Long cycles in graphs on a fixed surface*

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Abstract

We prove that there exists a function $a : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N}$ such that

(i) If G is a 4-connected graph embedded on a surface of Euler genus g such that the face-width of G is at least $a(g, \varepsilon)$, then G can be covered by two cycles each of which has length at least $(1 - \varepsilon)\, n$.

We apply this to derive lower bounds for the length of a longest cycle in a graph G on any fixed surface. Specifically, there exist functions $b : \mathbb{N}_0 \to \mathbb{N}$ and $c : \mathbb{N}_0 \to \mathbb{R}_+$ such that for every graph G on n vertices that is embedded on a surface of Euler genus g the following statements hold:

(ii) If G is 4-connected, then G contains a collection of at most $b(g)$ paths which cover all vertices of G, and G contains a cycle of length at least $n/b(g)$.

(iii) If G is 3-connected, then G contains a cycle of length at least $c(g)\, n^{\log 2/ \log 3}$.

Moreover, for each $\varepsilon > 0$, every 4-connected graph G with sufficiently large face-width contains a spanning tree of maximum degree at most 3 and a 2-connected spanning subgraph of maximum degree at most 4 such that the number of vertices of degree 3 or 4 in either of these subgraphs is at most $\varepsilon |V(G)|$.

1 Introduction

The notation and terminology in this paper is the same as in [12, 18, 21].

In 1956 Tutte [22] proved that every planar graph G which is not a forest contains a cycle C such that every component of $G-V(C)$ has at most three neighbors on C. We call such a cycle a *Tutte cycle*. Tutte proved that C can be chosen to contain any prescribed edge if G is 2-connected. For a short proof see [17]. Thomas and Yu [16] extended Tutte's theorem to projective planar graphs. It follows that every 4-connected planar or projective planar graph has a Hamiltonian cycle.

This result does not extend to 3-connected planar graphs since there exist planar triangulations on n vertices whose longest cycle is of length $O(n^{\alpha})$, where $\alpha = \log 2/\log 3 \approx 0.63$; cf. [13]. In fact, Grünbaum and Walther [8] conjectured that every 3-connected planar graph of order n contains a cycle of length at least cn^{α} for some positive constant c. Jackson and Wormald [10] proved the existence of a cycle of length at least cn^{β} where c is a positive constant and $\beta \approx 0.2$. Gao and Yu [9] improved their result by showing that every 3-connected planar graph G contains a cycle of length at least $\frac{1}{6}|V(G)|^{0.4}+1$. Recently, Chen and Yu [5] proved the conjecture of Grünbaum and Walther. Both aforementioned results hold also for graphs on the projective plane, the torus, and the Klein bottle.

As every graph can be embedded on some surface, these results do not generalize to surfaces of higher genera even for 1000-connected graphs. An additional modest condition on the face-width does not help either. Archdeacon, Hartsfield, and Little [1] proved that for each k there exists a k-connected triangulation of an orientable surface having face-width k in which every spanning tree has a vertex of degree at least k. In particular, such graphs are far from being Hamiltonian.

If the surface is fixed and the face-width is large, the situation changes. Thomassen [20] proved that large face-width of a triangulation of a fixed orientable surface implies the existence of a spanning tree of maximum degree at most 4 and that 4 cannot be replaced by 3. It was conjectured in [20] that the additional condition that the triangulation is 5-connected implies that the graph is Hamiltonian, and this was verified by Yu [23]. It was also observed in [20] that "5-connected" cannot be replaced by "4-connected". However, we show in this paper that the cutting technique used in [19, 21] to prove a 5-color theorem for each fixed surface can be used to prove the existence of long cycles in 4-connected or 3-connected graphs on a fixed surface. Specifically, we prove the following theorems.

Theorem 1.1 *There is a function $a : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N}$ such that for every $\varepsilon > 0$ and every 4-connected graph G that has an embedding of Euler genus g and face-width at least $a(g, \varepsilon)$, there are two cycles C_1, C_2 in G such that*

(1) *$V(C_1) \cup V(C_2) = V(G)$, and*

(2) *$|V(C_i)| \geq (1 - \varepsilon)|V(G)|$, for $i = 1, 2$.*

We apply Theorem 1.1 to prove

Theorem 1.2 *There exists a function $b : \mathbb{N}_0 \to \mathbb{N}$ such that, if G is a 4-connected graph of Euler genus g, then G contains a collection of paths $P_1, \ldots, P_k$, where $k \leq b(g)$, which cover all vertices of G, and G contains a cycle of length at least $2n/(5b(g))$.*

Theorem 1.3 *There exists a function $c : \mathbb{N}_0 \to \mathbb{R}_+$ such that, if G is a 3-connected graph of Euler genus g, then G has a cycle of length at least $c(g)\,|V(G)|^{\log 2/\log 3}$.*

Barnette [2, 3] proved that every 3-connected planar graph contains a spanning tree of maximum degree at most 3 and a 2-connected spanning subgraph of maximum degree at most 16, and Gao [7] improved the bound 16 to 6 (which is best possible). Sanders and Zhao [15] extended these results to higher surfaces.

Ellingham and Gao [6] modified the method from [20] to prove that large face-width of a 4-connected triangulation on a fixed surface implies the existence of a spanning tree of maximum degree at most 3, and Yu [23] extended this to nontriangulations. Theorem 1.1 implies the following extension of Yu's result.

Corollary 1.4 *There exists a function $a : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{N}$ such that, if G is a 4-connected graph embedded with face-width at least $a(g, \varepsilon)$ on a surface of Euler genus g, then G contains a spanning tree T of maximum degree at most 3, a 2-connected spanning subgraph H of maximum degree at most 4, and a path P such that*

(a) *$P \subseteq T \subseteq H$ and*

(b) *the number of vertices of degree 3 or 4 in T and in H is at most $\varepsilon|V(G)|$ and all such vertices are in $V(P)$.*

Proof. Let C_1, C_2 be the cycles in Theorem 1.1. Then we take $H = C_1 \cup C_2$ minus those edges of C_2 which are chords of C_1. Let e be an arbitrary edge of C_1 and let $P = C_1 - e$. Then G has a spanning tree T of maximum degree 3 which is obtained from $H - e$ by deleting only edges in $E(C_2) \backslash E(C_1)$ incident with vertices in C_1. It is obvious that H, T, P have the stated properties. $\square$

We shall use the following lemmas.

Lemma 1.5 *If G is a disconnected graph on a surface S, then $S \backslash G$ contains a simple, closed, twosided curve C which is either noncontractible in S, or contractible in S such that each of* $\mathrm{int}(C)$ *and* $\mathrm{ext}(C)$ *contains a connected component of G.*

Proof. Add on S an edge e between two components G_1, G_2 of G. The facial walk F containing e must contain e twice and in opposite directions because e is a cutedge. Therefore, S has a simple closed twosided curve C (close to $F \cap G_1$) such that $C \cap G = \emptyset$ and C crosses e once. If C is contractible, then $\mathrm{int}(C)$ contains one of G_1, G_2 and $\mathrm{ext}(C)$ contains the other. $\square$

Lemma 1.6 *Let G be a connected graph embedded on a surface S, and let A be a set of vertices such that $G - A$ is disconnected. Then S has a simple closed curve C such that $C \cap G \subseteq A$, and either C is noncontractible or else C is contractible and each of* $\mathrm{int}(C)$ *and* $\mathrm{ext}(C)$ *contains a connected component of $G - A$.*

Proof. Apply the proof of Lemma 1.5 to $G - A$. Let C_0 be the corresponding curve. We may assume that C_0 intersects G only in edges joining a component of $G - A$ and A and that C_0 intersects each such edge at most once and that each such intersection is a crossing. Now we modify C as follows. For each edge $e = uv$ $(v \in A)$ where C intersects G, we replace a short segment of C around that intersection with a simple curve which follows e to v, crosses through v and returns back on the other side of e. The resulting curve C' is homotopic to C_0 and is composed of one or more simple closed curves $C_1, \ldots, C_k$ which intersect G only at A. If all of these curves are contractible, so is C_0. Then each of $\mathrm{int}(C_0)$ and $\mathrm{ext}(C_0)$ contains a connected component of $G - A$ (by the assumption on C_0). It is easy to see (by induction on k) that the same must hold for at least one of the curves $C_1, \ldots, C_k$. $\square$

2 Proof of Theorem 1.1

First we introduce some notation.

If G is a plane 2-connected graph with outer cycle C_1 and another facial cycle C_0 disjoint from C_1, then we call G a *cylinder* with *outer cycle* C_1 and *inner cycle* C_0. If H is a graph on a surface of Euler genus g, with disjoint facial cycles C_0', C_1' of the same lengths as C_0 and C_1 (respectively), then we can identify C_0 and C_0' into a cycle C_0'' and identify C_1 and C_1' into a cycle C_1''. Let M be the graph obtained from the union of G and H after these identifications. The embeddings of G and H determine an embedding of M into a surface of Euler genus $g+2$. We also say that H is obtained from M by *cutting* C_0'' and C_1'' and by *deleting* the cylinder G. The *cylinder-width* of G is the largest integer q such that G has q pairwise disjoint cycles $R_0, \dots, R_{q-1}$ such that $C_0 \subseteq \mathrm{int}(R_0) \subseteq \mathrm{int}(R_1) \subseteq \cdots \subseteq \mathrm{int}(R_{q-1})$. The paper [21, Theorem 9.1] has a short proof of the following result:

> For any natural numbers g and r there exists a natural number $f(g,r)$ such that any 2-connected graph H on $\mathbb{S}_g$ (the orientable surface of Euler genus $2g$) having face-width $\geq f(g,r)$, contains g pairwise disjoint cylinders $Q_1, \dots, Q_g$ of cylinder-width at least r whose cutting and deletion results in a connected plane graph.

In [21] this was proved for triangulations but the proof extends to all (2-connected) graphs by standard techniques: If H is not a triangulation, we form a triangulation $H_1 \supseteq H$ by adding a new vertex in each face of size at least 4 and joining it to all vertices of H on that face. Then it is easy to see that, if H_1 contains a cylinder of cylinder-width q, then H contains a cylinder of cylinder-width at least $q/2-1$.

Suppose, in addition, that H is 4-connected. Let us focus on one of the g cylinders, say Q_j, and suppose its cylinder-width is $> 10q$. Let $R_0, R_1, \dots, R_{10q}$ be the cycles in the definition of the cylinder-width. We select an $i \in \{0, 1, \dots, q-1\}$ such that the number of vertices in the sub-cylinder between R_{5i} and R_{5i+5} is smallest possible. Then we cut R_{5i+2} and R_{5i+3} and delete the cylinder between these two cycles. We repeat this procedure for each of the cylinders $Q_1, \dots, Q_g$. The resulting graph H' is planar, 2-connected and has therefore a Tutte cycle C containing an edge which is not contained in any of the g cylinders.

We claim that any vertex $v \in V(H) \backslash V(C)$ is in one of the cylinders, say Q_j, and in Q_j, v is between R_{5i} and R_{5i+5}. (In particular, v is on neither R_{5i} nor R_{5i+5}.) To see this, let B be the C-bridge of H' containing v. (That is, B is the component B' of $H' - V(C)$ containing v together with the set A of vertices on C joined to B' and all edges between A and B'.) We apply Lemma 1.6 to the plane graph H' and let Q be the resulting simple closed curve intersecting H' only in A. Now, Q must intersect some face of H' which is not a face of H since otherwise A would separate H (which is impossible since H is 4-connected and $|A| \leq 3$) or Q would be noncontractible on $\mathbb{S}_g$ (which is impossible because H has face-width > 3). So we may assume without loss of generality that both C and B intersect R_{5i+2} in some Q_j. Hence C contains at least two vertices of each of R_{5i+1} and R_{5i}. Since B has at most three vertices of attachment, B cannot intersect R_{5i}. So, B is between R_{5i} and R_{5i+2}. Our choice of i (in each of the g cylinders) implies that C misses at most $|V(H)|/q$ vertices of H.

Suppose now that we select the indices i in $\{q, q+1, \ldots, 2q-1\}$ (one for each of the cylinders). Then we can find another cycle C' in H missing at most $|V(H)|/q$ vertices of H such that $V(C) \cup V(C') = V(H)$. This completes the proof of Theorem 1.1 in the orientable case.

We now turn to the nonorientable case. Let g, q be any natural numbers. Now draw any specific graph H_0 on $\mathbb{N}_g$ (the nonorientable surface of Euler genus g) such that H_0 contains $\lfloor g/2 \rfloor$ pairwise disjoint cylinders of cylinder width $10q+1$ whose removal results in a connected graph in the projective plane (if g is odd) or the sphere (if g is even). Robertson and Seymour [14] proved that, if the face-width of a graph H on $\mathbb{N}_g$ is sufficiently large, then one can delete edges and contract edges of H such that one obtains H_0 on $\mathbb{N}_g$. In particular, H also contains $\lfloor g/2 \rfloor$ pairwise disjoint cylinders of cylinder width $10q+1$ whose removal results in a connected graph in the projective plane or the sphere. If g is even, we repeat the proof in the orientable case. If g is odd, the same proof works, except that we use the extension of Tutte's theorem obtained by Thomas and Yu [16] that every 2-connected graph in the projective plane has a Tutte cycle containing any prescribed edge.

3 Proof of Theorem 1.2

Bondy and Locke [4] proved that, if a 3-connected graph has a path of length k, then it has a cycle of length at least $2k/5$. So, it suffices to prove the first statement in Theorem 1.2. We prove this by induction on the Euler genus.

By the theorems of Tutte [22] and Thomas and Yu [16], $b(0) = b(1) = 1$. Suppose that $b(0) \leq b(1) \leq \cdots \leq b(g-1)$ exist. We shall prove that $b(g) \leq 4\, a(g, 1/2) \cdot b(g-1) + 2g + 100$.

Let G be any 4-connected graph on a surface S of Euler genus $g \geq 2$. Let w_0 denote the face-width of G on S. We may assume that $w_0 < a(g, 1/2)$, since otherwise $V(G)$ is covered by two paths by Theorem 1.1.

Consider first the case where $w_0 \geq 4$. Let C_0 be a noncontractible simple closed curve intersecting G in w_0 vertices. We think of C_0 as a cycle in the graph obtained from G by adding ($\leq$) w_0 edges, and then we cut that graph along C_0. Then C_0 is cut into a cycle C_1, say, and (if C_0 is twosided) a cycle C_2. The resulting graph G_1 is embedded in a surface S' (possibly disconnected) of Euler genus $g-1$ or $g-2$ (if C_0 is onesided or twosided, respectively). We add a new vertex x_1 in the face bounded by C_1 and join it to all vertices of C_1. If C_2 exists, we also add a new vertex x_2 in the face bounded by C_2.

We assume that S' is connected. (The case where S' is disconnected is similar and easier.) We claim that the resulting graph G_1' is 4-connected. Suppose (reductio ad absurdum) that G_1' has a (smallest) vertex set A such that $G_1' - A$ is disconnected and $|A| \leq 3$. If A contains x_1, then A also contains two vertices of C_1 by the minimality of A. We now apply Lemma 1.6. It is easy to modify the resulting simple closed curve in S' into a noncontractible curve in S having only a proper subset of $V(C_0)$ in common with G, a contradiction to the definition of the face-width. So, we may assume that A contains neither x_1 nor x_2. Again, we apply Lemma 1.6 and, if necessary, modify the resulting simple closed curve R such that it does not intersect the interior of any of the faces having x_1 or x_2 on the boundary. Then R determines a simple closed curve R' on S such that $R' \cap G \subseteq A$. Since G is 4-connected, R' is noncontractible on S, contradicting the assumption that $w_0 \geq 4$. So, G_1' is 4-connected.

By the induction hypothesis, $V(G_1')$ is covered by at most $b(g-1)$ paths in G_1'. After removing x_1, x_2 and some vertices of C_1 (or C_2) from these paths, we obtain at most $2a(g, 1/2)b(g-1)$ paths in G which cover $V(G)$.

Consider next the case where $2 \leq w_0 \leq 3$. We let C_0 be a noncontractible simple closed curve on S intersecting G in at most 3 points. If possible, we choose C_0 such that it is onesided and $|C_0 \cap G|$ is smallest possible subject to that condition. If there are no onesided closed curves intersecting G in ≤ 3 points, then we select a twosided curve C_0 such that $|C_0 \cap G| = w_0$. As in the case $w_0 \geq 4$, we think of C_0 as a cycle (of length 2, or 3) and we cut S along C_0 such that C_0 becomes one cycle C_1 of length 4 or 6 (if C_0 is onesided) or two cycles C_1, C_2 of length 2 or 3 if C_0 is twosided. If C_0 is onesided we add a new vertex x_1 and join it to C_1. If C_0 is twosided, we do not add any of x_1, x_2. The resulting graph is called G_1. If G_1 is 4-connected, we apply induction as in the case $w_0 \geq 4$. It is easy to see that G_1 is 4-connected if

C_0 is onesided. (For, if a separating set A of at most three vertices contains x_1 and two vertices on C_1, then some component of $G_1 - A$ is a path on C_1 and we obtain a contradiction to the minimality of C_1.) Therefore we may assume that C_0 is twosided and that G_1 is not 4-connected. Now we apply Lemma 1.6 where A is a separating vertex set of G_1 with at most three vertices. The resulting simple closed curve C_3 is twosided (otherwise we would have taken that curve as C_0). If necessary, we modify C_3 so that it does not cross C_1 or C_2. (This is possible since $|V(C_0)| \leq 3$.) Now we cut C_3 into two cycles C_4 and C_5. If possible, we select a noncontractible curve C_6 in S which does not cross any of C_1, C_2, C_4, C_5 such that C_6 has less than 4 vertices in common with G and we cut C_6 into cycles C_7 and C_8. We continue like this as often as possible. Thus we cut S into surfaces and G into graphs $G_1, \ldots, G_p$. By Lemma 1.6, each of $G_1, \ldots, G_p$ is 4-connected or complete. We define an auxiliary multigraph J_1 whose vertices are the graphs $G_1, \ldots, G_p$. Each of the curves C_{3i} ($i = 0, 1, 2, \ldots$) that we have cut along belongs to two (or one) of the graphs $G_1, \ldots, G_p$, and J_1 will have an edge (or a loop) between these graphs. We say that the curve C_{3i} *corresponds to* that edge of J_1. As G is 4-connected, J_1 has no cutedge.

Next we define a multigraph J_0 with $V(J_0) \subseteq V(J_1)$ as follows. If J_1 is a cycle, we let J_0 consist of a vertex (corresponding to a surface of Euler genus > 0 if possible) and a loop. If J_1 is not a cycle, we let J_0 be the unique multigraph without vertices of degree 2 such that J_1 is a subdivision of J_0. Then J_0 has an edge e such that $J_0 - e$ has no cutedge. Let P be the path in J_1 which corresponds to the edge e. If P has length 1, then cutting S and G along the curve corresponding to P results in a 4-connected graph, and we complete the proof by induction. So assume that P has length at least 2. Assume that the notation is such that the first edge of P corresponds to C_0, and the last edge of P corresponds to one of $C_3, C_6, \ldots$, say R.

When we cut C_0 into C_1 and C_2, then S becomes a surface with boundaries C_1 and C_2. If we also cut R into R_1 and R_2, then we disconnect S into surfaces S' and S'' with boundaries C_1, R_1 and C_2, R_2, respectively. We make S', S'' into closed surfaces S_1' and S_1'' (respectively) by adding a cylinder (handle) with the outer and inner cycle R_1, C_1 and R_2, C_2, respectively. On each of these handles we add edges and possibly one new vertex so that the two graphs on the two handles are either complete graphs with four vertices or 4-connected graphs with 6 vertices (see Figure 1). Hence the resulting graphs on S_1' and S_1'' are 4-connected. If these graphs have Euler genus less than g, we complete the proof by induction (similarly as in the case $w_0 \geq 4$). So assume that at least one of them has Euler genus g. Hence S' or S'', say S', is a cylinder. By the choice of J_0 and P, S' corresponds to

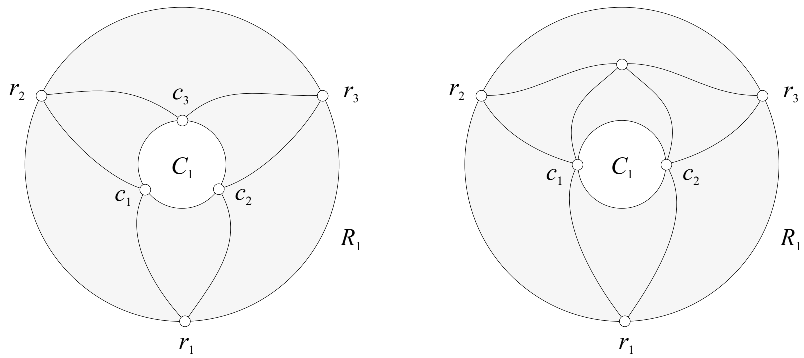

Figure 1: The cylinder added to S'

P. To each of S', S'' we add two discs so that C_1, C_2, R_1, R_2 become facial cycles. The graph on S'' is 4-connected or complete and of Euler genus less than g, so we apply induction to that graph. The graph on S' is planar with facial cycles C_1, R_1, each of length 2 or 3. We add a vertex y joined to all vertices of C_1 and a vertex z joined to all vertices of R_1. By [17], the resulting graph M has a path P from y to z such that each P-bridge has at most 3 vertices of attachment. (In [17] it is required that the graph is 2-connected. If M is not 2-connected, we apply [17] to each block of M.) Since G is 4-connected, P contains all vertices of M (except possibly some on C_1 or R_1) and the proof is complete when $2 \leq w_0 \leq 3$.

Consider finally the case where $w_0 = 1$. In each face which is not bounded by a cycle, we add a vertex joined to all vertices on the boundary. As G is 4-connected, we add at most $2g$ new vertices. For, in each augmented face we can draw a simple noncontractible curve having precisely one vertex in common with G, and no two of these curves are homotopic. Hence there are at most $2g$ such curves, see, e.g., [11]. Now we repeat the proof of the case when $2 \leq w_0 \leq 3$.

4 Proof of Theorem 1.3

If the Euler genus is at most 2, we apply the result of [5]. For the general case we repeat the inductive proof of Theorem 1.2. The only essential difference is that instead of using [17] at the end of that proof, we apply [5]. Note that we only need to show that G contains a path of length $cn^{\log 2/\log 3}$, by the aforementioned result of Bondy and Locke [4]. We leave the details to the reader.

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