E(R)|$. Since S contains no cutvertices, no facial walk of R has length 1. If a facial walk corresponding to a 2-cell face in $\mathcal{F}_B$ has length 2, then the corresponding S-bridge in that face is planar, hence just an edge joining two vertices of S. The number of such faces is $\leq \binom{s}{2}$. By (2), the sum of the lengths of faces in $\mathcal{F}_B$ is q. This implies that $2\binom{s}{2} + 3(|\mathcal{F}_B| - \binom{s}{2}) \leq q$, hence $3|\mathcal{F}_B| \leq q + \binom{s}{2}$. Similarly, the sum of the lengths of faces in $\mathcal{F}_A$ is q. Therefore, $|\mathcal{F}_A| \leq q/2$. Now, Euler's formula implies:

$$2 - g \leq s - q + |\mathcal{F}_A| + |\mathcal{F}_B| \leq s - \frac{q}{6} + \frac{1}{3}\binom{s}{2},$$

which yields (4.1).

After these preliminary results, we are ready for the proof of Theorem 4.5. Suppose that $G \in \mathbf{Forb}_0(\mathbb{S})$ and that $\mathbf{tw}(G) < w$. By the additivity of the genus (and using induction on g), we may assume that G is 2-connected. Let (T, Y) be a tree decomposition of G of width $< w$ such that (T4)–(T5) hold. Let $S = Y_t$ be a vertex separating set in G. By contracting an edge in one of the S-bridges, a graph embeddable in $\mathbb{S}$ is obtained. Claims (1)–(3) and the upper bound on $|\mathcal{F}_B|$ in the proof of (3) show that there are $\leq d := 2g + 2w + \binom{w}{2} - 4$ S-bridges in G. (T2) and (T5) imply that every vertex of the tree T has degree $\leq d$. By (T1), $|V(T)| \geq \frac{|V(G)|}{w}$. So, assuming G may have as many vertices as we like, T contains a path which is as long as we like. Applying Menger's theorem and the pigeonhole principle to the longest path in T and its subpaths one or more (but at most w) times, one can conclude that there exists an integer $s \leq w$ and there exist separating sets $S_0, \ldots, S_r$ (where r is as large as we want) such that the following hold:

(i) $|S_i| = s$, $i = 0, \ldots, r$.

(ii) There exist disjoint paths $P_1, \dots, P_s$ from S_0 to S_r which intersect $S_0, S_1, \dots, S_r$ in that order.

(iii) The path P_1 is *everywhere nontrivial* [8], i.e., P_1 has an edge e_i strictly between its intersection with S_{i-1} and S_i, $i = 1, \dots, r$.

For $i = 1, \dots, r$, let G_i be the graph obtained from G by contracting the edge e_i of P_1. Let Π_i be an embedding of G_i in $\mathbb{S}$, and let R_i be the corresponding graph on vertices of the mixed angles in Π_i with respect to the separator S_i of G_i. Since every vertex of S_i is incident with at least two S_i-bridges, $V(R_i) = S_i =: \{u_1^i, \dots, u_s^i\}$, where $u_l^i \in V(P_l)$, $l = 1, \dots, s$.

For $i = 1, \dots, r-1$, let $B^{(i)}$ be the S_i-bridge in G_i which contains the segment of P_1 from S_0 to S_i. Note that $B^{(i)}$ is obtained from the S_i-bridge $B_0^{(i)}$ in G_i containing the same segment of P_1 by contracting the edge e_i.

Let Π_i^R be the embedding of R_i in $\mathbb{S}$. We say that (R_i, Π_i^R) is *strongly homeomorphic* to (R_j, Π_j^R) if there is a homeomorphism $\mathbb{S} \to \mathbb{S}$ whose restriction to R_i induces an isomorphism of the Π_i^R-embedded graph R_i onto the Π_j^R-embedded graph R_j such that $u_l^i \mapsto u_l^j$, $l = 1, \dots, s$, and such that the face of R_i corresponding to the bridge $B^{(i)}$ is mapped onto the face of R_j corresponding to $B^{(j)}$.

Claim (3) combined with the surface classification theorem implies that the number of strong homeomorphism types of pairs (R_i, Π_i^R) is bounded in terms of g and w. As r can be arbitrarily large, there are indices i and $j > i$ such that (R_i, Π_i^R) and (R_j, Π_j^R) are strongly homeomorphic.

Take the embedding Π_i and delete the S_i-bridge $B^{(i)}$. Let F denote the resulting face in $\mathbb{S}$. Since (R_i, Π_i^R) and (R_j, Π_j^R) are strongly homeomorphic, the S_j-bridge $B^{(j)}$ can be embedded in F so that any vertex u_l^j of $B^{(j)}$ is identified with u_l^i $(l = 1, \dots, s)$ on the boundary of F. This gives rise to an embedding in $\mathbb{S}$ of the graph G' which is obtained from $G_i \setminus B^{(i)}$ by adding a disjoint copy of $B^{(j)}$ and identifying each $u_l^i \in V(G_i \setminus B^{(i)})$ with the vertex $u_l^j \in V(B^{(j)})$, $l = 1, \dots, s$. Although $B^{(j)}$ is a bridge in G_j but not a bridge in G, it contains as a minor a copy of the S_i-bridge $B_0^{(i)}$ of G. In order to get $B_0^{(i)}$ as a minor, we contract all edges of the paths P_l $(l = 1, \dots, s)$ between S_i and S_j in the copy of $B^{(j)}$ in G'. Now it is clear that the graph G' contains G as a minor. Since G' is embedded in $\mathbb{S}$, also its minor G admits an embedding in $\mathbb{S}$. This contradiction completes the proof. □

The above proof crystallized as a side result in the search of an efficient algorithm for determining the genus of graphs of bounded tree-width. It turned out that some of the main ingredients in this proof can also be found in the aforementioned work of Seymour [55].

It is well-known that testing planarity [20], constructing embeddings in the sphere $\mathbb{S}_0$ [12], or finding subgraphs that are subdivisions of Kuratowski graphs [62] can be performed by algorithms whose worst case running time is linear. Although the construction of minimum genus embeddings is **NP**-hard (by

Thomassen [58]), Filotti, Miller, and Reif [16] proved that for every fixed surface $\mathbb{S}$, there is a polynomial time algorithm for embedding graphs in $\mathbb{S}$. For every fixed surface $\mathbb{S}$, Robertson and Seymour's theory gives an $O(n^3)$ algorithm for testing embeddability in $\mathbb{S}$ using graph minors [37, 52]. Robertson and Seymour recently improved their $O(n^3)$ algorithms to $O(n^2 \log n)$ [42, 50, 51]. An embeddability testing algorithm can be extended to an algorithm which also constructs an embedding in polynomial time (with estimated complexity $O(n^6)$; see Archdeacon [2]). Mohar [25] (and the papers cited therein) improved these results by showing:

Theorem 4.6 (Mohar [25]) *Let $\mathbb{S}$ be a fixed surface. There is a linear time algorithm that for an arbitrary graph G either:*

(a) *finds an embedding of G in $\mathbb{S}$, or*

(b) *finds a subgraph $K \subseteq G$ which is a subdivision of some graph in* $\mathbf{Forb}(S)$.

A simpler linear time algorithm for embedding graphs in the projective plane is described by Mohar [23], while a simpler algorithm for the torus was developed recently by Juvan and Mohar [21].

5 Surface minors and the face-width

Given a Π-embedded graph G, every minor H of G can be considered as being obtained by deleting edges and contracting edges on the surface, so that the embedding of G determines an embedding Π' of H. In that case we say that the pair (H, Π') is a *surface minor* of (G, Π). If the embeddings Π and Π' are clear from the context, then we also say that H is a surface minor of G.

The grid graphs $P_k \square P_k$ can serve as a generic class for planar graphs in the following sense:

Theorem 5.1 (Robertson and Seymour [34]) *Let G_0 be a plane graph. Then there is an integer k such that G_0 is a surface minor of the $k \times k$ grid $P_k \square P_k$.*

Proof There is a plane graph G_1 with maximum degree 3 such that G_0 is a surface minor of G_1. It is well-known that every planar graph, hence also G_1 has a straight line embedding in the plane. Now, every edge can be modified so that it becomes a polygonal arc whose segments are all vertical or horizontal. Then it is easy to see that, for some large k, the $k \times k$ grid contains a subdivision of G_1. This completes the proof. $\square$

The proof of Theorem 5.1 does not give an explicit bound on the size of the grid. However, it is not difficult to show that the $O(n) \times O(n)$ grid suffices where n is the number of vertices of G_0; see Di Battista, Eades, Tamassia, and Tollis [7] for references.

Let G be a Π-embedded graph. If $\mathbf{eg}(G,\Pi) \geq 1$, the *face-width* $\mathbf{fw}(G,\Pi)$ of Π is the smallest integer r such that G has a Π-noncontractible cycle which is the union of r paths each of which is contained in a single Π-facial walk. If $g(G,\Pi) = 0$, we let $\mathbf{fw}(G,\Pi) = \infty$.

Theorem 5.1 has the following analogue for general surfaces.

Theorem 5.2 (Robertson and Seymour [36]) *Let G_0 be a graph that is Π_0-embedded in a surface $\mathbb{S} \neq \mathbb{S}_0$. Then there is a constant k such that for any graph G which is Π-embedded in $\mathbb{S}$ with face-width at least k, (G_0,Π_0) is a surface minor of (G,Π).*

Theorem 5.2 does not give explicit bounds on the face-width k that guarantees the presence of (G_0,Π_0) as a surface minor. Quantitative versions for many special cases are known, cf. [26]. Let us consider some of them.

D. Barnette and X. Zha (private communication) proposed the following conjectures.

Conjecture 5.3 (Barnette, 1982) *Every triangulation of a surface of genus $g \geq 2$ contains a noncontractible surface separating cycle.*

Ellingham and Zha (private communication) proved Conjecture 5.3 for triangulations of the double torus.

Conjecture 5.4 (Zha, 1991) *Every graph embedded in a surface of genus $g \geq 2$ with face-width at least 3 contains a noncontractible surface separating cycle.*

It follows from Theorem 5.2 that large face-width forces the existence of noncontractible surface separating cycles (where "large" may depend on the surface). Zha and Zhao [63] and Brunet, Mohar, and Richter [11] proved that face-width 6 (even 5 for nonorientable surfaces) is sufficient.

If Conjecture 5.3 is true, also the following may hold as suggested in Mohar and Thomassen [26].

Conjecture 5.5 *Let T be a triangulation of an orientable surface of genus g, and let h be an integer such that $1 \leq h < g$. Then T contains a surface separating cycle C such that the two surfaces separated by C have genera h and $g-h$, respectively.*

It is even possible that Conjecture 5.5 extends to all embeddings of face-width at least 3.

Suppose that the embedding of the graph G_0 in Theorem 5.2 is a minimum genus embedding. If G_0 is a surface minor of another embedded graph G (in the same surface), then also the embedding of G is a minimum genus embedding. Therefore, a consequence of Theorem 5.2 is that large face-width of an embedding implies that this is a minimum genus embedding.

Suppose now that G_0 is uniquely embeddable in $\mathbb{S}$ and that its embedding has face-width at least three. (Such graphs are easy to find.) If G is a 3-connected graph embedded in $\mathbb{S}$ such that G_0 is a surface minor of G, then also the embedding of G in $\mathbb{S}$ is unique. Consequently, sufficiently large face-width of a 3-connected graph implies uniqueness of the embedding. Both of theses results are treated in Seymour and Thomas [56] and Mohar [24] who proved that face-width of order $O(g \log g)$ ($g = \mathbf{eg}(G, \Pi)$) is sufficient, and this is essentially best possible (Archdeacon [3]).

There are numerous other results where Theorem 5.2 is used. However, the most surprising seems to be the flow-coloring duality on general surfaces discovered recently by Devos, Goddyn, Mohar, Vertigan, and Zhu [13]. The requirement is that the *edge-width* (which is defined as the length of a shortest noncontractible cycle on the surface) is large enough.

Let G be a 2-connected multigraph. The *circular flow number* $\phi_c(G)$ of G is the minimum real number r such that some orientation of G admits a real-valued flow whose absolute values all lie between 1 and $r - 1$. It is easy to see that $\lceil \phi_c(G) \rceil$ is the usual flow number, i.e., the smallest integer k such that G admits a nowhere-zero k-flow.

Let G be a loopless multigraph. The *circular chromatic number* $\chi_c(G)$ is the smallest real number r such that there exists a real-valued function $c : V(G) \to [0, r)$ such that for every edge uv of G, $1 \le |c(u) - c(v)| \le r - 1$. We refer to the recent survey article by Zhu [64] for additional details on circular colorings and flows.

Theorem 5.6 (Devos, Goddyn, Mohar, Vertigan, Zhu [13]) *There exists a function $w : \mathbb{R}^+ \times \mathbb{N} \to \mathbb{N}$ such that the following holds. If $\varepsilon > 0$ is a real number and G is a graph embedded in the orientable surface of genus g with edge-width $\ge w(\varepsilon, g)$, then*

$$\chi_c(G) - \varepsilon \le \phi_c(G^*) \le \chi_c(G),$$

where G^* is the geometric dual graph of G in $\mathbb{S}$.

Proof (sketch). The second inequality can be proved in the same way as the well-known flow-coloring duality result of Tutte [61], and so we sketch only the proof of the first inequality.

Suppose that G is a graph embedded in $\mathbb{S}_g$ and that its dual graph G^* admits a circular r-flow. If the edge-width of G is w, there is a graph $\tilde{G}$ in $\mathbb{S}_g$ which contains G as an induced subgraph such that $\mathbf{fw}(\tilde{G}) = w$. Moreover, $\tilde{G}$ can be chosen in such a way that the circular r-flow of G^* extends to a circular r-flow φ of $\tilde{G}^*$. If w is large enough, then by Theorem 5.2, $\tilde{G}$ contains cycles $C_1, \ldots, C_g$ such that after cutting the surface along these cycles (and pasting discs on the resulting holes), one obtains $g + 1$ surfaces, one homeomorphic to the sphere, all others homeomorphic to the torus such that each C_i corresponds to a face in the sphere and to a face in the ith torus. Moreover, we may assume that the face-width of all the torus embeddings is as large as we may need in the sequel. Let $G_0, G_1, \ldots, G_g$ be the corresponding graphs (where G_0 is the planar one), and let $G_0^*, G_1^*, \ldots, G_g^*$ be their dual graphs.

Fix an $i \in \{1, \ldots, g\}$. Since C_i is a surface separating cycle of $\tilde{G}$, the edges of $\tilde{G}^*$ dual to $E(C_i)$ form a cut in $\tilde{G}^*$. Therefore, their φ-sum is equal to 0. This implies that the restriction φ_i of φ in G_i^* is a circular r-flow in G_i^*.

Similarly, the restriction φ_0 of φ to G_0^* is a circular r-flow. Since G_0 is a plane embedding, the circular flow-coloring duality [64] shows that there is a circular $(r+\varepsilon)$-coloring c_0 of G_0 which is dual to the circular $(r+\varepsilon)$-flow $\frac{r+\varepsilon}{r}\varphi_0$.

As the face-width of G_i is large enough, Theorem 5.2 can be used to show that the toroidal $q \times q$ grid R_q is a surface minor in G_i, where $q = \lceil 2r^2/\varepsilon \rceil$. (As proved by Graaf and Schrijver [19], it is sufficient that the face-width is $\geq \frac{3}{2}q + 3$.) The toroidal grid consists of pairwise disjoint "vertical" cycles $A_1, \ldots, A_q$ and pairwise disjoint "horizontal" cycles $B_1, \ldots, B_q$. Let D_{kl} be the disk between A_k, A_{k+1}, B_l, and B_{l+1} (indices modulo q). By taking a slightly larger grid and omitting its part intersecting C_i, we may assume that C_i is disjoint from the grid.

Let D be the plane graph obtained by cutting G_i along A_1 and B_1. The flow φ gives rise to a circular r-flow in the planar dual of D. By the circular flow-coloring duality in the plane [64], there is a circular r-coloring c of D which is dual to φ.

Denote by α the φ-sum (mod r) of the edges dual to $E(A_1)$ (all considered to be oriented so that they cross A_1 from "left" to "right"). By choosing the direction of A_1, we may assume that $\alpha < r/2$. Similarly, we may assume that $\beta < r/2$, where β is the φ-sum (mod r) corresponding to B_1 (or to any B_l). It is not difficult to see that the following assignment defines a circular $(r+\varepsilon)$-coloring c_i of G_i:

$$c_i(v) := \frac{r+\varepsilon}{r}\Big(\Big(c(v) - \frac{(k-1)\beta}{q} - \frac{(l-1)\alpha}{q}\Big) \bmod r\Big)$$

if v is a vertex of D_{kl} which is not in $A_{k+1} \cup B_{l+1}$. (Recall that $x \bmod r$ is defined as $x - \lfloor \frac{x}{r} \rfloor r$ and that $0 \leq x \bmod r < r$.)

Observe that the coloring c_i is dual to the circular $(r+\varepsilon)$-flow $\varphi' := \frac{r+\varepsilon}{r}\varphi$ on all edges which are not part of the $q \times q$ grid in G_i. In particular, this is satisfied on the edges of C_i. Therefore, we may assume that c_i coincides on C_i with c_0 (by possibly replacing c_i with its cyclic shift). Then, the combination of circular $(r+\varepsilon)$-colorings $c_0, c_1, \ldots, c_g$ gives rise to a circular $(r+\varepsilon)$-coloring of G. □

References

[1] D. Archdeacon, A Kuratowski theorem for the projective plane, J. Graph Theory 5 (1981) 243–246.

[2] D. Archdeacon, The complexity of the graph embedding problem, in "Topics in Combinatorics and Graph Theory," R. Bodendiek and R. Henn (Eds.), Physica-Verlag, Heidelberg, 1990, pp. 59–64.

[3] D. Archdeacon, Densely embedded graphs, J. Combin. Theory Ser. B 54 (1992) 13–36.

[4] D. Archdeacon, P. Huneke, A Kuratowski theorem for nonorientable surfaces, J. Combin. Theory Ser. B 46 (1989) 173–231.

[5] Efficient algorithms and partial k-trees, Edited by S. Arnborg, S. Hedetniemi and A. Proskurowski, Discrete Appl. Math. 54 (1994), pp. i–ii and 97–280.

[6] S. Arnborg, A. Proskurowski, Linear time algorithms for NP-hard problems restricted to partial k-trees, Discrete Appl. Math. 23 (1989) 11–24.

[7] G. Di Battista, P. Eades, R. Tamassia, I. G. Tollis, Algorithms for drawing graphs: An annotated bibliography, Comput. Geom. 4 (1994) 235–282.

[8] T. Böhme, J. Maharry, and B. Mohar, $K_{a,k}$ minors in graphs of bounded tree-width, submitted.

[9] T. Böhme, J. Maharry, and B. Mohar, Unavoidable minors in large 7-connected graphs, in preparation.

[10] T. Böhme, B. Mohar, and B. Reed, Unavoidable minors in large t-tough graphs, in preparation.

[11] R. Brunet, B. Mohar, R. B. Richter, Separating and nonseparating disjoint homotopic cycles in graph embeddings, J. Combin. Theory Ser. B 66 (1996) 201–231.

[12] N. Chiba, T. Nishizeki, S. Abe, T. Ozawa, A linear algorithm for embedding planar graphs using PQ-trees, J. Comput. System Sci. 30 (1985) 54–76.

[13] M. Devos, L. Goddyn, B. Mohar, D. Vertigan, and X. Zhu, Duality of circular colorings and flows in orientable surfaces, in preparation.

[14] R. Diestel, Graph theory, Second edition, Springer-Verlag, New York, 2000.

[15] R. Diestel, T. R. Jensen, K. Y. Gorbunov, and C. Thomassen, Highly connected sets and the excluded grid theorem, J. Combin. Theory, Ser. B 75 (1999) 61–73.

[16] I. S. Filotti, G. L. Miller, J. Reif, On determining the genus of a graph in $O(v^{O(g)})$ steps, in "Proc. 11th Ann. ACM STOC," Atlanta, Georgia (1979) pp. 27–37.

[17] J. F. Geelen, A. M. H. Gerards, and G. Whittle, Branch width and well-quasi-ordering in matroids, manuscript, 2000.

[18] H. H. Glover, J. P. Huneke, C.-S. Wang, 103 graphs that are irreducible for the projective plane, J. Combin. Theory Ser. B 27 (1979) 332–370.

[19] M. de Graaf, A. Schrijver, Grid minors of graphs on the torus, J. Combin. Theory Ser. B 61 (1994) 57–62.

[20] J. E. Hopcroft, R. E. Tarjan, Efficient planarity testing, J. Assoc. Comput. Mach. 21 (1974) 549–568.

[21] M. Juvan, B. Mohar, A simplified algorithm for embedding graphs in the torus, preprint.

[22] A. Malnič, B. Mohar, Generating locally cyclic triangulations of surfaces, J. Combin. Theory Ser. B 56 (1992) 147–164.

[23] B. Mohar, Projective planarity in linear time, J. Algorithms 15 (1993) 482–502.

[24] B. Mohar, Uniqueness and minimality of large face-width embeddings of graphs, Combinatorica 15 (1995) 541–556.

[25] B. Mohar, A linear time algorithm for embedding graphs in an arbitrary surface, SIAM J. Discrete Math. 12 (1999) 6–26.

[26] B. Mohar, C. Thomassen, Graphs on Surfaces, Johns Hopkins University Press, Baltimore, 2001.

[27] B. Oporowski, J. Oxley, R. Thomas, Typical subgraphs of 3- and 4-connected graphs, J. Combin. Theory Ser. B 57 (1993) 239–257.

[28] B. A. Reed, Tree width and tangles: a new connectivity measure and some applications, in: "Surveys in Combinatorics 1997," Ed. R. A. Bailey, Cambridge Univ. Press, Cambridge, 1997, pp. 87–162.

[29] N. Robertson, P. D. Seymour, Graph minors — a survey, in "Surveys in Combinatorics 1985," Ed. I. Anderson, Cambridge Univ. Press, Cambridge, 1985, pp. 153–171.

[30] N. Robertson, P. D. Seymour, Graph minors. I. Excluding a forest, J. Combin. Theory Ser. B 35 (1983) 39–61.

[31] N. Robertson, P. D. Seymour, Graph minors. II. Algorithmic aspects of tree-width, J. Algorithms 7 (1986) 309–322.

[32] N. Robertson, P. D. Seymour, Graph minors. III. Planar tree-width, J. Combin. Theory Ser. B 36 (1984) 49–64.

[33] N. Robertson, P. D. Seymour, Graph minors. IV. Tree-width and well-quasi-ordering, J. Combin. Theory Ser. B 48 (1990) 227–254.

[34] N. Robertson, P. D. Seymour, Graph minors. V. Excluding a planar graph, J. Combin. Theory Ser. B 41 (1986) 92–114.

[35] N. Robertson, P. D. Seymour, Graph minors. VI. Disjoint paths across a disc, J. Combin. Theory Ser. B 41 (1986) 115–138.

[36] N. Robertson, P. D. Seymour, Graph minors. VII. Disjoint paths on a surface, J. Combin. Theory Ser. B 45 (1988) 212–254.

[37] N. Robertson, P. D. Seymour, Graph minors. VIII. A Kuratowski theorem for general surfaces, J. Combin. Theory Ser. B 48 (1990) 255–288.

[38] N. Robertson, P. D. Seymour, Graph minors. IX. Disjoint crossed paths, J. Combin. Theory Ser. B 49 (1990) 40–77.

[39] N. Robertson, P. D. Seymour, Graph minors. X. Obstructions to tree-decomposition, J. Combin. Theory Ser. B 52 (1991) 153–190.

[40] N. Robertson, P. D. Seymour, Graph minors. XI. Circuits on a surface, J. Combin. Theory Ser. B 60 (1994) 72–106.

[41] N. Robertson, P. D. Seymour, Graph minors. XII. Distance on a surface, J. Combin. Theory Ser. B 64 (1995) 240–272.

[42] N. Robertson, P. D. Seymour, Graph minors. XIII. The disjoint paths problem, J. Combin. Theory Ser. B 63 (1995) 65–110.

[43] N. Robertson, P. D. Seymour, Graph minors. XIV. Extending an embedding, J. Combin. Theory Ser. B 65 (1995) 23–50.

[44] N. Robertson, P. D. Seymour, Graph minors. XV. Giant steps, J. Combin. Theory Ser. B 68 (1996) 112–148.

[45] N. Robertson, P. D. Seymour, Graph minors. XVI. Excluding a non-planar graph, preprint, 1986.

[46] N. Robertson, P. D. Seymour, Graph minors. XVII. Taming a vortex, J. Combin. Theory Ser. B 77 (1999) 162–210.

[47] N. Robertson, P. D. Seymour, Graph minors. XVIII. Tree-decompositions and well-quasi-ordering, preprint, 1988.

[48] N. Robertson, P. D. Seymour, Graph minors. XIX. Well-quasi-ordering on a surface, preprint, 1989.

[49] N. Robertson, P. D. Seymour, Graph minors. XX. Wagner's conjecture, preprint,1988.

[50] N. Robertson, P. D. Seymour, Graph minors. XXI. Graphs with unique linkages, preprint, 1992.

[51] N. Robertson, P. D. Seymour, Graph minors. XXII. Irrelevant vertices in linkage problems, preprint, 1992.

[52] N. Robertson, P. D. Seymour, An outline of a disjoint paths algorithm, in: "Paths, Flows, and VLSI-Layout," B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver Eds., Springer-Verlag, Berlin, 1990, pp. 267–292.

[53] N. Robertson, P. D. Seymour, R. Thomas, A survey of linkless embeddings, in "Graph Structure Theory (Seattle, WA, 1991)", N. Robertson and P. Seymour Eds., Contemp. Math. 147, Amer. Math. Soc., Providence, RI, 1993, pp. 125–136.

[54] N. Robertson, R. P. Vitray, Representativity of surface embeddings, in: "Paths, Flows, and VLSI-Layout," B. Korte, L. Lovász, H. J. Prömel, and A. Schrijver Eds., Springer-Verlag, Berlin, 1990, pp. 293–328.

[55] P. D. Seymour, A bound on the excluded minors for a surface, submitted.

[56] P. D. Seymour, R. Thomas, Uniqueness of highly representative surface embeddings, J. Graph Theory 23 (1996) 337–349.

[57] R. Thomas, Recent excluded minor theorems for graphs, in "Surveys in Combinatorics 1999," J. D. Lamb and D. A. Preece Eds., Cambridge Univ. Press, Cambridge, 1999, pp. 201–222.

[58] C. Thomassen, The graph genus problem is NP-complete, J. Algorithms 10 (1989) 568–576.

[59] C. Thomassen, Embeddings and minors, Handbook of Combinatorics, Vol. 1, 2, 301–349, Elsevier, Amsterdam, 1995.

[60] C. Thomassen, A simpler proof of the excluded minor theorem for higher surfaces, J. Combin. Theory Ser. B 70 (1997) 306–311.

[61] W. T. Tutte, A contribution to the theory of chromatic polynomials, Canadian J. Math. 6 (1954) 80–91.

[62] S. G. Williamson, Depth-first search and Kuratowski subgraphs, J. Assoc. Comput. Mach. 31 (1984) 681–693.

[63] X. Zha, Y. Zhao, On nonnull separating circuits in embedded graphs, in "Graph Structure Theory (Seattle, WA, 1991)", N. Robertson and P. Seymour Eds., Contemp. Math. 147, Amer. Math. Soc., Providence, RI, 1993, pp. 349–362.

[64] X. Zhu, Circular chromatic number, a survey, Discrete Math., to appear.

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