Coloring Eulerian triangulations of the projective plane

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Abstract

A simple characterization of the 3, 4, or 5-colorable Eulerian triangulations of the projective plane is given.

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A graph is *Eulerian* if all its vertices have even degree. It is well known that Eulerian triangulations of the plane are 3-colorable. However, Eulerian triangulations on other surfaces may have arbitrarily large chromatic number. It is easy to find examples on the projective plane whose chromatic number is equal to 3, 4, or 5, respectively, and it is easy to see that the chromatic number of an Eulerian triangulation of the projective plane cannot be more than 5. In this paper we give a simple characterization of when an Eulerian triangulation of the projective plane is 3, 4, or 5-colorable.

The class of graphs embedded in some surface S such that all facial walks have even length (called *locally bipartite embeddings*) is closely related to Eulerian triangulations of S. Namely, if we insert a new vertex in each of the faces of a locally bipartite embedded graph G, and join it to all vertices on the corresponding facial walk, we obtain an Eulerian triangulation $F(G)$ which contains G as a subgraph. We say that $F(G)$ is a *face subdivision* of G and that the set of added vertices $U = V(F(G)) \setminus V(G)$ is a *color factor* of $F(G)$. Since U is an independent set, $\chi(G) \leq \chi(F(G)) \leq \chi(G) + 1$, where $\chi(\cdot)$ denotes the chromatic number of the corresponding graph.

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Youngs [7] proved that a quadrangulation Q of the projective plane which is not 2-colorable is neither 3-colorable, and its chromatic number is 4. Youngs' proof also implies that in any 4-coloring of a nonbipartite quadrangulation of the projective plane, there is a 4-face with all four vertices of distinct colors. This fact appears in a sligtly extended version (where 4-colorings are replaced by k-colorings, $k \geq 3$) in [5]. For our purpose, a strengthening of that result will be important:

Theorem 1 *Let G be a nonbipartite quadrangulation of the projective plane, and k an integer. If G is k-colored, then there are at least 3 faces of G whose vertices are colored with four distinct colors. In particular, $k \geq 4$.*

Proof. Suppose that G is not bipartite, that it is k-colored, that the set $\mathcal{F}_1$ of *multicolored faces* (i.e. those whose vertices have distinct colors) contains at most two elements, and that $|V(G)|$ is minimum subject to these conditions. Denote by $\mathcal{F}$ the set of all faces which are not in $\mathcal{F}_1$.

Suppose first that G has a facial walk $xyzwx \in \mathcal{F}$ such that x and z have the same color. If $x \neq z$, then we delete the edges xy and xw, and identify x and z. The resulting multigraph is a loopless nonbipartite k-colored quadrangulation of the projective plane with ≤ 2 multicolored faces, a contradiction to the minimality of G.

From now on we may assume that every facial walk in $\mathcal{F}$ has only three (or two) distinct vertices. Again, let $F = xyzwx \in \mathcal{F}$ be a facial walk and assume that $x = z$. Then there is a simple closed curve C in F which has precisely x in common with G and which has y and w on distinct sides. If C is contractible, then x is a cutvertex of G. We choose the notation such that y is in the interior of C. The subgraph of G in the interior of C is bipartite. Now we delete that part of the graph and also remove one of the edges between x and w. The resulting nonbipartite graph contradicts the minimality of G. So, we may assume that C is noncontractible. As no facial walk in $\mathcal{F}$ is a cycle, such a curve C can be chosen in any other face of $\mathcal{F}$ as well. Since the projective plane has no two disjoint noncontractible curves, it follows that any such curve contains the same vertex x and that every edge on a face in $\mathcal{F}$ is incident with x. If $\mathcal{F}_1 = \emptyset$, then every edge of G is incident with x, a contradiction to the assumption that G is nonbipartite. Hence $\mathcal{F}_1 \neq \emptyset$.

Let $F = abcd$ be a face in $\mathcal{F}_1$, where $b, c, d \neq x$. As shown above, the edges bc and cd cannot lie on faces of $\mathcal{F}$. Since every edge is in two facial walks, there is another face $F' \in \mathcal{F}_1$ containing bc and there is a face in $\mathcal{F}_1 \setminus \{F\}$ containing cd. Since $|\mathcal{F}_1| \leq 2$, these two faces are the same. Since $F' \notin \mathcal{F}$, it is a 4-cycle $a'bcd$. This implies that c has degree 2 in G and therefore $G - c$ is a nonbipartite quadrangulation of the projective plane. This contradicts the minimality of G. $\square$

Theorem 1 for a quadrangulation Q implies that the chromatic number of the Eulerian triangulation $F(Q)$ is equal to 5. Theorem 1 also implies that $F(Q)$ is not 5-critical since the removal of any two vertices of degree 4 in $F(Q)$ leaves a graph which is not 4-colorable.

Eulerian triangulations of the projective plane with chromatic number 5 may have arbitrarily large face-width and they show that nonorientable surfaces behave differently than the orientable ones. Namely, Hutchinson, Richter, and Seymour [5] proved that Eulerian triangulations of orientable surfaces of sufficiently large face-width are 4-colorable.

Gimbel and Thomassen [4] observed that Youngs' result [7] implies:

Theorem 2 (Gimbel and Thomassen [4]) *Let G be a graph embedded in the projective plane such that no 3-cycle of G is contractible. Then G is 3-colorable if and only if G does not contain a nonbipartite quadrangulation of the projective plane.*

Our main result will follow from

Proposition 3 (Fisk [3]) *Let G be an Eulerian triangulation of the projective plane. Then G contains a color factor U. In particular, G is a face subdivision of the locally bipartite projective planar graph $G - U$.*

Proof. Choose a face T_0 of G. Let R be the dual cubic graph of G, so that T_0 is one of its vertices. Every walk $W = T_0T_1 \dots T_k$ in the graph R determines a bijection $\sigma(W) : V(T_k) \to V(T_0)$ (where we consider T_i as a face in G and $V(T_i)$ as a subset of $V(G)$). These bijections are defined recursively (depending on k). If $k = 0$, then we set $\sigma(W) = id$; for $k > 0$, $\sigma(W)$ coincides with $\sigma(T_0T_1 \dots T_{k-1})$ on $V(T_{k-1}) \cap V(T_k)$.

For $x \in V(T_0)$, denote by $U(x)$ the set of all vertices of G which are mapped to x by some $\sigma(W)$ where W is a walk in R. The motivation for introducing these bijections is the following obvious fact:

(1) A vertex set $U \subseteq V(G)$ is a color factor in G if and only if U contains precisely one vertex of T_0, say x, and $U = U(x)$. This is further equivalent to the condition that for every closed walk W in R, $\sigma(W)$ fixes x.

Since G is 3-colorable if and only if $V(G)$ can be partitioned into three color factors, (1) implies

(2) G is 3-colorable if and only if $\sigma(W) = id$ for every closed walk W in R.

Suppose that $W = T_0T_1 \dots T_k$ and that $T_{i+1} = T_{i-1}$ for some i, $1 \le i < k$. Then $\sigma(W) = \sigma(W')$ where $W' = T_0 \dots T_{i-1}T_{i+2} \dots T_k$. We say that W' is obtained from W by an *elementary reduction*. Suppose now that $v \in V(G)$ and that $W = T_0T_1 \dots T_i \dots T_j \dots T_k$ is a walk in R such that the triangles $T_i, \dots, T_j$ contain the vertex v. Suppose that $T_i, \dots, T_j, T'_1, \dots, T'_r$ are the triangles in the local order around v. Let $W' = T_0 \dots T_i T'_r T'_{r-1} \dots T'_1 T_j \dots T_k$ be the walk in R which "goes around v in the other direction" than W. We say that W' has been obtained from W by a *homotopic shift* over v. Since v is of even degree, it is easy to see that $\sigma(W') = \sigma(W)$. It is well known that (on every surface) any two homotopic closed walks (where walks in R are considered as closed curves in the surface) can be obtained from each other by a sequence of homotopic shifts and elementary reductions and their inverses (cf., e.g., [6]). This shows that $\sigma(W) = \sigma(W')$ if W and W' are homotopic closed walks in R.

The projective plane has only two homotopy classes of closed walks. One of them contains contractible closed walks. Since they are homotopic to the trivial walk $W_0 = T_0$, we have $\sigma(W) = \sigma(W_0) = id$ for every contractible closed walk W. The other homotopy class contains noncontractible closed walks. Pick one of them, say W_1. Let W_2 be the square of W_1. Then $\sigma(W_2)$ is equal to $\sigma(W_0)$ since W_2 is contractible. Therefore $\sigma(W_1)$ has a fixed vertex, say x. This implies that x is a fixed point of $\sigma(W)$ for every closed walk W in R. By (1), $U(x)$ is a color factor in G. □

The main result of this note is:

Theorem 4 *Let G be an Eulerian triangulation of the projective plane. Then $\chi(G) \leq 5$ and G has a color factor. Moreover, if U is any color factor of G, then:*

(a) $\chi(G) = 3$ *if and only if $G - U$ is bipartite.*
(b) $\chi(G) = 4$ *if and only if $G - U$ is not bipartite and does not contain a quadrangulation of the projective plane.*
(c) $\chi(G) = 5$ *if and only if $G - U$ is not bipartite and contains a quadrangulation of the projective plane. Such a quadrangulation is necessarily nonbipartite.*

Proof. The existence of U follows by Proposition 3. Observe that $G - U$ is 3-degenerate, i.e., every subgraph of $G - U$ contains a vertex of degree ≤ 3. (This is an easy consequence of Euler's formula.) Thus, it is 4-colorable, and so $\chi(G) \leq 5$.

Now, (a) is obvious, so assume that $G - U$ is not bipartite. If $G - U$ does not contain a quadrangulation of the projective plane, then $G - U$ is 3-colorable by Theorem 2. Hence, G is 4-colorable. So, suppose that $Q \subseteq G - U$ is a quadrangulation of the projective plane. We claim that Q is not bipartite. For each face C of Q, the subgraph Q_C of $G - U$ inside C is a locally bipartite plane graph since the degrees (in G) of removed vertices in U are even. Therefore, Q_C is bipartite. Consequently, if Q were bipartite, then also $G - U = Q \cup (\cup_C Q_C)$ would be bipartite.

To show that G is not 4-colorable, assume (reductio ad absurdum) that there is a 4-coloring c of G. Consider the restriction of c to Q. By Theorem 1, Q has a face $C = v_1v_2v_3v_4$ on which all four colors are used. Let G_C be the subgraph of G inside C. Since G_C is obtained from Q_C by face subdivision (except for the face C), the degrees of $v_1, \ldots, v_4$ in G_C are all odd. The degrees of other vertices of G_C are even. We may assume that $v_1v_3 \notin E(G_C)$. Now, adding the edge v_1v_3 to G_C gives rise to a 4-colored triangulation of the plane in which precisely two vertices v_2 and v_4 are of odd degree. It is well known (cf., e.g., [2]) that the colors of v_2 and v_4 must be the same in any 4-coloring. This contradiction to our assumption that $c(v_2) \neq c(v_4)$ completes the proof. □

Corollary 5 *There is a polynomial time algorithm to compute the chromatic number of Eulerian triangulations of the projective plane.*

Proof. By the proof of Proposition 3, it suffices to take an arbitrary non-contractible walk W_1 in the dual graph R and compute $\sigma = \sigma(W_1)$. If $\sigma = id$, then G is 3-colorable. Otherwise, let $U = U(x)$ be a color factor, where x is (the unique) vertex of T_0 which is fixed by σ.

All it remains to do, is to check if $H := G - U$ contains a quadrangulation. This can be done in polynomial time as follows. For $v \in V(H)$, repeat the breadth-first-search starting at v (up to distance 2 from v). This way, all 4-cycles containing v are discovered. For each such 4-cycle C, one can check (in constant time, after an overall $O(n)$ preprocessing) if it is contractible and if it is nonfacial. If this happens, remove from H all vertices and edges inside the disk bounded by C. After repeating this procedure for all vertices of H, the resulting graph Q is a quadrangulation if and only if $G - U$ contains a quadrangulation. The overall time spent in this algorithm is easily seen to be $O(n^3)$, where $n = |V(G)|$, and it is not hard to improve it to an $O(n^2)$ algorithm. The details are left to the reader. □

At the end we would like to mention list colorings. Let G be an Eulerian triangulation of the projective plane. Denote by $\chi_l(G)$ the list chromatic number of G. By Theorem 4, G does not contain K_6 as a subgraph. As proved in [1], this implies that $\chi_l(G) \leq 5$. Therefore, $\chi_l(G) = \chi(G)$ if $\chi(G) = 5$. This raises the following

Question: Let G be an Eulerian triangulation of the projective plane. Is it possible that $\chi_l(G) > \chi(G)$?

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