Light structures in infinite planar graphs without the strong isoperimetric property

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Abstract

It is shown that the discharging method can be successfully applied on infinite planar graphs of subexponential growth and even on those graphs that do not satisfy the strong edge isoperimetric inequality. The general outline of the method is presented and the following applications are given: Planar graphs with only finitely many vertices of degree ≤ 5 and with subexponential growth contain arbitrarily large finite submaps of the tessellation of the plane or of some tessellation of the cylinder by equilateral triangles. Every planar graph with isoperimetric number zero and with essential minimum degree ≥ 3 has infinitely many edges whose degree sum is at most 15. In particular, this holds for all graphs with minimum degree ≥ 3 and with subexponential growth. The cases without infinitely many edges whose degree sum is ≤ 14 (≤ 13 or ≤ 12, respectively) are also considered. Several further results are obtained.

1 Introduction

An infinite graph G satisfies the strong edge isoperimetric inequality if there is a positive constant h such that every finite vertex set X has at least $h|X|$ outgoing edges. The supremum of all such constants h is called the isoperimetric number of G. Graphs with positive isoperimetric number have many

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interesting properties and have been studied in relation to various other problems. For example, Dodziuk [6], Gerl [8] and Soardi [20] considered them in relation to random walks and their transience, Mohar [17, 18, 19] and Žuk [23] were interested in their spectral properties, Gromov in their hyperbolic geometric [9] and hyperbolic groups structure [10], Lyons et al. [14, 16] considered applications in theoretical physics (phase transition and percolation), Woess [22] was motivated by the study of harmonic functions, while He and Schramm [13] investigated circle packings.

It is easy to see that every finite planar graph contains a vertex of degree at most 5. This is not true for infinite planar graphs whose minimum vertex degree can be arbitrarily large. However, if an infinite planar graph has minimum degree greater than 6, it has exponential growth and satisfies the strong isoperimetric inequality. This fact was proved by Dodziuk [6] who used the strong isoperimetric inequality to prove transience of random walks on such graphs. Extensions of Dodziuk's result were obtained by various authors in different contexts. See, e.g., Mohar [19], Soardi [20], Calogero [5], Žuk [23], Woess [22], and Baues and Peyerimhoff [3]. Another extension is obtained in this paper where it is proved that every planar graph with only finitely many vertices of degree ≤ 5 and of subexponential growth contains every finite submap of T_0 or of T_0/Γ where T_0 is the infinite triangular lattice $[6^3]$, and $\Gamma \leq \mathrm{Aut}([6^3])$ is an infinite cyclic group.

If uv is an edge of a graph, the *weight* of uv is defined as the sum of degrees of its endvertices. Kotzig [15] proved that every (finite) 3-connected planar graph contains an edge uv of weight at most 13. Grünbaum and Shephard [11] extended Kotzig's theorem to graphs of periodic tilings of the plane by proving that there is an edge of weight at most 15. Such graphs are universal covering spaces of graphs embedded in the torus. Stehling [21] extended the result of Grünbaum and Shephard to normal tilings of the plane (where all tiles are uniformly bounded). Theorems 5.1 and 5.2 and Corollary 5.5 in this paper are extensions of that result to arbitrary infinite planar graphs of minimum degree ≥ 3 and with subexponential growth. Let us observe that the graphs considered by Grünbaum and Shephard in [11] or by Stehling [21] have quadratic growth, have only one end and have only faces of finite length.

Moreover, we prove that every connected infinite planar graph whose isoperimetric number is zero contains infinitely many edges of weight at most 15, and we also show that graphs without infinitely many edges of weight ≤ 14 (≤ 13 and ≤ 12, respectively) contain arbitrarily large submaps of certain tessellations of the plane or the flat cylinder.

More generally, it is shown that the well-known discharging method

(used, for example, in the proof of the Four Color Theorem [2, 12]) can be extended from finite planar graphs to infinite planar graphs of subexponential growth, and usually also to planar graphs whose edge isoperimetric number is zero. The main result (Theorem 3.1) and its proof give rise to explicit positive lower bounds on the isoperimetric number.

It is worth mentioning that planar graphs treated in this paper are much more general than graphs of normal tilings since they may have any of the following:

(a) Growth rates are arbitrary.

(b) Vertex degrees need not be bounded.

(c) Face lengths may be unbounded and faces of infinite length may exist.

(d) The graph may have more than one end (even uncountably many are possible).

In most of previous works (e.g., [3, 5, 11, 14, 21, 22, 23]), (a)–(d) are not allowed.

Graphs in this paper are simple (no loops, no multiple edges) and locally finite, i.e., finite or countably infinite and the degree of every vertex is finite.

We follow standard graph theory terminology and notation. If G is a graph and $v \in V(G)$, then $\deg_G(v)$ is the degree of v. If G is embedded in the plane and f is its face, then $\deg_G(f)$ denotes the length of f. If $X \subseteq V(G)$, then $G(X)$ is the subgraph of G induced on the vertex set X, and δX is the set of all edges with one endvertex in X and the other end in $V(G) \setminus X$. For a positive integer r, $N_r(X)$ denotes the set of all vertices in $V(G) \setminus X$ whose distance from X is at most r.

2 Strong isoperimetric property

Let G be a connected graph and let $v \in V(G)$. Denote by $B_n(v)$ the set of all vertices of G at distance at most n from v (the *ball* of radius n centered at v), and let $b_n(v) := |B_n(v)|$. If the graph G is not clear from the context, we write $b_n(G, v)$, or $B_n(G, v)$. The graph G has *exponential growth* (from the vertex v) if $b_n(v) \geq Cq^n$ for some constants $C > 0$ and $q > 1$, and for every $n \geq 0$. It has *polynomial growth* if $b_n(v) \leq p(n)$ where $p(\cdot)$ is a polynomial. G has *subexponential growth* if it does not grow exponentially. Let us observe that the type of the growth is independent of v since G is connected. Let

$$\tau(G) = \liminf_{n\to\infty} \left(b_n(G, v)\right)^{1/n} .$$

If v and u are vertices of G and $d = \mathrm{dist}(u,v)$, then $b_n(u) \le b_{n+d}(v) \le b_{n+2d}(u)$. It follows that $\tau(G)$ does not depend on the choice of v. The following is an easy exercise.

Proposition 2.1 *Let G be a connected graph. Then $\tau(G) > 1$ if and only if G has exponential growth.*

The (*edge*) *isoperimetric number* $h_1(G)$ of G is the number

$$h_1(G) = \inf \Big\{ \frac{|\delta X|}{|X|} \;\Big|\; X \subset V(G),\ 0 < |X| < \infty \Big\}.$$

Similarly, the (*vertex*) *isoperimetric number* $h_0(G)$ is defined as

$$h_0(G) = \inf \Big\{ \frac{|N_1(X)|}{|X|} \;\Big|\; X \subset V(G),\ 0 < |X| < \infty \Big\}.$$

Since every vertex in $N_1(X)$ is incident with an edge in δX, $h_0(G) \le h_1(G)$.

Proposition 2.2 *Let G be a connected graph. Then $\tau(G) \ge 1 + h_0(G)$. If G is a planar graph, then $\tau^2(G) \ge 1 + \frac{1}{2}h_1(G)$.*

Proof. Let $v \in V(G)$. For $n \ge 1$, let $b_n = b_n(v)$ and $s_n = b_n - b_{n-1}$. By selecting $X = B_{n-1}(G,v)$, we conclude that $s_n = |N_1(X)| \ge h_0(G)|X| = h_0(G)b_{n-1}$. This easily implies that $b_n \ge (1 + h_0(G))^n$. Consequently, $\tau(G) \ge 1 + h_0(G)$.

To prove the second inequality, let X be as above and let $A = N_1(X)$ and $B = N_1(X \cup A)$. Then $A \cup B$ and the edges joining A and B form a bipartite planar graph. Euler's formula and standard counting arguments imply that this graph has less than $2|A \cup B| = 2s_n + 2s_{n+1}$ edges. On the other hand, the edge set of this graph is $\delta(X \cup A)$ and hence its cardinality is at least $h_1(G)|X \cup A|$. As a consequence, $h_1(G)b_n \le 2(b_{n+1} - b_{n-1})$. Therefore, $b_n \ge (\frac{h_1(G)}{2} + 1)^{(n-1)/2}$. This implies that $\tau^2(G) \ge \frac{h_1(G)}{2} + 1$. □

There are planar graphs G such that $h_0(G) = h_1(G) = 0$ and $\tau(G) > 1$, and there are planar graphs for which $h_0(G) = 0$ and $h_1(G) > 0$ (and hence also $\tau(G) > 1$).

We say that G satisfies the *strong vertex isoperimetric inequality* if $h_0(G) > 0$, and that G satisfies the *strong edge isoperimetric inequality* if $h_1(G) > 0$.

A graph G has *bounded degrees* if there is an integer M such that $\deg_G(v) \le M$ for every $v \in V(G)$. Clearly,

Proposition 2.3 *If G has bounded degrees, then $h_0(G) > 0$ if and only if $h_1(G) > 0$.*

3 Discharging in infinite planar graphs

Let G be a locally finite plane graph. Let $F(G)$ be the set of faces of G. If H is a finite subgraph of G, consider its induced embedding in the plane. Let n_H, e_H, and f_H denote the number of vertices, edges, and faces of H, respectively. For $i \geq 0$, let n_i be the number of vertices of degree i in H, and let f_i be the number of faces of H of length i. Let us observe that $n_H = \sum_i n_i$, $2e_H = \sum_i i\, n_i = \sum_i i\, f_i$, and $f_H = \sum_i f_i$. Let $\alpha \in \mathbb{R}$ be a real number ($0 \leq \alpha \leq 3$), and let $\alpha' = 3 - \alpha$. Euler's formula implies that

$$\begin{aligned} 12 &\leq 6(n_H - e_H + f_H) \\ &= 6\sum_{i\geq 0} n_i - \alpha \sum_{i\geq 0} i\, n_i - \alpha' \sum_{i\geq 0} i\, f_i + 6\sum_{i\geq 0} f_i \\ &= \sum_{i\geq 0}(6 - \alpha\, i)n_i + \sum_{i\geq 0}(6 - \alpha' i) f_i \\ &= \sum_{v\in V(H)} (6 - \alpha \deg_H(v)) + \sum_{f\in F(H)} (6 - \alpha' \deg_H(f)). \qquad (1) \end{aligned}$$

Next, assign to each vertex $v \in V(G)$ and each face $f \in F(G)$ a *charge*

$$c(v) = \alpha \deg_G(v) - 6 \qquad \text{and} \qquad c(f) = \alpha' \deg_G(f) - 6,$$

respectively. (If $\deg(f) = \infty$, then we set $c(f) = \infty$.)

Similarly, denote by $c_H(v) = \alpha \deg_H(v) - 6$ ($v \in V(H)$) and $c_H(f) = \alpha' \deg_H(f) - 6$ ($f \in F(H)$) the corresponding charges of vertices and faces of H. Inequality (1) shows that the sum of all charges in H is negative.

Suppose that there is a set $\mathcal{R}$ of *discharging rules* that defines a redistribution of charges in G in the following way. The rules determine that each vertex and each face sends some of its charge to other vertices and faces. Let $\varphi(x, y) \geq 0$ be the charge that is sent from x to y ($x, y \in V(G) \cup F(G)$). Let $c^-(x) = \sum_y \varphi(x, y)$ be the charge sent from x to other vertices and faces, and let $c^+(x) = \sum_y \varphi(y, x)$ be the charge received at x from other vertices and faces when applying the rules. The new value of the charge at x is then equal to $c^*(x) = c(x) - c^-(x) + c^+(x)$ and is called the *final charge* at x.

Discharging rules not only determine the values $\varphi(x, y)$ but also determine one or more paths $P(x, y)$ along which the charge is sent from x to y. If $P(x, y)$ contains more than one path, it is also determined which proportion of $\varphi(x, y)$ is sent from x to y along particular paths in $P(x, y)$. Each path in $P(x, y)$ is a path in G except that its first (and last) edge is an auxiliary edge joining the "center" of a face with an incident vertex if x (or y) is a

face of G. The maximum length of paths in $P(x,y)$, taken over all pairs x,y such that $\varphi(x,y) \neq 0$, is called the *impact range* of $\mathcal{R}$.

Let e be an (oriented) edge of G, or an auxiliary edge joining the center of a face with an incident vertex. Let $\varphi(e)$ be the sum of all charges $\varphi(x,y)$ (or corresponding parts of these charges) for which a path in $P(x,y)$ uses the edge e in the given direction. The value $\varphi_0 = \sup\{\varphi(e)\}$ is called the *flow value* of $\mathcal{R}$.

Let $F \in F(G)$, and let x be a vertex incident with F. Let $\varphi'(F,x)$ denote the total charge sent from F through x, i.e. $\varphi'(F,x) = \varphi(e)$ where e is the auxiliary edge joining F and x. The discharging rules $\mathcal{R}$ are said to be *smooth* if there is a constant c_0' such that for every $F \in F(G)$ and every segment S on the facial walk of F the following condition holds:

$$\varphi'(F, V(S)) = \sum_{x \in V(S)} \varphi'(F,x) \leq \alpha' |V(S)| + c_0'.$$

The condition of smoothness is needed in case when there are arbitrarily large or even infinite faces.

If X and Y are finite subsets of $V(G) \cup F(G)$, then $c(X) = \sum_{x \in X} c(X)$ and $\varphi(X,Y) = \sum\{\varphi(x,y) \mid x \in X,\ y \in Y\}$. The same notation is used for other functions defined on vertices (and faces) of G. For example, $\deg(X) = \sum_{x \in X} \deg(x)$.

Theorem 3.1 *Let G be a connected infinite planar graph and let $0 < \alpha < 3$ and $\varepsilon > 0$ be real numbers. Let c and c^* be the charge and the final charge, respectively, with respect to a set of smooth discharging rules of impact range at most r and with flow value φ_0. Suppose that there is a finite set $\mathcal{F} \subset F(G)$ of faces such that the final charge of every face in $F(G) \setminus \mathcal{F}$ is nonnegative, and that there is a finite vertex set $U \subset V(G)$ such that $c^*(v) \geq \varepsilon$ for every vertex $v \in V(G) \setminus U$. Then the following holds:*

(a) *G has exponential growth.*

(b) *If G has bounded degree, then $h_0(G) > 0$.*

(c) *If $r \leq 1$, then $h_1(G) > 0$.*

Proof. Let X be a finite set of vertices of G and let $\overline{X} = X \cup N_r(X)$. Let $H = G(X)$ be the subgraph of G induced on X. Let $\mathcal{F}_0 \subseteq F(H) \cap F(G)$ be the set of faces of H that are also faces of G. Let $\mathcal{F}_1' = F(H) \setminus \mathcal{F}_0$, and let

$\mathcal{F}_1 \subseteq F(G)$ be the set of faces of G that are not in $\mathcal{F}_0$ but are incident with an edge with its endvertices in $\overline{X}$. Let

$$\nu = -\sum\{c^*(v) \mid v \in V(G),\ c^*(v) < 0\}$$

and

$$\phi = -\sum\{c^*(f) \mid f \in F(G),\ c^*(f) < 0\}.$$

The assumptions of the theorem imply:

$$c^*(X) \geq \varepsilon(|X| - |U|) - \nu \tag{2}$$

and

$$c^*(\mathcal{F}_0) \geq -\phi. \tag{3}$$

Every face in $\mathcal{F}_1$ is incident with at least two edges in $\delta X \cup E(G(N_r(X)))$ (possibly twice with the same edge), and every such edge is contained in at most two faces in $\mathcal{F}_1$. Therefore,

$$\begin{aligned} |\mathcal{F}_1'| \ \leq\ |\mathcal{F}_1| \ &\leq\ |\delta X| + |E(G(N_r(X)))| \\ &\leq\ |\delta X| + 3|N_r(X)|. \end{aligned} \tag{4}$$

In the second inequality we used a well-known corollary of Euler's formula that a finite planar graph on n vertices contains less than $3n$ edges. Next,

$$\begin{aligned} c_H(F(H)) &= c_H(\mathcal{F}_0) + c_H(\mathcal{F}_1') \\ &= c(\mathcal{F}_0) + \alpha' \deg_H(\mathcal{F}_1') - 6|\mathcal{F}_1'| \\ &\geq c(\mathcal{F}_0) + \alpha' \deg_H(\mathcal{F}_1') - 6(|\delta X| + 3|N_r(X)|). \end{aligned} \tag{5}$$

Since $c_H(v) = c(v) - \alpha \deg_G(v) + \alpha \deg_H(v)$ $(v \in X)$, we have:

$$c_H(X) = c(X) - \alpha \deg_G(X) + \alpha \deg_H(X) = c(X) - \alpha|\delta X|. \tag{6}$$

We will need estimates on the charge increase in $\mathcal{F}_0$ and in X. Since the discharging rules have impact range $\leq r$, the faces in $\mathcal{F}_0$ and vertices in X send (and receive) nonzero charge only to (from) $\overline{X} \cup \mathcal{F}_0 \cup \mathcal{F}_1$. Therefore,

$$\begin{aligned} c^+(\mathcal{F}_0) - c^-(\mathcal{F}_0) &= \varphi(\mathcal{F}_0, \mathcal{F}_0) + \varphi(\overline{X}, \mathcal{F}_0) + \varphi(\mathcal{F}_1, \mathcal{F}_0) \\ &\quad - \varphi(\mathcal{F}_0, \mathcal{F}_0) - \varphi(\mathcal{F}_0, \overline{X}) - \varphi(\mathcal{F}_0, \mathcal{F}_1) \\ &\leq \varphi(\mathcal{F}_1, \mathcal{F}_0) + \varphi(\overline{X}, \mathcal{F}_0) - \varphi(\mathcal{F}_0, X) \end{aligned} \tag{7}$$

and

$$\begin{aligned} c^+(X) - c^-(X) &= \varphi(X,X) + \varphi(\mathcal{F}_0,X) + \varphi(N_r(X)\cup\mathcal{F}_1,X) \\ &\quad - \varphi(X,X) - \varphi(X,\mathcal{F}_0) - \varphi(X,N_r(X)\cup\mathcal{F}_1) \\ &\leq \varphi(\mathcal{F}_0,X) + \varphi(N_r(X)\cup\mathcal{F}_1,X) - \varphi(X,\mathcal{F}_0). \end{aligned} \tag{8}$$

Smoothness of the discharging rules is needed in the following estimate. Let us observe that the charge sent from $N_r(X)\cup\mathcal{F}_1$ to $X\cup\mathcal{F}_0$ either flows through some edge in δX or goes from a face $F\in\mathcal{F}_1$ through a vertex in X incident with F. This implies

$$\begin{aligned} \varphi(N_r(X)\cup\mathcal{F}_1, X\cup\mathcal{F}_0) &\leq \varphi_0|\delta X| + \sum_{F\in\mathcal{F}_1}\sum_{x\in X\cap V(F)} \varphi'(F,x) \\ &\leq \varphi_0|\delta X| + \alpha' \deg_H(\mathcal{F}_1') + c_0'|\delta X| \end{aligned} \tag{9}$$

where c_0' is the constant from the definition of the smoothness of $\mathcal{R}$. In the last inequality we used the fact that the number of facial segments S of faces in $\mathcal{F}_1$ which have all their vertices in X is at most $|\delta X|$, and that their total length is smaller than $\deg_H(\mathcal{F}_1')$.

Starting with (1) and applying inequalities (2)–(9) we get:

$$\begin{aligned} -12 &\geq c_H(X) + c_H(F(H)) \\ &\geq c(X) - \alpha|\delta X| + c(\mathcal{F}_0) - 6(|\delta X| + 3|N_r(X)|) + \alpha' \deg_H(\mathcal{F}_1') \\ &= c^*(X) - c^+(X) + c^-(X) + c^*(\mathcal{F}_0) - c^+(\mathcal{F}_0) + c^-(\mathcal{F}_0) \\ &\quad - (\alpha+6)|\delta X| - 18|N_r(X)| + \alpha' \deg_H(\mathcal{F}_1') \\ &\geq c^*(X) - \varphi(\mathcal{F}_0,X) - \varphi(N_r(X)\cup\mathcal{F}_1,X) + \varphi(X,\mathcal{F}_0) \\ &\quad + c^*(\mathcal{F}_0) - \varphi(\mathcal{F}_1,\mathcal{F}_0) - \varphi(\overline{X},\mathcal{F}_0) + \varphi(\mathcal{F}_0,X) \\ &\quad - (\alpha+6)|\delta X| - 18|N_r(X)| + \alpha' \deg_H(\mathcal{F}_1') \\ &= c^*(X) + c^*(\mathcal{F}_0) - \varphi(N_r(X)\cup\mathcal{F}_1, X\cup\mathcal{F}_0) \\ &\quad - (\alpha+6)|\delta X| - 18|N_r(X)| + \alpha' \deg_H(\mathcal{F}_1') \\ &\geq \varepsilon|X| - \varepsilon|U| - \nu - \phi \\ &\quad - 18|N_r(X)| - (\varphi_0 + c_0' + \alpha + 6)|\delta X|. \end{aligned} \tag{10}$$

Suppose now that X is large, say $|X| \geq \frac{2}{\varepsilon}(\varepsilon|U| + \nu + \phi)$. Then (10) implies

$$|\delta X| + |N_r(X)| \geq \frac{\varepsilon}{2\rho}|X|. \tag{11}$$

where $\rho = \max\{18, \varphi_0 + c_0' + \alpha + 6\}$.

If $r \leq 1$ then $|N_r(X)| \leq |\delta X|$. Now, (11) shows that $|\delta X| \geq \frac{\varepsilon}{4\rho}|X|$. Since this holds for every vertex set X which is large enough and since G is connected, it follows that $h_1(G) > 0$. This proves (c). By Propositions 2.2 and 2.3, this proves also (a) and (b) in the case when $r \leq 1$.

Suppose now that $r \geq 1$. Inequality (11) holds for every finite vertex set X with at least $\frac{2}{\varepsilon}(\varepsilon|U| + \nu + \phi)$ elements. In particular, it applies to the set $X_1 = X \cup N_1(X)$:

$$|\delta X_1| + |N_r(X_1)| \geq \frac{\varepsilon}{2\rho}|X_1|. \tag{12}$$

As in the proof of Proposition 2.2, we see that $|\delta X_1| \leq 2(|N_1(X)|+|N_1(X_1)|)$ $= 2|N_2(X)| \leq 2|N_{r+1}(X)|$. Since $|N_r(X_1)| \leq |N_{r+1}(X)|$, (12) gives:

$$|N_{r+1}(X)| \geq \frac{\varepsilon}{6\rho}|X|.$$

Consequently,

$$\tau(G) \geq \left(1 + \frac{\varepsilon}{6\rho}\right)^{1/(r+1)} > 1. \tag{13}$$

This completes the proof of (a).

Suppose now that $\deg_G(v) \leq M$ (where $M \geq 3$) for every $v \in V(G)$. Then

$$\begin{aligned} |N_{r+1}(X)| &\leq |N_1(X)| \cdot (1 + (M-1) + (M-1)^2 + \cdots + (M-1)^r) \\ &\leq M^{r+1}|N_1(X)|. \end{aligned}$$

Therefore, $|N_1(X)| \geq M^{-(r+1)}\frac{\varepsilon}{6\rho}|X|$ (for every finite vertex set X with sufficiently many vertices). Since G is connected, it follows that $h_0(G) > 0$.

□

Let us observe that the proof of Theorem 3.1 gives explicit positive lower bounds on the exponential growth rate $\tau(G)$ and on the isoperimetric constants $h_0(G)$ and $h_1(G)$, respectively.

If $r = 0$ (i.e., no discharging rules are used), the inequalities in the proof of Theorem 3.1 can be strengthened as follows. We shall assume that $U = \emptyset$ and $\mathcal{F} = \emptyset$, so that $\nu = \phi = 0$. First of all, (4) can be replaced by

$$|\mathcal{F}_1'| \leq |\delta X| \tag{14}$$

and (5) by

$$c_H(F(H)) \geq c(\mathcal{F}_0) + (3\alpha' - 6)|\delta X|. \tag{15}$$

Inequality (9) is void since $\varphi(N_r(X) \cup \mathcal{F}_1, X \cup \mathcal{F}_0) = 0$. Then (10) reduces to

$$-12 \geq \varepsilon|X| - (\alpha + 6 - 3\alpha')|\delta X| = \varepsilon|X| - (9 - 4\alpha')|\delta X|. \tag{16}$$

Assuming that $\alpha' < \frac{9}{4}$, this inequality implies

$$h_1(G) \geq \frac{\varepsilon}{9 - 4\alpha'}. \tag{17}$$

4 Planar graphs of minimum degree 6

Let $T_0 = [6^3]$ be the tessellation of the plane with equilateral triangles (see Figure 1). We may assume that the origin of $\mathbb{R}^2$ is a vertex of T_0. Denote that vertex by z_0. Let z be a vertex of T_0 distinct from z_0. Let $\Gamma \leq \mathrm{Aut}(T_0)$ be the infinite cyclic group generated by the translation of $\mathbb{R}^2$ which maps z_0 to z. Then $T_0^z = T_0/\Gamma$ is a tessellation of the "flat" cylinder by equilateral triangles. (The graph of T_0^z may have loops or multiple edges if $\mathrm{dist}_{T_0}(z, z_0) \leq 2$.) If r is an integer, denote by $T_0(r)$ and $T_0^z(r)$ the submap of T_0 or T_0^z, respectively, consisting of all vertices and faces that are at distance $\leq r$ from z_0.

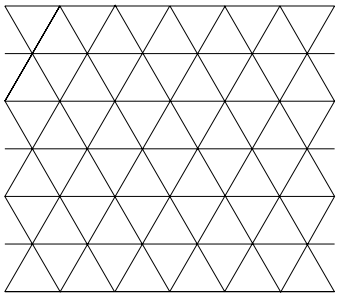

Figure 1: A fragment of the tessellation $[6^3]$

Let G_0 and G be plane graphs and let H_0 be a subgraph of G_0. Suppose that G contains a subgraph H isomorphic to H_0 such that a closed walk in H bounds a face of G if and only if the corresponding closed walk in H_0 bounds a face of G_0. Then we say that G *contains* H_0 (as a submap).

Theorem 4.1 *Let G be a finite planar graph, r a positive integer, and let $m \geq (3r+1)^2$. Suppose that v is a vertex of G such that all vertices at distance $\leq m$ from v are of degree* 6 *and are incident only with triangular faces. Then one of the following holds:*

(a) *G contains $T_0(r)$ as a submap.*

(b) *G contains $T_0^z(3r)$ as a submap where* $\mathrm{dist}_{T_0}(z, z_0) \leq 3r - 1$.

Proof. Let $U \subseteq V(G)$ be the set of vertices of G at distance at most m from v. Fix a triangle A_0 incident with v. Let C be a cycle of G with $V(C) \subseteq U$. Denote by $\mathrm{int}(C)$ the component of $\mathbb{R}^2 \setminus C$ which does not contain A_0. For $u \in V(C)$, let $\delta(u, C)$ be the number of edges incident with u which are embedded in $\mathrm{int}(C)$, and let $\eta(C) = \sum_{u \in V(C)} (\delta(u, C) - 2)$.

The following property of $\eta(C)$ is easy to prove:

(P1) If P is a path in $\mathrm{int}(C)$ with $V(P) \subseteq U$ such that $P \cap C$ are the endvertices of P, let C_1 and C_2 be the two cycles of $C \cup P$ distinct from C. Then
$$\eta(C_1) + \eta(C_2) = \eta(C) - 6.$$

If C_2 in (P1) is a facial triangle, then (P1) implies that $\eta(C_1) = \eta(C)$. Repetitive use of this property shows that:

(P2) If a cycle C_1 with $V(C_1) \subseteq U$ is obtained from C by consecutively adding facial triangles with at least one edge in common with the current cycle, then $\eta(C_1) = \eta(C)$.

Properties (P1) and (P2) can be used to prove:

(P3) Suppose that $|V(C)| = k$ and that all vertices in $\mathrm{int}(C)$ that are at distance at most $k - 2$ from C are in U. If $\eta(C) < 0$, then $\eta(C) = -6$ and all vertices in $\mathrm{int}(C)$ are at distance at most $k - 3$ from C.

The proof of (P3) is by induction on k. Suppose first that $k = 3$. Since $\eta(C) < 0$, there is a vertex $u \in V(C)$ with $\delta(u, C) \leq 1$. It is easy to see that $\delta(u, C) = 1$ is not possible. Therefore, $\delta(u, C) = 0$. This implies that C is a facial triangle, and hence (P3) holds.

Suppose now that $k \geq 4$. Let A be the set of facial triangles in $\mathrm{int}(C)$ that have a vertex in $V(C)$. Let E' be the set of edges in $\mathrm{int}(C)$ that are contained in precisely one triangle in A. If $E' = \emptyset$ then A partitions $\mathrm{int}(C)$ into triangles, and (P2) implies that $\eta(C)$ is equal to $\eta(T) = -6$ for some $T \in A$. Hence, we may assume that $E' \neq \emptyset$. Then E' can be partitioned into cycles $C_1, \ldots, C_r$ ($r \geq 1$) with disjoint interiors. The cycles C_i are boundaries of faces of $C \cup A$ in $\mathrm{int}(C)$ that are not triangles in A. Since $C_1, \ldots, C_r$ can be obtained from C by successively applying operations in (P1) and (P2),

$$\eta(C_1) + \cdots + \eta(C_r) = \eta(C) - 6(r - 1). \tag{18}$$

A simple calculation shows that $|E'| \leq k + \eta(C) < k$. This implies that $|V(C_i)| < k$ for $i = 1, \ldots, r$. By the induction hypothesis, either $\eta(C_i) = -6$ or $\eta(C_i) \geq 0$. Since $\eta(C) < 0$, (18) implies that $\eta(C_i) = -6$ for every i, and, consequently, $\eta(C) = -6$. Moreover, all vertices in $\mathrm{int}(C_i)$ are at distance at most $k-4$ from C_i, $i = 1, \ldots, r$. Therefore, all vertices in $\mathrm{int}(C)$ are at distance at most $k-3$ from C. This completes the proof of (P3).

Now we are prepared to prove the theorem. Let q be the largest integer $\leq m$ such that the q-ball $B_q(G, v)$ is isomorphic to the q-ball $B_q(T_0, z_0)$ in T_0. We are done if $q \geq r$, so suppose that $1 \leq q \leq r-1$.

Denote by C the outer cycle of $B_q(G, v)$. Note that $|V(C)| = 6q$ and that $\eta(C) = 6$. Since $B_{q+1}(G, v)$ is not isomorphic to $B_{q+1}(T_0, z_0)$, there are distinct vertices $u_1, u_2 \in V(C)$ that are not adjacent on C, and there are edges e_1, e_2 incident with u_1 and u_2, respectively, such that either $e_1 = e_2$, or e_1 and e_2 have a vertex u in $\mathrm{int}(C)$ in common. Let C_1 and C_2 be the cycles in $C + e_1 + e_2$ distinct from C. For $i = 1, 2$, $|V(C_i)| \leq 6q$ and all vertices at distance at most $6q-2$ from C_i are in U. By (P2), $\eta(C_1) + \eta(C_2) = \eta(C) - 6 = 0$. If $\eta(C_i) < 0$, then by (P3), $\eta(C_i) = -6$. This implies that $\delta(u_1, C_i) = \delta(u_2, C_i) = \delta(u, C_i) = 0$ since all vertices u' on $C \cap C_i$ distinct from u_1 and u_2 have $\delta(u', C_i) = \delta(u', C) \geq 2$. Clearly, this is not possible. Consequently, $\eta(C_1) = \eta(C_2) = 0$.

Let R be a shortest cycle in G with the following properties:

(a) All vertices of R are at distance at most $(3r+1)(3r - |V(R)|)$ from v. (In particular, $V(R) \subseteq U$ and $|V(R)| \leq 3r-1$.)

(b) $\eta(R) = 0$.

Since $q \leq r-1$, $|V(C)| = 6q \leq 6r-6$. Assuming $|V(C_1)| \leq |V(C_2)|$, we have $|V(C_1)| \leq \frac{1}{2}|V(C)| + 2 \leq 3r-1$. Since all vertices on C_1 are at distance at most $q+1$ from v, C_1 satisfies (a) and (b). Therefore, R exists.

Suppose that there exists a path P in G, such that P intersects R precisely at its endvertices u_1, u_2, and such that the length of P is smaller than the distance from u_1 to u_2 on R. Consider the cycles $R_1, R_2 \subset R \cup P$ distinct from R. (P1) and (P3) imply that $\eta(R_1) = 0$ and $\eta(R_2) = -6$ (or vice versa). Since R_1 is shorter than R and every vertex on R_1 is at distance at most $3r(3r-k) + \frac{k}{2}$ from v, R_1 satisfies (a)–(b) and contradicts the minimality of R. This shows that R is an isometric subgraph of G.

Fix any vertex v_0 on R. Then R determines a path $P(R)$ in T_0 from z_0 to some vertex $z \in V(T_0)$ such that v_0 corresponds to the ends of the path and such that at each other vertex on $P(R)$, the number of edges on the "left" side of the path is equal to $\delta(u, R)$, where u is the corresponding

vertex on R. Since R is an isometric subgraph in G, the path $P(R)$ is also isometric in T_0. The path $P(R)$ can be extended to a two-way-infinite path $P_\infty(R)$ in T_0. Since $\eta(R) = 0$, it is easy to see that $P_\infty(R)$ does not cross itself and that it gives rise to a cycle R' in T_0^z.

We claim that the vertices of G at distance at most $3r$ from R form a submap of G isomorphic to the distance-$3r$ neighborhood of R' in T_0^z. If not, then we proceed in the same way as above when we proved existence of C_1 and C_2. Here, we obtain cycles R_1 and R_2 in G whose length is not larger than the length k and such that $\eta(R_1) + \eta(R_2) = \eta(R) - 6 = -6$. Then $\eta(R_1) = 0$ and $\eta(R_2) = -6$ (or vice versa). If the length of R_1 is smaller than k, we get a contradiction to the minimality of R. Otherwise, the length of R_2 is either 3 or 4. Since R_2 is not a facial triangle and since $\eta(R_2) = -6$, we easily get a contradiction. This completes the proof. □

If G is a graph, its *minimum degree* is denoted by $\delta(G)$. We also define its *minimum essential degree* $\delta_{\mathrm{ess}}(G)$ as the minimum integer d such that there are infinitely many vertices of degree d. If such d does not exist then $\delta_{\mathrm{ess}}(G) = \infty$.

A simple application of Theorem 3.1(c) (with $\alpha = 1$ and $r = 0$, i.e., no discharging rules at all) shows that $h_1(G) > 0$ if $\delta_{\mathrm{ess}}(G) \geq 7$. This result was proved by Dodziuk [6]. Here we show that a similar result holds when $\delta_{\mathrm{ess}}(G) = 6$.

Theorem 4.2 *Let G be a connected infinite planar graph with $\delta_{\mathrm{ess}}(G) \geq 6$. If G has subexponential growth, then either G contains every finite submap of T_0, or there exists a vertex $z \in V(T_0)$ such that G contains every finite submap of T_0^z.*

Proof. Let U be the set of vertices whose degree is at most 5, and let U_6 be the set of vertices of degree 6. Let m be an arbitrary positive integer. We shall apply the discharging method with $\alpha = 1$ and use the following discharging rule.

Let x be a face of length ≥ 4 or a vertex which is not in U_6. Let $P(x)$ be the set of all paths of length at most $m+1$ that start at x and whose all other vertices are in U_6. (If x is a face then the first edge of every such path is an auxiliary edge joining the "center" of x with an incident vertex of degree 6 in G.) There are at most $\deg_G(x) \cdot 6^{m+1}$ such paths, and the discharging rule determines that x sends charge $\varepsilon = (2 \deg_G(x) \cdot 6^{m+1})^{-1}$ along every path in $P(x)$ to the terminal vertex of the path. Clearly, $c^*(x) \geq c(x) - \frac{1}{2} \geq \frac{1}{2}$ if $x \in V(G) \setminus (U \cup U_6)$. Also, every face has nonnegative final charge and every vertex that belongs to a path in some $P(x)$ has its final charge $\geq \varepsilon$.

Theorem 3.1 can be applied to the given discharging process, showing that either G has exponential growth, or there is a vertex $v \in U_6$ that has not received any charge. Since G has subexponential growth, v exists, and every vertex at distance at most m from v is of degree 6 and incident only with triangular faces.

As m is arbitrarily large, Theorem 4.1 implies that there is an infinite sequence of pairs $(r_i, z_i) \in \mathbb{N} \times V(T_0)$, where $r_i \to \infty$ as $i \to \infty$ and $\mathrm{dist}_{T_0}(z_i, z_0) \leq 3r_i - 1$, such that G either contains $T_0(r_i)$ or $T_0^{z_i}(3r_i)$. If the first case occurs infinitely often or if $\limsup \mathrm{dist}_{T_0}(z_i, z_0) = \infty$, then G contains every finite submap of T_0. Otherwise, there exists $z \in V(T_0)$ such that $z_i = z$ for infinitely many values of i. In that case, G contains every finite submap of T_0^z. This completes the proof. □

5 Kotzig's theorem for infinite graphs

Let $uv \in E(G)$. The sum of degrees, $\deg(u) + \deg(v)$, of the ends of uv is called the *weight* of the edge uv. If $\deg(u) \leq \deg(v)$, then the edge uv is said to be of *type* $(\deg(u), \deg(v))$. The edge is *very light* if its weight is at most 11 or if it is of type (3,9) or (3,10). The edge is *light* if it is either very light or of one of the following types: (3,11), (3,12), (4,8), or (6,6).

Kotzig [15] proved that every finite 3-connected planar graph contains a very light edge. Borodin [4] showed that the same result holds under a weaker hypothesis that the minimum degree is ≥ 3. Grünbaum and Shephard [11] and Stehling [21] extended Kotzig's theorem to graphs of doubly periodic tilings and of normal tilings of the plane, respectively, by showing that they always contain light edges. Let us observe that the graphs of normal tilings have quadratic growth.

The main result of this section is an extension of Grünbaum and Shephard's result to arbitrary planar graphs that do not satisfy the strong isoperimetric inequality. In particular, this holds for all graphs of subexponential growth.

Theorem 5.1 *Let G be a connected infinite planar graph with $\delta_{\mathrm{ess}}(G) \geq 3$. If G has only finitely many light edges, then $h_1(G) > 0$.*

Proof. Suppose that G is a connected infinite planar graph with $l < \infty$ light edges. We define the charge for vertices and faces of G as in Section 3 with $\alpha = 1$. Then all faces have nonnegative charge and all vertices of degree ≥ 7 have positive charge $c(v) = \deg(v) - 6$. Let $\phi(3) = \frac{21}{20}$, $\phi(4) = \frac{3}{5}$,

$\phi(5) = \frac{1}{4}$, and $\phi(6) = \frac{1}{20}$. We shall apply the following set of discharging rules.

Rule 1: Suppose that f is a face where $\deg(f) \geq 4$ and that v is a vertex incident with f. If $4 \leq \deg(v) \leq 6$, then $\varphi(f, v) = \phi(\deg(v))$. If $\deg(v) = 3$, then $\varphi(f, v) = 1$. Otherwise, $\varphi(f, v) = 0$.

Rule 2: Suppose that $uv \in E(G)$ is an edge of G where $\deg(u) \geq 7$ and $\deg(v) \leq 6$. If both faces containing uv are triangles, then $\varphi(u, v) = \phi(\deg(v))$. If only one face containing uv is a triangle, then $\varphi(u, v) = \frac{1}{2}\phi(\deg(v))$. If uv is not contained in a triangular face and $\deg(v) = 3$, then $\varphi(u, v) = \frac{1}{20}$. Otherwise, $\varphi(u, v) = 0$.

Rules 1 and 2 clearly satisfy the premises of Theorem 3.1 with impact range $r = 1$ and with flow value $\varphi_0 = \frac{21}{20}$. Therefore, it suffices to see that the final charge is nonnegative for all faces that are not incident with light edges, and that the final charge is $\geq \varepsilon = \frac{1}{10}$ for all vertices of degree ≥ 3 that are not incident with a light edge.

Let $f \in F(G)$. If $\deg(f) = 3$ then $c^*(f) = c(f) = 0$. Otherwise, suppose that $\deg(f) \geq 4$ and that f is not incident with a light edge. Clearly, for any two consecutive vertices u, v on the facial walk of f,

$$\varphi(f, u) + \varphi(f, v) \leq 1.$$

This implies that $c^-(f) \leq \frac{1}{2}\deg(f)$, and hence $c^*(f) = c(f) - c^-(f) \geq \frac{3}{2}\deg(f) - 6 \geq 0$.

Suppose now that v is a vertex of degree $d \geq 3$ that is not incident with a light edge. If $d = 3$, then it is easy to see that $c^+(v) \geq 3 + \frac{1}{10}$, and hence $c^*(v) \geq \frac{1}{10}$. If $4 \leq d \leq 6$, then $c^+(v) = d \cdot \phi(d)$. Therefore, $c^*(v) = \frac{2}{5}$ if $d = 4$, $c^*(v) = \frac{1}{4}$ if $d = 5$, and $c^*(v) = \frac{3}{10}$ if $d = 6$.

If $7 \leq d \leq 8$, then v sends charge only to vertices of degrees 5 and 6. To any two consecutive neighbors u, w (with respect to the clockwise orientation around v), v sends at most charge $\varphi(5) = \frac{1}{4}$. Therefore, $c^*(v) \geq d - 6 - \frac{d}{2} \cdot \frac{1}{4} \geq \frac{1}{8}$.

If $9 \leq d \leq 12$, then v sends charge only to vertices of degrees 4, 5, and 6. To any two consecutive neighbors it sends at most $\varphi(4) = \frac{3}{5}$. Therefore, $c^*(v) \geq d - 6 - \frac{d}{2} \cdot \frac{3}{5} \geq \frac{3}{10}$.

If $d \geq 13$, then v sends at most $\varphi(3)$ to any two consecutive neighbors. Thus, $c^*(v) \geq d - 6 - \frac{d}{2} \cdot \frac{21}{20} \geq \frac{7}{40}$. This completes the proof. □

Theorem 5.1 is best possible as shown by the following examples. Let $T_0 = [6^3]$ be the tessellation of the plane by equilateral triangles, let $T_1 = [4.8^2]$ be the triangulation of the plane obtained from the tessellation with

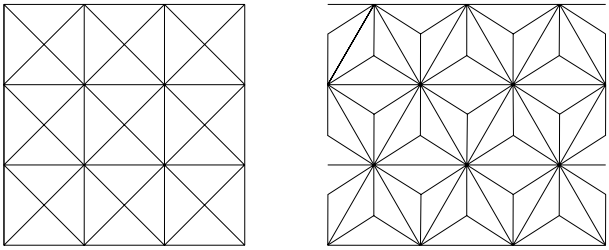

Figure 2: Fragments of tessellations $[4.8^2]$ and $[3.12^2]$

squares by adding a vertex of degree 4 in each square, and let $T_2 = [3.12^2]$ be the triangulation of the plane obtained from $[6^3]$ by inserting a vertex of degree 3 into each triangle of $[6^3]$. See Figure 2. These examples contain only edges of types (6,6), (4,8), and (3,12), respectively. We also define the cylindrical quotients T_1^z and T_2^z in the same way as T_0^z (cf. Section 4).

We need some further notation. Let $T = T_0$ or $T = T_0^z$, and let $\mathcal{F}$ be a set (possibly infinite) of facial triangles of T such that every vertex of T is incident with at most one triangle in $\mathcal{F}$. Next, add into each face of T which is not in $\mathcal{F}$ a new vertex of degree 3 joined to the vertices on the boundary of that face. Let $T(\mathcal{F})$ be the resulting map. Note that $T_0(\emptyset) = T_2$ and $T_0^z(\emptyset) = T_2^z$.

Theorem 5.2 *Let G be a connected infinite planar graph of subexponential growth and with $\delta_{\text{ess}}(G) \geq 3$.*

(a) *If G has only finitely many very light edges and finitely many edges of types $(4,8)$, $(6,6)$, and $(3,11)$, then either G contains every finite submap of T_2, or G contains every finite submap of some T_2^z.*

(b) *If G has only finitely many very light edges and finitely many edges of types $(4,8)$ and $(6,6)$, then G contains arbitrarily large finite submaps of some map of the form $T_0(\mathcal{F})$ or $T_0^z(\mathcal{F})$.*

(c) *If G has only finitely many very light edges, then either G contains arbitrarily large finite submaps of some map of the form $T_0(\mathcal{F})$ or $T_0^z(\mathcal{F})$, or G contains every finite submap of a map of the form T_0, T_0^z, T_1, or T_1^z.*

Proof. The proof is a continuation of the proof of Theorem 5.1 by adding similar discharging rules as used in the proof of Theorem 4.2, with an application of Theorem 4.1 and its extension from T_0 to T_1 and T_2. We leave the

details to the reader. Let us only observe that in the case (b), one cannot guarantee that G contains every finite submap of $T_0(\mathcal{F})$ or of some $T_0^z(\mathcal{F})$. The reason that this is possible in the other cases is that T_0, T_1, T_2 and their quotients have automorphism group with only finitely many orbits. □

The following results are proved analogously:

Corollary 5.3 *Let G be a connected infinite planar graph of minimum essential degree $\delta_{\text{ess}}(G) \geq 4$ and with $h_1(G) = 0$. Then G has infinitely many edges of weight at most 12. If G has subexponential growth and does not have infinitely many edges of weight at most 11, then there is a map T isomorphic to T_0, T_0^z, T_1, or T_1^z such that G contains every finite submap of T.*

Corollary 5.4 *Let G be a connected infinite planar graph of minimum essential degree $\delta_{\text{ess}}(G) \geq 5$ and with subexponential growth. If G does not have infinitely many edges of weight at most 11, then either G contains every finite submap of T_0, or G contains every finite submap of some T_0^z.*

It is easy to construct examples which show that Theorem 5.2 and Corollaries 5.3 and 5.4 cannot be improved to include infinite submaps of $[6^3]$, $[4.8^2]$, or $[3.12^2]$ (or their cylindrical quotients).

The following corollary of Theorem 5.2(b) is an extension of Grünbaum and Shephard's result [11] which states that a doubly periodic tiling of the plane has an edge of weight at most 14 except when it is isomorphic to $[3.12^2]$. Our result below extends Grünbaum and Shephard's result to doubly periodic tilings without edges of weight at most 13. Such tilings are obtained as follows. Let Q be a 6-regular triangulation of the torus. (Such graphs have a simple three-parameter description. See, e.g., [1].) Select a set $\mathcal{F}$ of facial triangles of Q such that every vertex of Q is incident with at most one triangle in $\mathcal{F}$. Then, add into each triangular face which is not in $\mathcal{F}$ a new vertex of degree 3 joined to the vertices on the boundary of that face. Let $Q(\mathcal{F})$ be the resulting toroidal map. Then a doubly periodic tiling of the plane with minimum weight 14 (or 15 if $\mathcal{F} = \emptyset$) is obtained as the universal cover of $Q(\mathcal{F})$. Note that the tiling is isomorphic to $[3.12^2]$ if $\mathcal{F} = \emptyset$.

Corollary 5.5 *A doubly periodic tiling of the plane has no edges of weight at most 13 if and only if it is a universal cover of some toroidal triangulation $Q(\mathcal{F})$ described above.*

The proof is left to the reader.

6 Hyperbolic tessellations

Let G be a planar graph. Then its *minimum face length* is denoted by $\delta^*(G)$. We also define its *minimum essential face length* $\delta^*_{\text{ess}}(G)$ as the minimum integer d such that there are infinitely many faces of length d. If such d does not exist, then $\delta^*_{\text{ess}}(G) = \infty$.

If G is an infinite planar graph with $\delta_{\text{ess}}(G) \geq p$ and $\delta^*_{\text{ess}}(G) \geq q$ where $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$, then we say that G is *strongly hyperbolic*. (This is a special case of more general hyperbolic maps as introduced, for example, by Woess [22].) Let $\alpha = 3 - \frac{6}{q}$, $\alpha' = \frac{6}{q}$, and let $\kappa = (\frac{1}{p} + \frac{1}{q}) - \frac{1}{2} < 0$. The value of κ can be viewed as an upper bound on the (negative) curvature, see, e.g., [3]. Then for all vertices $v \in V(G)$ of degree at least p

$$c(v) = \alpha \deg(v) - 6 \geq \alpha p - 6 = -6p\kappa > 0$$

and for all faces $f \in F(G)$ of length at least q

$$c(f) = \alpha' \deg(f) - 6 \geq \alpha' q - 6 = 0.$$

If $\delta(G) \geq p$ and $\delta^*(G) \geq q$, then inequality (17) in the remark after Theorem 3.1 shows that

$$h_1(G) \geq \frac{2pq|\kappa|}{3q - 8}. \tag{19}$$

This bound compares well with the result of Baues and Peyerimhoff [3], whose bound is just slightly better than (19).

More generally, the above method can be used to get explicit positive lower bounds on $h_1(G)$ in more general hyperbolic cases introduced by Woess [22]. We have to remark, though, that our results are more general than those in [3, 22] since we do not require tessellations to be normal or to have only one end.

7 Light subgraphs

Kotzig's theorem (cf. Section 5) motivated further research on finite 3-connected planar graphs. Fabrici and Jendrol' [7] proved that for every integer k, every finite 3-connected planar graph G that contains at least one k-path, also contains a k-path whose vertices all have degree at most $5k$ in G. Therefore, the k-path P_k is said to be *light* in the set of 3-connected planar graphs. On the other hand, no other connected planar graph is light [7].

We extend these results to infinite graphs. Let us observe that the assumption on 3-connectivity is necessary already for the path P_3 on three vertices (while for P_2 a weaker condition $\delta(G) \geq 3$ suffices as shown by Theorem 5.1).

Theorem 7.1 *Let $k \geq 1$ be an integer and let G be a 3-connected infinite planar graph of subexponential growth. Then G contains infinitely many (disjoint) k-paths whose vertices have degree at most $6k$.*

Proof. A vertex of G of degree at most $6k$ is called a *minor* vertex. Other vertices are *major*. Let $M(G)$ denote the set of major vertices of G.

Let $\ldots v_1 v_2 \ldots v_n \ldots$ be a facial walk of a face F and i, j integers such that $i + 2 \leq j \leq i + k$, v_i and v_j are major vertices and $v_{i+1}, v_{i+2}, \ldots, v_{j-1}$ are minor vertices. Then we say that v_i and v_j are *joinable* in F. Let G' be the graph that is obtained from G by adding all edges between all pairs of vertices that are joinable in some face. Since G is 3-connected, G' is a planar graph (without multiple edges). Clearly, G' has subexponential growth and $M(G') = M(G)$. Therefore, it suffices to prove that G' has infinitely many k-paths whose vertices are not in $M(G')$.

Let $G'' = G'(M(G'))$ be the induced subgraph of G' on major vertices. We may assume that $M(G')$ is infinite. Since G'' has subexponential growth, Theorem 4.2 implies that G'' has infinitely many vertices of degree at most 6.

Let v be any major vertex of G' whose degree in G'' is at most 6. Since G and hence also G' is 3-connected, the *link* of v (which consists of all vertices and edges that are on a common face with v but disjoint from v) is a cycle in G'. Since $\deg_{G'}(v) \geq 6k + 1$ and $\deg_{G''}(v) \leq 6$, there is a sequence of k consecutive neighbors of v (with respect to the clockwise ordering around v in the plane), none of which is a major vertex. By the definition of G' it follows that the corresponding segment on the link of v contains no major vertices. Therefore, this segment contains a k-path $P(v)$.

This proves that there is an infinite set of k-paths $P(v)$ that contain only minor vertices, and each $P(v)$ contains a neighbor of v. Every such k-path corresponds to only finitely many distinct vertices v. This easily implies that there is an infinite subset of these paths whose members are distinct and pairwise disjoint. □

Let $P = v_1 v_2 v_3 \ldots$ be a one-way-infinite path in G. We say that P has

average degree $\leq D$ if for every positive integer k:

$$\sum_{i=1}^{k} \deg_G(v_i) \leq kD.$$

Not every planar graph of polynomial growth contains an infinite path with bounded average degree. We found a complicated example in which for every infinite path $P = v_1v_2v_3\ldots$ and every $k \geq 1$

$$\sum_{i=1}^{k} \deg_G(v_i) = \Theta(k \log k).$$

We propose the following

Conjecture 7.2 *There is a constant C such that every connected infinite planar graph with subexponential growth contains a one-way-infinite path $P = v_1v_2v_3\ldots$ such that for every $k \geq 1$*

$$\sum_{i=1}^{k} \deg_G(v_i) \leq Ck \log k.$$

A proof of Conjecture 7.2 would solve an open problem about finite graphs (cf., e.g., [7, Problem 3]).

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