K_6-minors in projective planar graphs*

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Abstract

It is shown that every 5-connected graph embedded in the projective plane with face-width at least 3 contains the complete graph on 6 vertices as a minor.

1 Introduction

Let G be a simple graph embedded in a surface Σ. For the most of the paper we will only concern the case where Σ is the projective plane $\mathbb{N}_1$. We use the terminology from [1], basic facts about graph embeddings can be found in [2] or [5]. If v is a vertex and f is a face of G then $\mathrm{d}(v)$ and $\mathrm{d}(f)$ stand for the *degree* of v and *length* of f, respectively. By $\delta(G)$ we denote the minimum degree of vertices of G.

A *separation* of a graph G is a pair of subgraphs (H, K) such that $E(H) \cap E(K) = \emptyset$, $H \cup K = G$ and $|V(H)| > |V(H) \cap V(K)| < |V(K)|$. The *order* of the separation (H, K) is $|V(H) \cap V(K)|$. A separation of order k is a *k-separation*. A graph G is *k-connected* if $|V(G)| > k$ and G has no k'-separation for any $k' < k$. If G is k-connected then $\delta(G) \geq k$.

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Let G be a graph embedded in Σ. We say that a vertex v *sees* a vertex u if u and v lie on a common face. Seeing is a symmetric relation among vertices of G that contains adjacency.

Let U, W be subsets of $V(G)$. A $U - W$ *path* is a path P in G such that one endvertex of P is in U the other endvertex is in W and no intermediate vertex is in $U \cup W$. We say that $S \subseteq V(G)$ *separates* U from W if every $U - W$ path uses a vertex from S. We define similarly $u - W$, $U - w$, and $u - w$ paths where $u, w \in V(G)$.

Let Σ denote any surface other than the sphere. Let C be a contractible simple closed curve in Σ. In that case we can define the *interior* of C, $\mathrm{Int}(C)$, as the component of $\Sigma \setminus C$ which is homeomorphic to an open disc. Similarly we define the *exterior* $\mathrm{Ext}(C)$ of C. If a simple closed curve C is not contractible it is called *essential*. A basic fact from algebraic topology states that any two essential closed curves in the projective plane are *homotopic*. The *face-width* or *representativity* of a graph G embedded in Σ is the minimum number of intersecting points of G with any essential simple closed curve and is denoted by $\mathrm{fw}(G)$. See [8] or [5] for an introduction.

A graph H is a *minor* of a graph G, $H \leq_m G$, if H can be obtained by a series of contractions of edges from a subgraph of G. A minor H of G can be described in the following way: a vertex v of H corresponds to a connected subgraph v_H of G, distinct vertices of H correspond to disjoint subgraphs of G and if two vertices v and u of H are adjacent there is an edge of G connecting a vertex of v_H with a vertex of u_H. Every vertex of G clearly belongs to at most one v_H. The phrase that a vertex of G *is* a certain vertex of the minor H will mean that it *belongs* to some v_H, $v \in V(H)$.

A *k-wheel* (W_k) is a graph with vertices $v_0, v_1, \ldots, v_k, \ldots$ such that v_0 is adjacent exactly to vertices $v_1, \ldots, v_k$ and the vertices of W_k distinct from v_0 induce a (simple) cycle; v_0 is called the *hub* of the wheel, the vertices $v_1, \ldots, v_k$ are called *spoke vertices*, the other vertices are called *rim* vertices and may or may not be present; edges incident with v_0 are called *spokes*, the remaining ones are called *rim* edges. Further on, we assume that spoke vertices appear along the cycle according to their indices.

Let G be a graph embedded in Σ, and let $v \in V(G)$. The *surface neighborhood* of v is the closure of the union of faces incident with v. If G is 3-connected and $\mathrm{fw}(G) \geq 3$ (or G is plane and 3-connected), then the surface neighborhood of any vertex is a closed disc with induced embedding of a wheel. This implies that in this case no two faces of G can both be incident with a pair of nonadjacent vertices of G.

Let

$$C = v_1 f_1 \dots v_k f_k v_1 \tag{1}$$

be a sequence such that v_i ($i = 1, \dots, k$) are distinct vertices of G, f_i ($i = 1, \dots, k$) are faces of G, and each face f_i is incident with vertices v_i and v_{i+1}. Then we say that C is a *face chain* (FC). We call v_i and f_i ($i = 1, \dots, k$) the *vertices* and *faces* of C, respectively, and write $V(C) = \{v_1, \dots, v_k\}$. We also call v_i and f_i the *beads* of C and define k to be the *length* of C. We define the length of a (contiguous) subsequence C' of C as one half r, where r is the number of beads of C' minus one. Note that the length of the subsequence is an integer if and only if the end beads are of the same type. The face chain C determines a closed curve $\Gamma = \Gamma(C)$ which intersects G in vertices $v_1, \dots, v_k$ and runs through faces $f_1, \dots, f_k$ (in this order). If each v_i occurs on the boundary of f_{i-i} and f_i at most once, then $\Gamma(C)$ is determined up to homotopy. We may assume that Γ is a simple closed curve unless there are indices $i < j$ such that $f_i = f_j$ and $v_i, v_j, v_{i+1}, v_{j+1}$ occur in this (interlaced) order along the boundary of f_i. If Γ is essential, then we say that C is an *essential face chain*. Suppose now that Γ is contractible and simple. The graph $G_p := G \cap (\mathrm{Int}(\Gamma) \cup \Gamma)$ is clearly a plane graph and will be referred to as the *plane part* of C. The graph $G_x := G \cap (\mathrm{Ext}(\Gamma) \cup \Gamma)$ will be called the *exterior part* of C. If $|V(G_p)| > k$ then C is called a *k-separating face chain* (k-SFC for short). If, furthermore, $|V(G_x)| > k$ then (G_p, G_x) is a k-separation.

Lemma 1 *Let G be a 5-connected graph and let $C = v_1 f_1 \dots v_5 f_5 v_1$ be a 5-SFC. Then either $V(G_p) = 6$ or $V(G_p) \geq 11$. In the latter case, there exists a vertex $v \in V(G_p)$ that sees no vertices of C.*

Proof. Let us call $v_1, \dots, v_5$ the outer vertices of G_p and call the remaining ones interior vertices. We have to show that there cannot be less than 6 interior vertices provided there are at least two of them. Without the loss of generality, we can assume that outer vertices induce a 5-cycle in the order of their indices. Suppose that G_p is a counterexample with a minimum number of vertices and let $r = |V(G_p)|$.

Suppose first that $r = 7$. Let u and w be the interior vertices. Since $\mathrm{d}(u) \geq 5$, u has at least four neighbors in $V(C)$. So has w. Then u and w have at least three common neighbors in $V(C)$ which is clearly not possible. Therefore $r \geq 8$.

If $u \in V(G_p)$ is the only interior vertex adjacent to v_1, the sequence u, v_2, v_3, v_4, v_5 would induce another 5-SFC, its plane part would have one vertex less than G_p and at least 7 vertices which is a contradiction to the minimality of G_p. Therefore, every outer vertex of G_p is adjacent to at least two interior vertices. Obviously, no interior vertex can be adjacent to three consecutive outer vertices and there can be at most one adjacent to three nonconsecutive ones. Since $\delta(G) \geq 5$, this implies that $G_p - V(C)$ is a plane graph with at most one vertex of degree ≤ 2; hence it is not outerplanar. Let v be an interior vertex of $G_p - V(C)$. Its neighborhood alone contains at least 6 vertices and these are distinct from vertices of C. This completes the proof. $\square$

We will use another technical lemma.

Lemma 2 *Let G be either a plane or projective plane graph with minimum degree 5. Then G contains a vertex v of degree 5 which is incident with at least four triangular faces and the fifth face f incident with v has length at most 5. Moreover, if $\mathrm{d}(f) = 4$, at least three vertices of f have degree 5. If $\mathrm{d}(f) = 5$, all vertices of f have degree 5.*

Proof. We will prove this fact using discharging method. Let $\mathrm{d}(v) - 6$ and $2\mathrm{d}(f) - 6$ be the initial charges of every vertex v and face f, respectively. Plugging

$$\sum_{v \in V(G)} \mathrm{d}(v) = 2|E(G)| \quad \text{and} \quad \sum_{f \in F(G)} \mathrm{d}(f) = 2|E(G)|$$

into Euler's formula we obtain

$$\sum_{v \in V(G)} (\mathrm{d}(v) - 6) + \sum_{f \in F(G)} (2\mathrm{d}(f) - 6) < 0.$$

The total initial charge is negative. Now we redistribute the charge from positively charged faces of G according to the following rule:

R. Every face f of length at least 4 distributes its positive charge uniformly among vertices of degree 5 incident with it.

After charge distribution, only vertices of degree five can be negatively charged. Since the total charge of G remains negative, there exists a negatively charged vertex v which can be incident with at most one nontriangular face. Moreover, if v is incident with a face f of length ≥ 4, then more than $2\mathrm{d}(f) - 6$ vertices of f have degree 5. This completes the proof. $\square$

Throughout the paper we will use the following two theorems. The first one can be proved using Menger's theorem (cf. [6] or [9]).

Theorem 3 *Let G be a plane graph and let $s_1, \ldots, s_l, t_l, \ldots, t_1$ be vertices along the outer face f_0 in that order. Then G contains disjoint $s_i - t_i$ paths P_i $(i = 1, \ldots, l)$ if and only if there is no k-separating face chain C such that f_0 is a bead of C and vertices of C separate more than k pairs s_i, t_i.*

The second result has been proven by Robertson and Seymour in [6].

Theorem 4 *Let G be a projective plane graph and f a face bounded by a facial cycle $s_1 s_2 s_3 t_1 t_2 t_3$ of length 6. Then G contains disjoint $s_i - t_i$ paths P_i $(i = 1, 2, 3)$ if and only if there exists no k-SFC with $k \leq 5$ containing f in its interior.*

2 Main theorem

Throughout this section, let G be a 5-connected graph embedded in the projective plane $\mathbb{N}_1$ with face-width at least 3. We will show that G contains K_6 as a minor.

(1) *There exist no k-SFC for $k \leq 4$.*

Proof. Let C be a k-SFC and $k \leq 4$. Let G_p and G_x be the plane part and the exterior part with respect to C. If $|V(G_x)| = k \leq 4$ we can find an essential curve Γ with $|\Gamma \cap G| \leq 2$. Otherwise $|V(G_x)| > k$ and (G_p, G_x) is a k-separation. A contradiction. $\square$

(2) *Suppose C is a 5-SFC with vertices $V(C) = \{v_1, v_2, v_3, v_4, v_5\}$. If $V(G_x) = V(C)$ then $v_1 v_3 v_5 v_2 v_4$ is a 5-cycle of G and is contained in G_x.*

Proof. With local deformation of C we may assume that G_p contains all the edges of G of the form $v_i v_{i+1}$. It is then enough to show that $\delta(G_x) = 2$. Suppose this is not the case. Then G_x is a proper subgraph of the 5-cycle $v_1 v_3 v_5 v_2 v_4$. We may assume that the edge $v_1 v_3$ is not present in G_x. In this case, vertices v_4 and v_5 cover all edges of G_x. Now the induced embedding of $G - v_4 - v_5$ is in the plane, which is a contradiction to $\mathrm{fw}(G) \geq 3$. $\square$

Choosing a 5-wheel minor W

Choose an arbitrary vertex v of degree 5 which satisfies the conclusion of Lemma 2. Let v_1, v_2, v_3, v_4 and v_5 be its neighbors and let g_i ($i = 1, \ldots, 5$) denote the face incident with v, v_i, and v_{i+1} (indices modulo 5). If v is incident with a nontriangular face, let g_5 be the one. Vertices v_i ($i = 1, \ldots, 5$) are spoke vertices of the 5-wheel induced by the surface neighborhood of v.

Let $C = u_1f_1u_2f_2u_3f_3u_4f_4u_5f_5u_1$ be a maximal (in the sense of containment of edges in its interior) 5-SFC that contains v in its interior. Clearly, C exists, because $v_1f_1v_2f_2 \ldots v_5f_5v_1$ is a 5-SFC which contains v in its interior. We define G_p and G_x with respect to C.

First suppose that v is the only vertex in $\mathrm{Int}(C)$. In this case $v_i = u_i$ for $i = 1, \ldots, 5$. Also observe that $f_i \neq g_i$ for $i = 1, \ldots, 4$ and $f_5 = g_5$ if and only if g_5 is not a triangle. We set W as the 5-wheel induced by the neighborhood of v.

Now, suppose $\mathrm{Int}(C)$ contains at least two vertices. By Lemma 1 there is an interior vertex u_0 of G_p which does not see any of the vertices u_i ($i = 1, \ldots, 5$). Since G is 5-connected, there are internally disjoint paths P_i joining u_0 with u_i, $i = 1, \ldots, 5$. It is easy to see that $P_i \subseteq G_p$ for all $i = 1, \ldots, 5$. The neighorhood of u_0 is strictly contained in $\mathrm{Int}(C)$. It is easy to observe that it contains internally disjoint $P_i - P_{i+1}$ paths Q_i for $i = 1, \ldots, 5$. Let W be a graph induced by the union $\bigcup_{i=1}^{5}(P_i \cup Q_i)$. Clearly W is contractible to a 5-wheel with hub u_0 and $u_1, \ldots, u_5$ are contained in the spoke vertices of the 5-wheel minor.

In both cases the following is true.

(3) *Let C be a maximal 5-SFC in G containing v in its interior and let $u_1, \ldots, u_5$ be the vertices of C. Then there is a 5-wheel minor W having vertices of C as its spoke vertices. Moreover, W is entirely in $\mathrm{Int}(C) \cup V(C)$ except in the case when $\mathrm{Int}(C)$ contains exactly one vertex of degree five which is not incident with triangles only. In this case, only the subdivided rim edge u_1u_5 is realized in the exterior of C.*

Finding a suitable cycle minor U in G_x

In this part we will mostly work with the outer part G_x of the 5-SFC $C = u_1f_1u_2f_2u_3f_3u_4f_4u_5f_5u_1$ chosen above. Let $U \subseteq G_x$ denote a graph which contracts to a 5-cycle with vertices u_1, u_3, u_5, u_2, u_4 in that order. Our goal is to show that U exists. Observe that if W is the 5-wheel minor from (3) and

W is contained in $\mathrm{Int}(C) \cup V(C)$, then $W \cup U$ is contractible to K_6. If the subdivided rim edge $u_1 - u_5$ of W is in the exterior part of C, then we shall argue that U can be chosen so that $W \cup U$ still contains K_6 minor. More precisely, let U_i be the connected subgraph of U which is contracted to u_i in order to get the 5-cycle. Then $W \cup U$ is contractible to K_6 if $U_3 \cap W = \{u_3\}$, i.e. U_3 does not contain a vertex of the $u_1 - u_5$ segment of W.

First we prove a statement concerning 6-SFCs in G.

(4) *At most one of the vertices u_i $(i = 1, \ldots, 5)$ can be a vertex of a 6-SFC D such that $\mathrm{Int}(C) \cup f_i \cup f_{i-1} \subseteq \mathrm{Int}(D)$. If such a 6-SFC exists, then D and C have no beads in common apart from u_i.*

Proof. First we show that apart from u_i the 6-SFC D which satisfies the assumptions contains no other bead of C. Suppose this is not the case. Let x be another common bead of C and D. Then u_i and x divide C and D into subsequences of lengths c_1, c_2 and d_1, d_2, respectively, see Figure 1. By the

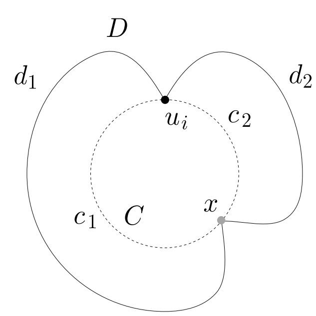

Figure 1: A 6-separating face chain D.

maximality of C we have $c_1 + d_2 \geq 6$ and $c_2 + d_1 \geq 6$. On the other hand, $c_1 + c_2 + d_1 + d_2 = 5 + 6 = 11$ which is a contradiction.

Let $i \neq j$ and suppose that there exist D_i and D_j, each satisfying the conditions of the proposition, containing vertices u_i and u_j, respectively. By the preceeding paragraph, u_i lies in the interior of D_j, and u_j lies in the interior of D_i. Therefore D_i and D_j share at least two beads. Denote them by x and y and label the lengths of segments according to Figure 2. We may assume that, apart from x and y, the segments labeled with e_1 and e_3 contain no bead of D_i. If $e_1 + e_3 \leq d_2$, we could replace the segment of D_i of length d_2 by the two segments of D_j. This would either contradict maximality of C or give a 6-SFC intersecting $V(C)$ in two vertices. Hence,

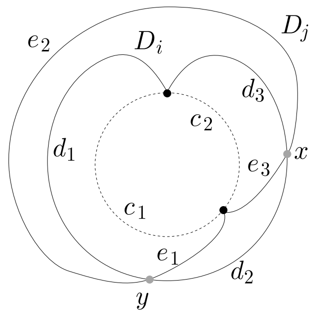

Figure 2: 6-separating face chains D_i and D_j.

$d_2 \le e_1 + e_3 - 1$. This easily implies that $d_2 + e_2 \le 5$. The corresponding segments of D_i and D_j thus form a 5-SFC C' (contradicting maximality of C) unless they have a vertex z in common. In that case, C' contains a face chain of length ≤ 2. Since G is 3-connected, this face chain is essential. This implies that $\mathrm{fw}(G) \le 2$, a contradiction. $\square$

We call a vertex u_i *nice* if it is not contained as a bead in a 6-SFC which satisfies the conditions of (4). Since at least one of u_2 and u_4 is nice we may assume that u_2 is a nice vertex. We say that the face f_i ($i = 1, \ldots, 5$) is *essential* if $f_i \cup W$ contains an essential cycle. This is true if f_i contains the vertex v_{i+3} (index modulo 5). There is another possibility for f_2 and f_3 to be essential without containing vertices u_5 and u_1, respectively. This is possible if W is not contained in G_p and f_2 (or f_3) intersects the $u_1 - u_5$ segment which is in G_x.

Suppose that f_i is an essential face. Then f_i or f_5 is a face of every 6-SFC D which contains $\mathrm{Int}(C)$ in its interior. Now, (4) implies:

(5) *If some face f_i ($i = 1, \ldots, 5$) is essential, then all vertices $u_1, \ldots, u_5$ are nice.*

We now have the necessary tools to construct the suitable cycle minor.

(6) *Suppose that neither f_1 nor f_2 are essential faces and that u_2 is a nice vertex. Then G_x contains a subgraph U which contracts to a 5-cycle through vertices u_1, u_3, u_5, u_2, u_4, respectively, and such that $U_3 \cap W = \{u_3\}$.*

Proof. Let x_1 and x_3 be the vertices of G_x incident with f_1 and f_2, respectively, and adjacent to u_2. Observe that $x_1 \neq u_1$ and $x_3 \neq u_3$ by the

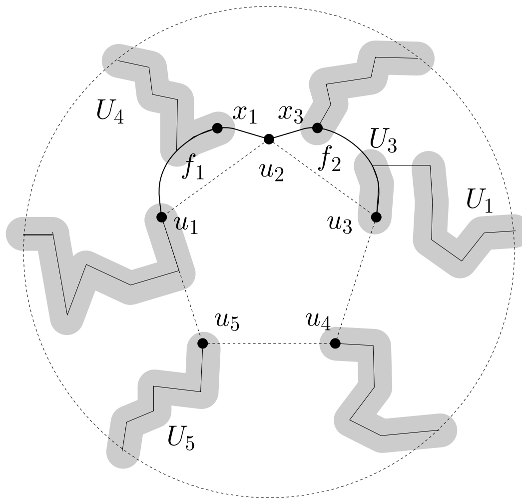

Figure 3: Contracting paths in U.

maximality of C. Delete u_2 from G_x and add an edge joining vertices x_1 and x_3. We add edges x_1u_1, x_3u_3 and u_1u_5 as well. Denote the resulting graph by G'_x. We want to find disjoint $x_1 - u_4$, $x_3 - u_5$ and $u_1 - u_3$ paths in G'_x. According to Theorem 4 the only obstruction for existence of such paths is a 5-SFC C' containing the endpoints of required paths in $\mathrm{Int}(C') \cup C'$. By the maximality of C, C' is not a 5-SFC in G. This is only possible if one of the beads of C' is a face incident with the edge x_1x_3. Such a C' would then induce a 6-SFC containing u_2 and satisfying the assumptions of (4). This is a contradiction since u_2 is a nice vertex.

Contracting appropriately the obtained paths as shown in Figure 3, the noncontracted edges induce a required cycle. We may assume that the u_1-u_3 path has connected intersection with f_2. Then we may achieve $U_3 \subseteq f_2$, so that $U_3 \cap W = \{u_3\}$. □

The following proposition settles another case.

(7) *Suppose that f_1 is an essential face but f_2 is not essential. Then we can find a subgraph U in G_x as stated in (6).*

Proof. The situation is roughly depicted in Figure 4. Since f_1 is an essential face, it contains u_4. We cut the projective plane along the corresponding essential curve (shown as the dotted line), split the vertex u_4 and obtain a 'planar' drawing of G_x. This drawing is represented in Figure 4 as the square

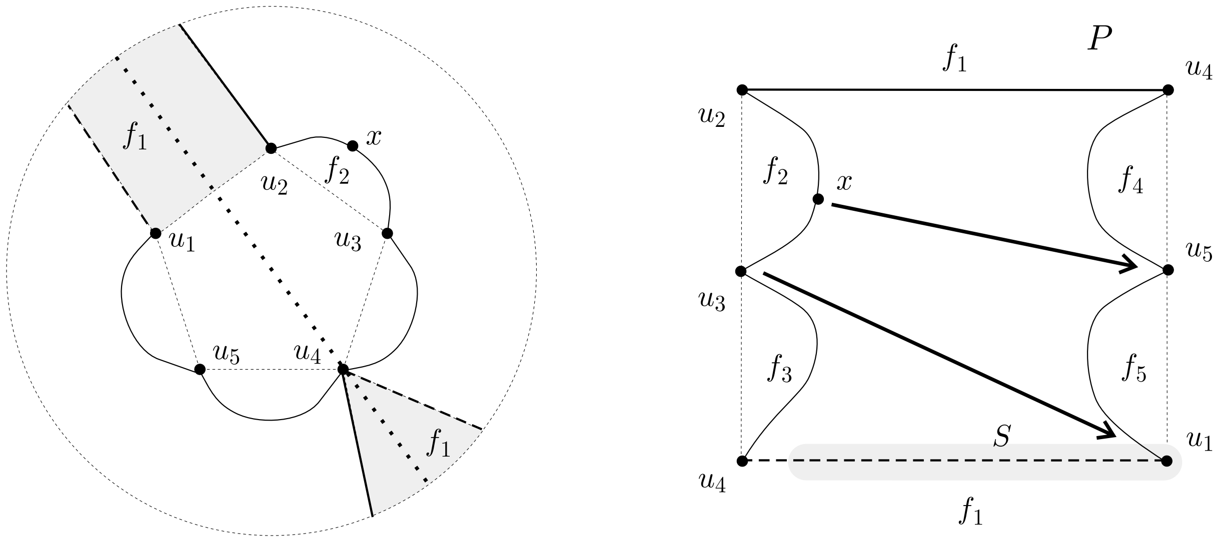

Figure 4: Cutting when f_1 is an essential face.

P. Let S denote the set of vertices of the $(u_4, u_1]$ segment along f_1 and let $x \in V(G_x)$ be the neighbor of u_3 incident with f_2. In order to find U it is enough to find pairwise disjoint paths Q_1 linking x with u_5, and Q_2 linking u_3 with u_1 (respectively) that are disjoint from vertices on the top segment of P and the left copy of u_4.

Suppose that f is a face lying inside the square P, where $f \neq f_i$ ($i = 1, \ldots, 5$). The following observations are true and they are easy to argue:

1. f is not incident with the base and the top of P at the same time.
2. f is not incident with f_4 and the left copy of u_4 at the same time.
3. If f is incident with the left copy of u_4 and f' is a face in P incident with the right copy of u_4, then f and f' do not share a vertex (distinct from u_4).

Concerning the above observations and using Theorem 3 (applied to P after removal of the left copy of u_4 and the top segment of P) the only possible obstruction for the required paths Q_1 and Q_2 is a pair of faces g and $g' \neq f_4$ in P sharing a common vertex y such that g is incident with the left copy of u_4 and g' is incident with a vertex $z \neq u_4$ (by observation 3) on

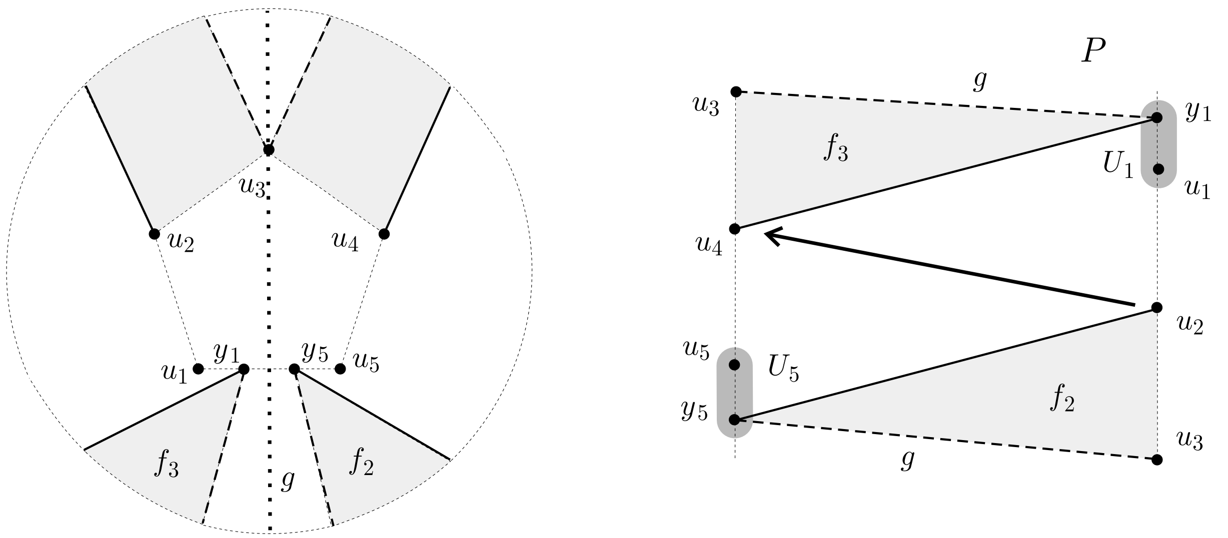

Figure 5: Cutting when both f_2 and f_3 are essential.

the segment $[u_2, u_4)$ in P. In this case the sequence $u_4 g y g' z f_1 u_1 f_5 u_5 f_4 u_4$ is a 5-SFC contradicting the maximality of C.

Now it is easy to observe, that we can choose U_3 to consist of the vertex u_3 only.

□

By (6) and (7) we may assume that f_2 is essential. By (5), all vertices u_i ($i = 1, \ldots, 5$) are nice. Hence, we may apply (6) to u_4, f_3, and f_4 if neither f_3 nor f_4 is essential. If f_4 is essential and f_3 is not essential, we apply the arguments of (7) to u_4, f_3 and f_4, by symmetry. Therefore we may assume, that both, f_2 and f_3 are essential. This case is considered in the next proposition.

(8) *Suppose that f_2 and f_3 are essential faces. Then G_x contains U as stated in (6).*

Proof. Since f_2 and f_3 are essential, they both contain at least one vertex from the $u_1 - u_5$ segment of W. Denote them by y_5 and y_1, respectively. The situation is depicted in Figure 5.

Note that if $W \subseteq G_p$, then $y_1 = u_1$ and $y_5 = u_5$. In the case $W \nsubseteq G_p$, f_2 and f_3 do not share a vertex apart from u_3, since $\mathrm{d}(u_3) \geq 5$. In both cases $y_5 \neq y_1$.

Now, consider the 3-FC $u_3 f_3 y_1 f_5 y_5 f_2 u_3$. Since it is not a separating face-chain, it bounds a triangular face. We denote the latter by g. This implies, that u_3 is adjacent to both y_1 and y_5 and we can set U_3, the connected subgraph which contracts to the third vertex of U, to be $\{u_3\}$.

Furthermore, we set $U_i := \{y_i, u_i\}$ $(i = 1, 5)$. We cut the projective plane along the dotted line splitting the vertex u_3 in order to obtain a 'planar' drawing P of G_x. Now, it suffices to find a path Q, linking u_2 with u_4, which is disjoint from both U_1 and U_5.

Applying Theorem 3 to the disk obtained from P by deleting u_3, U_1, and U_5, yields that the only possible obstruction for Q would be a face $h \neq f_5$ containing vertices from both, U_1 and U_5. However, such a face h does not exist since it would give rise to an essential curve intersecting G only in two vertices, contradicting $\mathrm{fw}(G) \geq 3$. □

Now we state the main theorem which is a direct consequence of (3), (6)–(8), and the remarks preceeding (8).

Theorem 5 *If G is a 5-connected graph embedded in the projective plane with* $\mathrm{fw}(G) \geq 3$ *then it contains a K_6 minor.*

3 Final remarks

Theorem 5 is in a sense best possible. We cannot relax any of the assumptions. Observe that K_6 triangulates the projective plane. Therefore every embedding of K_6 in $\mathbb{N}_1$ has face-width at least 3. Since the face-width is minor-monotone, no projective graph of face-width less than 3 contains a K_6-minor. Neither can we relax the connectivity assumption.

Proposition 6 *There is an infinite family of 4-connected projective planar graphs with minimum degree 5 and face-width 3 which do not contain a K_6-minor.*

Proof. Let G be a graph shown on Figure 6. The frame of G is a graph H depicted in black which consists of 7 vertices and 13 edges. H_1 and H_2 are plane graphs attached to the frame such that G is 4-connected, has minimum degree 5 and face-width 3. Such graphs H_1 and H_2 obviously exist; any 4-connected planar graph with minimum degree 5 and with a face of length 4 is a candidate.

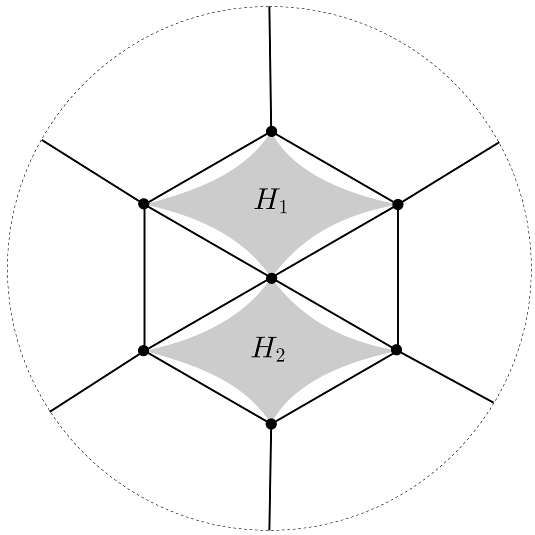

Figure 6: A graph with no K_6 minor.

Suppose G contains a K_6 minor. Since G is connected we can obtain a K_6 using contractions and deletions of edges only. First, contract and delete as many edges of $E(G) \setminus E(H)$ as possible such that the resulting graph G' still contains a K_6 minor. Denote by H_1' and H_2' the resulting minors of H_1 and H_2, respectively. Observe that the subset of vertices of G' that corresponds to a vertex of K_6 is either a connected set of frame vertices or a single vertex.

Let $U_i' = V(H_i') \setminus V(H)$, $i = 1, 2$. If $U_1' = U_2' = \emptyset$, then H_1' and H_2' contribute to the K_6 minor at most one diagonal edge each. Hence G' contains 7 vertices and at most 15 edges. Since K_6 has 15 edges as well, it cannot be a minor of G'.

Suppose now that $r := |U_1'| > 0$. The vertices of U_1' induce the complete graph K_r, hence $r \leq 4$. Each of these vertices is joined to the vertices of the K_6 minor outside U_1'. This implies that $r \neq 4$. If $1 \leq r \leq 3$, each vertex $u \in U_1'$ has $6 - r$ edges joining u with the frame. This is easily seen to be impossible. The proof is complete. $\square$

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