$K_{a,k}$ minors in graphs of bounded tree-width *

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It is shown that for any positive integers k and w there exists a constant $N = N(k,w)$ such that every 7-connected graph of tree-width less than w and of order at least N contains $K_{3,k}$ as a minor. Similar result is proved for $K_{a,k}$ minors where a is an arbitrary fixed integer and the required connectivity depends only on a. These are the first results of this type where fixed connectivity forces arbitrarily large (nontrivial) minors.

Key Words: Graph theory, graph minor, connectivity.

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1. INTRODUCTION

In this paper, all graphs are finite and may have loops and multiple edges. A graph H is a *minor* of a graph G, $H \leq_{\mathrm{m}} G$, if H can be obtained from a subgraph of G by contracting connected subgraphs. There are many results concerning the structure of graphs that do not contain a certain graph as a minor. These excluded graphs include K_5 and $K_{3,3}$ [13], V_8 [8], the 3-cube [6] and the octahedron [7]. See also [2] and [12]. There are well-known structures which guarantee a certain minor exists for large graphs. For instance, any 5-connected graph on at least 11 vertices contains the 3-cube as a minor [6]. Any 5-connected non-planar graph on at least 8 vertices contains a V_8 minor [8]. In addition, there are Ramsey-type results similar to the fact that any sufficiently large connected graph contains either a k-path or a k-star. Oporowski, Oxley and Thomas [11] proved that any large 4-connected graph must have a large minor from a set of four families of graphs. Ding [3] has characterized large graphs that do not contain a $K_{2,k}$ minor. A corollary of his result is that any large 5-connected graph contains a $K_{2,k}$ minor.

Our results are a cross section of all of these types of results:

THEOREM 1.1. *For any positive integers k and w there exists a constant $N = N(k,w)$ such that every 7-connected graph of tree-width at most w and of order at least N contains $K_{3,k}$ as a minor.*

THEOREM 1.2. *There is a function $c : \mathbb{N} \to \mathbb{N}$ such that for any $a \geq 3$ the following holds. For any positive integers k and w there exists a constant $N = N(k,w)$ such that every $c(a)$-connected graph of tree-width at most w and of order at least N contains $K_{a,k}$ as a minor.*

Theorem 1.1 is sharp in the sense that the 7-connectivity condition cannot be relaxed. Moreover, the function $c(a)$ in Theorem 1.2 must be at least $2a+1$. These facts follow from the following construction of a family of arbitrarily large $2a$-connected graphs (of tree-width $3a-1$) none of which contain a $K_{a,2a+1}$-minor.

Let m and a be integers greater than 3. Define the graph $N_{m,a}$ as follows. Let the vertices be indexed $v_{x,y}$ where $1 \leq x \leq m$ and $1 \leq y \leq a$. The vertex $v_{x,y}$ is adjacent to another vertex $v_{w,z}$ if and only if $w \in \{x-1, x, x+1\}$ where $x \pm 1$ is considered modulo m.

PROPOSITION 1.1. *For any integers $a \geq 3$ and $m \geq 3$, $K_{a,2a+1} \not\leq_{\mathrm{m}} N_{m,a}$.*

Proof. Suppose the theorem is false for some $a \geq 3$. Let m be the least integer such that $N_{m,a} \geq_{\mathrm{m}} K_{a,2a+1}$. Let the *clasps* of $N_{m,a}$ be defined as $CL_i = \{v_{i,y} \mid y = 1, 2, \ldots, a\}$ for $i = 1, 2, \ldots, m$.

As $N_{m,a} \geq_{\mathrm{m}} K_{a,2a+1}$, there is a set of $2a+1$ connected subgraphs, $\mathcal{S} = \{S_1, S_2, \ldots, S_{2a+1}\}$, and a set of a connected subgraphs of $N_{m,a}$, $\mathcal{T} = \{T_1, T_2, \ldots, T_a\}$, such that for every i, j there is an edge from some vertex in T_i to some vertex in S_j and such that all these subgraphs are pairwise disjoint. Assume that the S_i and T_i are chosen with $l := \sum_{i=1}^{2a+1} |V(S_i)| + \sum_{i=1}^{a} |V(T_i)|$ minimum. Then it is easy to see that each of the subgraphs in $\mathcal{S} \cup \mathcal{T}$ is a path meeting each clasp in at most one vertex. Let $\mathcal{S}_1$ be the set of single vertex subgraphs contained in $\mathcal{S}$. It is easy to see that $\mathcal{T}$ cannot contain any single vertex subgraphs.

Claim 1: *For every $1 \leq i \leq m$, there is a subgraph $S_j \in \mathcal{S}_1$ such that $S_j \subseteq CL_i$.*

Suppose CL_i does not contain any of the subgraphs in $\mathcal{S}_1$. Then contracting a matching of size a between CL_i and $CL_{i-1} \cup CL_{i+1}$ (indices taken modulo m) using as many edges of $\mathcal{S} \cup \mathcal{T}$ as possible gives a subgraph of $N_{m-1,a}$ that still contains $K_{a,2a+1}$ as a minor. This contradiction to the minimality of m proves the claim.

Claim 2: *If there is a subgraph in $\mathcal{S}$ that contains at least two vertices, then there is a clasp that contains no member of $\mathcal{S}_1$.*

Suppose S_1 (say) intersects CL_1 and CL_2. By the minimality of l, we may assume that $S_1 \cap CL_m = \emptyset$. Moreover, there is a subgraph T_j that does not intersect $CL_1 \cup CL_2 \cup CL_3$. Otherwise, the intersection of S_1 with CL_1 could be removed from S_1. Therefore, a single vertex subgraph $S_i \in \mathcal{S}_1$ contained in CL_2 would not be adjacent to T_j. Hence, the clasp CL_2 is as stated in the claim.

Claims 1 and 2 imply that all subgraphs in $\mathcal{S}$ are single vertices. To complete the proof, notice that if every clasp of $N_{m,a}$ contains one of the single vertex subgraphs of $\mathcal{S}_1$, then each T_j must must contain at least $m-2$ vertices in order to be adjacent to all of the subgraphs in $\mathcal{S}$. Hence $|V(\mathcal{S})| + |V(\mathcal{T})| \geq |\mathcal{S}| + (m-2)|\mathcal{T}| \geq 2a+1+(m-2)a > ma = |V(N_{m,a})|$. This contradiction completes the proof. ▮

In our proof of Theorem 1.2, $c(3) = 7$ and $c(a) = 264a+1$ for $a \geq 4$, and we have no intention to find the best possible value for $c(a)$. However, the previous example shows that $c(a)$ must be at least $2a+1$ for $a \geq 3$. It is worth remarking that our proof of Theorem 1.2 works also for $c(a) = 3a-1$ if we assume that the minimum degree is at least $264a+1$.

2. BOUNDED TREE-WIDTH STRUCTURE

A *tree decomposition* of a graph G is a pair (T, Y), where T is a tree and Y is a family $\{Y_t \mid t \in V(T)\}$ of vertex sets $Y_t \subseteq V(G)$, such that the following two properties hold:

(W1) $\bigcup_{t\in V(T)} Y_t = V(G)$, and every edge of G has both ends in some Y_t.

(W2) If $t, t', t'' \in V(T)$ and t' lies on the path in T between t and t'', then $Y_t \cap Y_{t''} \subseteq Y_{t'}$.

The *width* of a tree decomposition (T, Y) is $\max_{t\in V(T)}(|Y_t| - 1)$. It was shown in [11] that if a graph G has a tree decomposition of width at most w then G has a tree decomposition of width at most w that further satisfies:

(W3) For every two vertices t, t' of T and every positive integer k, either there are k disjoint paths in G between Y_t and $Y_{t'}$, or there is a vertex t'' of T on the path between t and t' such that $|Y_{t''}| < k$.

(W4) If t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$.

(W5) If $t_0 \in V(T)$ and B is a component of $T - t_0$, then $\bigcup_{t\in V(B)} Y_t \backslash Y_{t_0} \neq \emptyset$.

In the rest of the paper we give the proof of Theorems 1.1 and 1.2. We let $a \geq 3$, k, and w be given positive integers. Let G be an $c(a)$-connected graph with a tree decomposition (T, Y) of width at most w that satisfies (W1)–(W5).

We will develop a structure that is similar to that used in [11]. First, we define the constants that will be used in the proofs.

$$\begin{aligned}
n_5 &= r^{n_4}, \quad \text{where } r = (k-1)\binom{w+1}{a} \\
n_4 &= n_3^{w+1} \\
n_3 &= (2n_2)^p, \quad \text{where } p = 2^{w+1} \\
n_2 &= n_1^q, \quad \text{where } q = 2^{\binom{w+1}{2}} \\
n_1 &= \begin{cases} 2k(2w+3)^2 & \text{if } a = 3 \\ 2k(c(a)+2a+2) - 4a - 2 & \text{if } a \geq 4 \end{cases}
\end{aligned}$$

We assume that $|V(G)| = N \geq (w+1)n_5$ and that G has no $K_{a,k}$-minor. By (W1) we have

Claim 2.1. $|V(T)| \geq n_5$.

Claim 2.2. Every vertex of T has degree at most $r = (k-1)\binom{w+1}{a}$.

Proof. Suppose $t_0 \in V(T)$ has degree at least $r+1$. Let $\mathcal{C}$ be the set of components of $G - Y_{t_0}$. By (W2) and (W5), it is clear that $|\mathcal{C}| \geq r+1$. For $C \in \mathcal{C}$, let $X(C)$ be the set of vertices of Y_{t_0} adjacent to some vertex of C. Clearly, $|X(C)| \geq a$ for every $C \in \mathcal{C}$ since G is $c(a)$-connected and $c(a) \geq a$. By the Pigeonhole Principle, there is a set $\mathcal{C}' \subseteq \mathcal{C}$ of k components for which $\bigcap_{C \in \mathcal{C}'} X(C)$ contains a (or more) vertices of Y_{t_0}. By contracting B to a vertex for each $B \in \mathcal{C}'$, we see that G contains a $K_{a,k}$ minor, a contradiction. ∎

From this it follows that

Claim 2.3. T contains a path R of length $|E(R)| \geq n_4$.

The proof of the following claim can be found in [11].

Claim 2.4. There is a subsequence of length n_3 of the vertices of $V(R)$, $r_1, r_2, \ldots, r_{n_3}$, such that for some $s \geq 1$, $|Y_{r_i}| = s$ for $i = 1, 2, \ldots, n_3$ and for every vertex of R between r_1 and r_{n_3}, $|Y_{r_i}| \geq s$.

From now on we replace R by the subpath from r_1 to r_{n_3}. Note that because of the $c(a)$-connectivity and (W5), $c(a) \leq s \leq w+1$.

By (W3) and Claim 2.4, there are s disjoint paths in G from Y_{r_1} to $Y_{r_{n_3}}$. Fix these paths, denote them by $P_1, P_2, \ldots, P_s$, and put $Z = P_1 \cup \cdots \cup P_s$. Since G is 3-connected, these paths can be chosen such that every Z-bridge in G is attached to at least two of the paths (cf., e.g., [4]), which we assume henceforth.

Notice that for any $t, t' \in \{r_1, \ldots, r_{n_3}\}$ and for every $j \in \{1, \ldots, s\}$ there is a unique subpath of P_j with one end in Y_t and the other end in $Y_{t'}$. Denote this subpath by $P_j(t, t')$.

The path P_j is said to be *trivial* if it consists of a single vertex, and it is said to be *everywhere nontrivial* (*almost nontrivial*) w.r.t. the sequence $r_1, \ldots, r_{n_3}$ if $P_j(r_i, r_{i+1})$ contains at least three (respectively, at least two) vertices for each $i = 1, \ldots, n_3 - 1$.

Claim 2.5. There is a subsequence $q_1, q_2, \ldots, q_{n_2}$ of $r_1, \ldots, r_{n_3}$ of length n_2 such that for each $j = 1, \ldots, s$, $P_j(q_1, q_{n_2})$ is either trivial or everywhere nontrivial (w.r.t. the subsequence).

Proof. Clearly, there is a subsequence of $r_1, \ldots, r_{n_3}$ of length $\sqrt{n_3}$ such that the corresponding segment of P_1 is either trivial or everywhere almost nontrivial with respect to the subsequence. By repeating this argument on the subsequence for $P_2, \ldots, P_s$, respectively, we end up with a sequence of length at least $2n_2$ such that every path is either trivial or everywhere almost nontrivial. By taking every second element of this sequence, the required subsequence $q_1, q_2, \ldots, q_{n_2}$ is obtained. ▮

The paths P_j and P_l are said to be *everywhere bridge connected* (resp. *everywhere bridge disconnected*) with respect to a sequence $p_1, \ldots, p_n$ of vertices of R if for every $i = 1, \ldots, n-1$, there exists (resp. does not exist) a Z-bridge which has a vertex of attachment in $P_j(p_i, p_{i+1})$ and a vertex of attachment in $P_l(p_i, p_{i+1})$.

Claim 2.6. There is a subsequence $p_1, p_2, \ldots, p_{n_1}$ of $q_1, \ldots, q_{n_2}$ of length n_1 such that for every distinct pair of indices $j, l \in \{1, \ldots, s\}$, $P_j(p_1, p_{n_1})$ and $P_l(p_1, p_{n_1})$ are either everywhere bridge connected or everywhere bridge disconnected (w.r.t. the new subsequence).

Proof. The proof is similar to the proof of Claim 2.5 except that we have to repeat the subsequence argument $\binom{s}{2} \le \binom{w+1}{2}$ times. ▮

3. THE AUXILIARY GRAPH A

Our next goal is to examine the structure of the *auxiliary graph* A which contains information about which pairs of the paths are everywhere bridge connected. The graph A has vertex set $V(A) = \{P_1, \ldots, P_s\}$, and the paths P_j and P_l are adjacent vertices in A if they are everywhere bridge connected w.r.t. $p_1, \ldots, p_{n_1}$ (cf. Claim 2.6).

Claim 3.1. Suppose that $U \subseteq V(A)$ contains only everywhere nontrivial paths. If the subgraph of A induced by U is connected, then $V(A) \setminus U$ contains at most $a-1$ vertices that are adjacent to U in A.

Proof. Suppose that $P_1, \ldots, P_a$ are vertices in $V(A) \setminus U$ adjacent to U in A. Contract each path P_j $(j = 1, \ldots, a)$ in G to a single vertex w_j. Next, for $i = 1, 3, 5, \ldots, 2k-1$, contract all segments $P_j(p_i, p_{i+1})$, where $P_j \in U$, and also contract all edges in bridges connecting these segments in G, to get k vertices $z_1, z_3, \ldots, z_{2k-1}$ in a minor of G. Clearly, $n_1 \ge 2k$, so $z_1, z_3, \ldots, z_{2k-1}$ exist. Since U is adjacent to $P_1, \ldots, P_a$ in A, it is easy to see that vertices $w_1, \ldots, w_a$ and $z_1, z_3, \ldots, z_{2k-1}$ give rise to a $K_{a,k}$ minor of G. ▮

We shall apply Claim 3.1 together with the help of the following lemma.

LEMMA 3.1. *Let H be a connected graph. If H has at least $2a^2$ vertices of degree ≥ 3, then H contains a tree T with $\geq a$ vertices of degree 1.*

Proof. Let d be the maximum vertex degree in H, and let v_0 be a vertex of degree d. If $d \geq a$, then T is the star centered at v_0. So, suppose that $d < a$. Then it is sufficient to prove the following. Assuming that H has at least $2a^2 - (d-1)^2$ vertices of degree ≥ 3, we shall prove by induction on $a - d$ that the tree T exists. Let N_1 be the set of all vertices of degree ≥ 3 which can be reached from v_0 on paths whose internal vertices all have degree 2. Then $1 \leq |N_1| \leq d$. Let N_2 be the "second neighborhood" of v_0, consisting of vertices of degree ≥ 3 which are not in $N_1 \cup \{v_0\}$ and which can be reached from v_0 on paths for which exactly one internal vertex has degree ≥ 3. Similarly, let N_3 be the "third neighborhood" of v_0. Then $1 \leq |N_2| \leq d(d-1)$ and $|N_3| \geq 1$ since H is connected and $2a^2-(d-1)^2 > 1+d+d(d-1) \geq 1+|N_1|+|N_2|$. Let $v_3 \in N_3$, and let W be a path from v_0 to v_3 which contains precisely two other vertices of degree ≥ 3. Now, contract W to a vertex $\tilde{v}_0$ and remove possible parallel edges. Denote the resulting graph by $\tilde{H}$. If a vertex of $\tilde{H}$ has degree smaller than in H, then it was adjacent to two (or three) vertices of W. This implies that $\tilde{H}$ has at least $2a^2-(d-1)^2-(2d-1) = 2a^2-((d+1)-1)^2$ vertices of degree ≥ 3. Since v_0 and v_3 have no common neighbors, $\tilde{v}_0$ is its vertex of maximum degree $\geq d+1$. By the induction hypothesis, $\tilde{H}$ contains a tree $\tilde{T}$ with at least a vertices of degree 1. Clearly, $\tilde{T}$ gives rise to the required tree T in H. ▮

At least one of the paths is everywhere nontrivial, say P_1. Let A_1 be the induced subgraph of A on the everywhere nontrivial paths. Let A_0 be the induced subgraph of A consisting of the connected component of A_1 containing P_1 together with (at most $a-1$) trivial paths adjacent to that component.

From now on we shall assume that G is $c(a)$-connected, where $c(3) = 7$ and $c(a) = 264a + 1$ for $a \geq 4$.

Claim 3.2. $A_0 \cap A_1$ has at least $\lceil \frac{c(a)-a+1}{2} \rceil$ vertices. If $a = 3$, A_0 is isomorphic to a path or a cycle on at least four vertices. If $a \geq 4$, then every vertex of $A_0 \cap A_1$ has degree at most $a-1$ and at most $2a^2$ of these vertices have degree more than 2 in $A_0 \cap A_1$.

Proof. Let $U = V(A_0 \cap A_1)$, $x = |U|$, and $y = |V(A_0)| - x$. By Claim 3.1 we see that $y \leq a-1$. Since the $2x+y$ endvertices of the paths in A_0 in Y_{p_1} and Y_{p_3} separate the graph G, we have $2x + y \geq c(a)$. This implies that $x \geq (c(a) - a + 1)/2$, and proves the first part of the claim.

By Claim 3.1, every vertex in $A_0 \cap A_1$ has degree at most $a-1$ in A. If $a = 3$, this implies that $A_0 \cap A_1$ is a path or a cycle, and the trivial paths in $V(A_0)$ can be adjacent only to vertices of degree ≤ 1 in $A_0 \cap A_1$. This and Claim 3.1 imply that A_0 is a path or a cycle. If $|V(A_0)| \leq 3$, then the endpoints of the paths in $V(A_0)$ would give a ≤ 6-separator in G.

Suppose now that $a \geq 4$. By Claim 3.1 every vertex of $A_0 \cap A_1$ has degree at most $a-1$. Suppose that there are more than $2a^2$ vertices of degree ≥ 3. By Lemma 3.1, $A_0 \cap A_1$ contains a tree T with $\geq a$ vertices of degree 1. Let U be the set of vertices of degree ≥ 2 in T. The subgraph of A induced by U is connected, and Claim 3.1 yields a contradiction. This completes the proof. ▮

Denote by $Z'(i)$ the union of $P_j(p_i, p_{i+1})$ where $P_j \in V(A_0)$, $i = 1, 2, \ldots, n_1 - 1$. Let Z_i be the subgraph of G obtained by taking the union of $Z'(i)$ and all those Z-bridges B that have all vertices of attachment in $Z'(i)$ such that there is no $i' < i$ for which B would have all its vertices of attachment in $Z'(i')$.

4. FINDING $K_{3,K}$ MINORS

In this section we consider the case when $a = 3$ since the best possible connectivity 7 requires more elaborate techniques than the general case treated in the next section. For $i = 1, 2, \ldots, n_1 - 2w - 2$, let $H_i = \bigcup_{k=0}^{2w} Z_{i+k}$. Let $R, R' \in V(A_0)$ be paths which are adjacent in A_0. For $i = 1, 2, \ldots, n_1 - 2w - 2$ define the graph $D_i = D_i(R, R')$ as follows. First, take $S = (R \cup R') \cap H_i$ together with all Z-bridges in H_i that have vertices of attachment on R and on R'. Finally, add two edges e_1, e_2, where e_1 joins the "left" endvertices, λ in $R \cap H_i$ and λ' in $R' \cap H_i$, and e_2 joins the "right" endvertices, ρ and ρ', of these two paths. Then $S + e_1 + e_2 =: C$ is a cycle in D_i. If R (R') is everywhere trivial, then $\lambda = \rho$ ($\lambda' = \rho'$).

Claim 4.1. Suppose that $a = 3$. Then for every i, there are adjacent vertices R, R' of A_0 such that $D_i(R, R')$ has no embedding in the plane where the vertices $\lambda, \lambda', \rho', \rho$ would lie on the outer face in the prescribed order.

Proof. Suppose that H_i is a planar graph. Let v_j be the number of vertices of degree j in H_i. By Euler's formula and standard counting arguments it follows that

$$L := \sum_{j \geq 0} (6 - j) v_j \geq 12. \qquad (1)$$

Observe that H_i has at most $2s$ vertices of degree ≤ 6 since the minimum degree in G is at least 7 (by the 7-connectivity of G). On the other hand, since at least three of the paths in H_i are nontrivial, these paths contain at least $3(2(2w+1)-1) = 12w+3$ vertices of degree ≥ 7 in H_i. Therefore,

$$L \leq 6 \cdot 2s - (12w + 3) \leq 12(w + 1) - 12w - 3 = 9.$$

This contradiction to (1) shows that H_i is not planar. Recall that A_0 is a path or a cycle on at least 4 vertices, $R_1, \ldots, R_d$, $d \geq 4$. This implies, in particular, that no Z-bridge in H_i is attached to more than two of the paths (otherwise, there would be a 3-cycle in A_0, and so A_0 would be equal to the 3-cycle). Moreover, if every $D_i(R_j, R_{j+1})$ ($j = 1, \ldots, d$, indices taken modulo d) has an embedding in the plane with the corresponding cycle C_j being the outer cycle, then $\bigcup_{j=1}^{d} D_i(R_j, R_{j+1}) \supseteq H_i$ would be planar as well, contrary to the above. Hence, there is an index j such that $D_i(R_j, R_{j+1})$ has no such embedding. Since there are no local Z-bridges, $D_i(R_j, R_{j+1})$ neither has an embedding in the plane where the vertices $\lambda, \lambda', \rho', \rho$ are on the outer face in the prescribed order. ▮

We shall need a result about crossing paths from from [9]. A *separation* of a graph G is a pair (A, B) of subraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and its *order* is $|V(A \cap B)|$. By a *society* we mean a pair (G, Ω), where G is a graph and Ω a cyclic permutation of a subset $\overline{\Omega}$ of $V(G)$. A *cross* in (G, Ω) is a pair of disjoint paths in G with ends s_1, t_1 and s_2, t_2, respectively, all in $\overline{\Omega}$, such that s_1, s_2, t_1, t_2 occur in Ω in that order (but not necessarily consecutive). The following formulation of a theorem of Robertson and Seymour [9] appears in [10].

Theorem 4.1 (Robertson and Seymour). *Let (G, Ω) be a society such that there is no separation (A, B) of G of order ≤ 3 with $\overline{\Omega} \subseteq V(A) \neq V(G)$. Then the following are equivalent:*

(a) *There is no cross in (G, Ω).*
(b) *G can be drawn in a disc with the vertices in $\overline{\Omega}$ drawn on the boundary of the disc in order given by Ω.*

Claim 4.2. If $D_i(R, R')$ is nonplanar, then one of the following holds:

(a) $D_i(R, R')$ contains disjoint paths Q_1, Q_2 connecting λ with ρ' and λ' with ρ, respectively.

(b) $D_i(R, R')$ contains a path Q (resp., Q') disjoint from R' (resp., R) which connects λ and ρ (resp., λ' and ρ') such that after replacing R (resp., R') by Q (resp., Q'), there is a Z-bridge in H_i which is attached to more than two of the paths $P_1, \ldots, P_s$.

Proof. Let $H = D_i(R, R')$. Let C be the cycle of H defined before Claim 4.1. Let $\overline{\Omega}$ be the set of vertices of C which are incident with an edge in $E(G) \setminus E(H)$. The cyclic order of $\overline{\Omega}$ on C defines the society (H, Ω). Since G is 4-connected and no vertex in $V(H) \setminus \overline{\Omega}$ is incident with an edge in $E(G) \setminus E(H)$, there is no separation (A, B) of H of order ≤ 3 with $\overline{\Omega} \subseteq V(A) \neq V(G)$. Since H is nonplanar, Theorem 4.1 implies that there is a cross R_1, R_2 in (H, Ω). Let α_i, β_i be the endvertices of R_i $(i = 1, 2)$. We may assume that:

(i) None of the vertices $\lambda, \lambda', \rho, \rho'$ is an internal vertex of R_1 or R_2.

Subject to (i) choose the cross R_1, R_2 such that

(ii) $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ contains as many vertices in $\{\lambda, \lambda', \rho, \rho'\}$ as possible and, subject to (i) and (ii)

(iii) as few edges in $E(H) \setminus E(R \cup R')$ as possible.

If $\lambda, \lambda', \rho, \rho'$ are all endvertices of R_1, R_2, then we have (a). Hence we may assume that λ is not an endvertex of R_1, R_2. If $R \cap (R_1 \cup R_2) \neq \emptyset$, let v be the first vertex of $R_1 \cup R_2$ on R (starting at λ towards ρ). We may assume that $v \in V(R_1)$. Let $R_1 = R_1' \cup R_1''$ where $V(R_1') \cap V(R_1'') = \{v\}$. By replacing one of the segments R_1' or R_1'' in R_1 by a segment from v to λ on R, a new cross is obtained which contradicts (ii) or (iii), except when R_1' or R_1'' is the segment of R from v to ρ. In particular, three of the endvertices of R_1, R_2 are on R'. The above proof implies that λ' and ρ' are the endvertices of the paths. Since R_1, R_2 cross, R_1 joins a vertex $x \in V(R') \setminus \{\lambda', \rho'\}$ with ρ, and R_2 joins λ' and ρ', where R_2 is disjoint from R. It is easy to see, that this gives (b).

Suppose now that $R \cap (R_1 \cup R_2) = \emptyset$. Condition (ii) implies that λ' and ρ' are the endvertices of R_1 and R_2, respectively. There is a C-bridge B in H such that $E(R_1 \cup R_2) \cap E(B) \neq \emptyset$. Since B is not a local bridge, it is attached to R as well. Therefore, there is a path L in B from R to $R_1 \cup R_2$ (say to R_2) which is internally disjoint from $C \cup R_1 \cup R_2$. Let y be the vertex of R_1 which is as close as possible to ρ' on R'. Let R_2' be the segment of R_2 from $R_2 \cap L$ to the end of R_2 distinct from ρ'. By (iii), R_2' is disjoint from the segment Q'' of R' from y to ρ'. Therefore, the path Q' composed of the segment of R_1 from λ' to y and Q'' can be taken as the path Q' in (b). Note that, after replacing R' by Q', the Z-bridge containing $L \cup R_2'$ will be attached to at least three paths in $\{P_1, \ldots, P_s\}$. ∎

We are ready to complete the proof of Theorem 1.1. Suppose that $a = 3$ and that A_0 is a path or a cycle on consecutive vertices $R_1, \ldots, R_d$, where $4 \leq d \leq w + 1$. Let $D_i^j = D_i(R_j, R_{j+1})$, $j = 1, \ldots, d$. We shall only consider the indices i of the form $i = 1 + t(2w + 2)$, $t = 0, 1, \ldots$, and we call them *admissible indices*.

Let us first assume that the case (b) of Claim 4.2 occurs less than $2kd$ times at admissible indices i. Since there are at least $4kd$ admissible indices, Claim 4.2(a) implies that there is an index $j \in \{1, \ldots, d\}$, and there are admissible indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n_1 - 2w - 2$ such that

(i) each of $D^j_{i_1}, D^j_{i_2}, \ldots, D^j_{i_k}$ contains paths as stated in Claim 4.2(a), and

(ii) for $l = 1, \ldots, k-1$, $i_{l+1} - i_l \geq 2w + 2$.

We can exchange the segments of the paths R_j and R_{j+1} in H_{i_l} by the two paths Q_1, Q_2 of Claim 4.2(a). In this way the new paths in $H_{i_l} \cup Z_{i_l+2w+2}$ would no longer satisfy the condition of Claim 3.1. Namely, if R_j and R_{j+1} have degrees d_1, d_2 in A_0, then they would be everywhere bridge connected (w.r.t. the sequence $p_{i_1-1}, p_{i_2-1}, \ldots, p_{i_k-1}$) with $d_1 + d_2 - 1$ other paths. If $d_1 = d_2 = 2$, this gives a $K_{3,k}$ minor in the same way as in the proof of Claim 3.1 (since one of R_j or R_{j+1} is everywhere nontrivial). If $d_1 = 1$ (say), then the path R_{j+2} has degree 2 in A_0 by Claim 3.2 and (in addition to R_{j+3}) it becomes everywhere bridge connected to the two new paths (w.r.t. the sequence $p_{i_1-1}, p_{i_2-1}, \ldots, p_{i_k-1}$). It is easy to see from the definition of A_0 that R_{j+2} cannot be trivial, so the proof of Claim 3.1 applies again.

Let us now assume that the case (b) of Claim 4.2 occurs $2kd$ or more times (for admissible indices i). Then there is an index $j \in \{1, \ldots, d\}$, and there are admissible indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n_1 - 2w - 2$ such that

(i) each of $D^j_{i_1}, D^j_{i_2}, \ldots, D^j_{i_k}$ contains a path Q (or each of $D^j_{i_1}, D^j_{i_2}, \ldots, D^j_{i_k}$ contains a path Q') as stated in Claim 4.2(b), and

(ii) for $l = 1, \ldots, k-1$, $i_{l+1} - i_l \geq 2w + 2$.

For any $D^j_{i_l}$ we replace the segment of R_j (resp., R_{j+1}) by the corresponding path Q (resp., Q') such that there is a Z-bridge (where Z is defined as the union of the new paths) attached to R_j, R_{j+1}, and R_{j+2} (or R_{j-1}). We may assume that k of these bridges, $B_1, \ldots, B_k$ are attached to R_j, R_{j+1}, and R_{j+2}. Now, there is a $K_{3,k}$-minor obtained by contracting R_j, R_{j+1}, R_{j+2} into single vertices and adding paths in $B_1, \ldots, B_k$ to these vertices. This completes the proof of Theorem 1.1.

5. FINDING $K_{A,K}$ MINORS FOR $A \geq 4$

Suppose now that $a \geq 4$ and $c(a) = 264a + 1$. Let $r = 2c(a) + 2$. For $i = 1, 2, \ldots, n_1 - r$, let $H_i = \bigcup_{j=0}^{r-1} Z_{i+j}$. We also write $S_i = Y_{p_i}$.

Claim 5.1. For every $1 \leq i \leq n_1 - r$, the average degree of vertices in H_i is at least $c(a) - \frac{1}{2}$.

Proof. Every vertex of G has degree at least $c(a)$. Let $s_0 = |V(A_0 \cap A_1)|$ be the number of everywhere nontrivial paths in $V(A_0)$. Then

$$|V(H_i)| \geq s_0(2r+1) > 4s_0c(a). \tag{2}$$

Each trivial path in $V(A_0)$ is everywhere bridge connected to some nontrivial path. Hence, the degree of the corresponding vertex in H_i is at least $r/2 \geq c(a)$. Only the ends of nontrivial paths can have degree less than $c(a)$ in H_i. This fact and inequality (2) imply that

$$2|E(H_i)| \geq c(a)(|V(H_i)| - 2s_0) \geq (c(a) - \frac{1}{2})|V(H_i)|.$$

This completes the proof. ▮

A graph L is said to be *q-linked* if it has at least $2q$ vertices and for any ordered q-tuples $(s_1, \ldots, s_q)$ and $(t_1, \ldots, t_q)$ of $2q$ distinct vertices of L, there exist pairwise disjoint paths $P_1, \ldots, P_q$ such that for $i = 1, \ldots, q$, the path P_i connects s_i and t_i. Such collection of paths is called a *linkage* of $(s_1, \ldots, s_q)$ and $(t_1, \ldots, t_q)$.

Claim 5.2. For every $1 \leq i \leq n_1 - r$, there exists a subgraph L_i of H_i which is $3a$-linked.

Proof. Mader [5] proved that every graph of average degree at least $4c$ contains a c-connected subgraph. Therefore, since H_i has average degree at least $c(a) - 1 \geq 264a$, H_i contains a $66a$-connected subgraph L_i. Bollobás and Thomason [1] have shown that every $22t$-connected graph is t-linked. Hence, the graph L_i is $3a$-linked. ▮

We will now construct a disjoint paths $\mathcal{P}_1^\circ, \ldots, \mathcal{P}_a^\circ$ by routing the paths $P_1, \ldots, P_s$ through L_i in at least k pairwise disjoint subgraphs H_i. In each graph L_i, there will also be an extra vertex linked to each of the a paths. Contracting these paths will then give a $K_{a,k}$-minor in G.

Claim 5.3. In H_i, there exist $2a$ pairwise disjoint paths, $Q_1^{(i)}, \ldots, Q_a^{(i)}$ and ${Q'}_1^{(i)}, \ldots, {Q'}_a^{(i)}$ such that the following hold:

(a) For $l = 1, 2, \ldots, a$, the path $Q_l^{(i)}$ starts in L_i and ends in S_{i+r}.
(b) For $l = 1, 2, \ldots, a$, the path ${Q'}_l^{(i)}$ starts in S_i and ends in L_i.
(c) Every path $Q_l^{(i)}$ and ${Q'}_l^{(i)}$ $(l = 1, 2, \ldots, a)$ has only its endvertices in $S_i \cup S_{i+r} \cup V(L_i)$.

Proof. Let $\Pi_0 = V(A_0)\backslash V(A_1)$ be the set of vertices of H_i corresponding to the trivial paths in A_0. Let $\mathcal{W} = \{W_1, \ldots, W_{2a}\}$ be a set of $2a$ pairwise disjoint paths joining $V(L_i)$ with $S_i \cup S_{i+r}$ such that:

(1) $W_l \subseteq H_i - \Pi_0$ for every $l = 1, 2, \ldots, 2a$.

(2) The number of edges in $\bigcup_{l=1}^{2a} E(W_l) \setminus \bigcup_{j=0}^{r-1} E(Z'(i+j))$ is minimum.

(3) Subject to (2), if n_L is the number of paths W_l ending in S_i, and n_R is the number of paths W_l ending in S_{i+r}, $|n_L - n_R|$ is minimum.

Disjoint paths satisfying (1) exist by large connectivity: Since $c(a) \geq 3a-1$, and $|V(L_i)| > 3a$, and $|S_i \cup S_{i+r}| \geq 3a - 1$, there exist $3a-1$ disjoint paths from $V(L_i)$ to $S_i \cup S_{i+r+1}$ by Menger's theorem. Since there are at most $a-1$ vertices in Π_0, the removal of those paths which intersect Π_0 leaves at least $2a$ paths satisfying condition (1).

If at least two paths of $\mathcal{W}$ intersect a path P_j, then let W and W' be the paths that intersect P_j as close as possible (on P_j) to S_i and S_{i+r}, respectively. If $W = W'$, suppose that the intersection u of W with P_j nearest S_i (say) comes before the intersection nearest S_{i+r}. By (2), W ends at S_i, i.e., its segment from u to its end coincides with the segment $P_j(u, S_i)$ of P_j. This shows that $W \neq W'$. Then the path W (resp. W') must end at S_i (resp. S_{i+r}) by (2).

Suppose that precisely one path, say $W \in \mathcal{W}$, intersects a path P_j. In this case we can elect to have W ending at $P_j \cap S_i$ or at $P_j \cap S_{i+r}$ by following the path P_j. This implies that the value $|n_L - n_R|$ in (3) can be made to be zero. Then $n_L = n_R = a$.

Now let the a paths in $\mathcal{W}$ that end in S_i be called ${Q'_1}^{(i)}, {Q'_2}^{(i)}, \ldots, {Q'_a}^{(i)}$ and the a paths in $\mathcal{W}$ that end in S_{i+r} be called $Q_1^{(i)}, Q_2^{(i)}, \ldots, Q_a^{(i)}$. It is easy to see that (c) may be requested. This completes the proof. ∎

Let T be a spanning tree of $A_0 \cap A_1$. By Claim 3.2, $|V(T)| \geq a$. This implies the following claim.

Claim 5.4. There are vertices $t_1, t_2, \ldots, t_a$ of T such that for $l = 1, 2, \ldots, a$, the vertex t_l is a leaf of the subtree $T \setminus \{t_1, \ldots, t_{l-1}\}$.

For each $i = 1, 2, \ldots, n_1 - r$ and each $l = 1, 2, \ldots, a$, let $J_l^{(i)} \in \{P_1, \ldots, P_s\}$ be the vertex of T such that $Q_l^{(i)}$ ends up on the corresponding path in G. Choose an enumeration of $Q_1^{(i)}, Q_2^{(i)}, \ldots, Q_a^{(i)}$ such that, for $l = 1, 2, \ldots, a$, the distance from $J_l^{(i)}$ to t_l in T is minimum (where smaller values of l have preference over the larger values).

Choose a similar enumeration of ${Q'_1}^{(i)}, \ldots, {Q'_a}^{(i)}$.

Define $\alpha = r + 4a + 2$ and for $t = 1, \ldots, k$ set $i_t = 1 + (t-1)\alpha$. Observe that $i_k = n_1 - r$.

To construct the path $\mathcal{P}_l^\circ$, we first link $Q_l^{(i_t)}$ to $Q_l'^{(i_{t+1})}$ for every $t = 1, \ldots, k-1$. Then each $Q_l'^{(i_t)}$ is linked to $Q_l^{(i_t)}$ inside L_{i_t} $(t = 1, \ldots, k)$. We do this as described below.

Let $i' = i + \alpha$. Link $Q_l^{(i)}$ with $Q_l'^{(i')}$ as follows: Follow the path $J_l^{(i)}$ from $J_l^{(i)} \cap S_{i+r}$ through $2l$ segments to the separator S_{i+r+2l}. Continue the path within Z_{i+r+2l} to the path t_l. This can be done by following the bridges between paths corresponding to the path in the spanning tree T from $J_l^{(i)}$ to t_l.

Construct a similar path from $Q_l'^{(i')}$ to t_l using bridges in $Z_{i'-2l}$. Then connect these paths along t_l, and denote by P_l^i the resulting path joining $Q_l^{(i)}$ with $Q_l'^{(i')}$.

Claim 5.5. The constructed paths P_l^i $(l = 1, \ldots, a)$ are pairwise disjoint.

Proof. Consider two of the paths, say P_l^i and P_m^i, where $l < m$. There are four possibilities where these two paths may intersect:

(1) P_l^i intersects $J_m^{(i)}$ inside Z_{i+r+2l}: This is not possible since $J_m^{(i)}$ would then be closer to t_l in T, and the path $Q_m^{(i)}$ would be indexed before $Q_l^{(i)}$.

(2) P_m^i intersects t_l inside Z_{i+r+2m}: This is not possible since t_l is a leaf in $T \setminus \{t_1, \ldots, t_{l-1}\}$.

The remaining cases, when P_l^i intersects P_m^i inside $Z_{i'-2l}$ or inside $Z_{i'-2m}$ (respectively) are handled similarly. This completes the proof. ∎

Let v_l be the vertex of $Q_l'^{(i)}$ in L_i, and let u_l be the vertex of $Q_l^{(i)}$ in L_i. Choose u_l' to be a neighbor of u_l in $L_i \setminus \{v_1, \ldots, v_a, u_1, \ldots, u_a\}$. Since L_i is $3a$-linked, the minimum degree of L_i is at least $3a$, so such neighbors exist. The vertices u_l' may even be chosen so that they are pairwise distinct. Let $v_1' = u_1'$, and let $v_2', \ldots, v_a'$ be distinct neighbors of v_1' in L_i. We may assume that if $v_\alpha' = u_\beta'$, then $\alpha = \beta$.

Since L_i is $2a$-linked, there is a linkage from $(v_1, \ldots, v_a, v_1', \ldots, v_a')$ to $(u_1, \ldots, u_a, u_1', \ldots, u_a')$. The resulting paths joining v_l and u_l $(l = 1, \ldots, a)$ are used to link $Q_l'^{(i)}$ and $Q_l^{(i)}$ inside L_i, for $i \in \{i_1, \ldots, i_k\}$. Together with the paths P_l^i, $i \in \{i_1, \ldots, i_{k-1}\}$, this determines the path $\mathcal{P}_l^\circ$. On the other hand, the paths in the linkage from $(v_1', \ldots, v_a')$ to $(u_1', \ldots, u_a')$ are disjoint from $\mathcal{P}_1^\circ, \ldots, \mathcal{P}_a^\circ$ and can be used to link v_1' to each of these paths.

Now, it can be shown that G contains a $K_{a,k}$ minor: For each $l = 1, \ldots, a$, contract the path $\mathcal{P}_l^\circ$ to a single vertex. For $i \in \{i_1, \ldots, i_k\}$, the vertex $v_1' \in V(L_i)$ is joined to $u_1', \ldots, u_a'$ and hence to each of the a paths $\mathcal{P}_1^\circ, \ldots, \mathcal{P}_a^\circ$. Since this is repeated k times, we get a $K_{a,k}$ minor in G.

The proof of Theorem 1.2 is complete.

6. CONCLUSION

Our more recent results show that the condition on bounded tree-width in Theorem 1.1 can be removed. The authors plan a second paper in which the large tree-width case is handled. This will prove the following, which was conjectured independently by Ding [3] and the authors:

There is a function $f : \mathbb{N} \to \mathbb{N}$ *such that any* 7*-connected graph on at least* $f(k)$ *vertices contains a* $K_{3,k}$ *minor.*

It seems reasonable to the authors that this result can be extended to $K_{4,k}$-minors and possibly even to $K_{a,k}$-minors. The logical conjectures would be the following:

Conjecture 6.1. There is a function $f : \mathbb{N} \to \mathbb{N}$ such that any 9-connected graph on at least $f(k)$ vertices contains a $K_{4,k}$ minor.

Conjecture 6.2. There are functions $f : \mathbb{N} \to \mathbb{N}$ and $c : \mathbb{N} \to \mathbb{N}$ such that any $c(a)$-connected graph on at least $f(k)$ vertices contains a $K_{a,k}$ minor.

Our final remark is that the sequence of graphs $K_{a,k}$, where a is fixed and k tends to infinity, is essentially the only family of graphs for which a result like our Theorem 1.2 holds. More precisely:

THEOREM 6.1. *Let* c *and* $w \geq c$ *be positive integers, and let* H_k $(k \geq 1)$ *be a sequence of graphs such that* $\lim_{k\to\infty} |V(H_k)| = \infty$. *Suppose that for any positive integer* k *there exists an integer* $N(k)$ *such that every* c*-connected graph of tree-width* $\leq w$ *and of order at least* $N(k)$ *contains* H_k *as a minor. Then* $H_k \leq_{\mathrm{m}} K_{c,N(k)}$ *for* $k \geq 1$.

Proof. Clearly, the graph $K_{c,N(k)}$ is c-connected and has tree-width $c \leq w$. By the assumption on the family H_k, $K_{c,N(k)}$ contains H_k as a minor. ▮

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