On a list-coloring problem*

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Abstract

We study the function $f(G)$ defined for a graph G as the smallest integer k such that the join of G with a stable set of size k is not $|V(G)|$-choosable. This function was introduced recently in order to describe extremal graphs for a list-coloring version of a famous inequality due to Nordhaus and Gaddum [1]. Some bounds and some exact values for $f(G)$ are determined.

1 Introduction

We consider undirected, finite, simple graphs. A *coloring* of a graph $G = (V, E)$ is a mapping $c\colon\ V \to \{1, 2, \ldots\}$ such that $c(u) \neq c(v)$ for every edge $uv \in E$. If $|c(V)| \leq k$, then c is also said to be a *k-coloring*. The *chromatic number* $\chi(G)$ is the smallest integer k such that G admits a k-coloring. A graph is *k-colorable* if it admits a k-coloring.

Vizing [4], as well as Erdős, Rubin and Taylor [2] introduced a variant of the coloring problem as follows. Suppose that each vertex v is assigned a list $L(v) \subseteq \{1, 2, \ldots\}$ of allowed colors; we then want to find a coloring c such that $c(v) \in L(v)$ for all $v \in V$. If such a coloring exists, we say that G is *L-colorable* and that c is an *L-coloring* of G. The graph is *k-choosable* if G is L-colorable for every assignment L that satisfies $|L(v)| \geq k$ for all $v \in V$. The *choice number* or *list-chromatic number* $Ch(G)$ of G is the smallest k such that G is k-choosable. Clearly, every graph satisfies $Ch(G) \geq \chi(G)$.

Let G_1, G_2 be two vertex-disjoint graphs. The graph $G_1 * G_2 = (V(G_1) \cup V(G_2),\ E(G_1) \cup E(G_2) \cup \{xy \mid x \in V(G_1), y \in V(G_2)\})$ is called the *join* of G_1 and G_2. It is easy to see that $\chi(G_1 * G_2) = \chi(G_1) + \chi(G_2)$ for

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any two vertex-disjoint graphs G_1, G_2. So, the chromatic number has a straightforward behavior with respect to the join operation. On the other hand, the choice number does not behave so simply. For instance, if G_1 and G_2 are edgeless graphs on n and n^n vertices, respectively, then obviously $Ch(G_1) = Ch(G_2) = 1$, but it is known (see [3]) that $Ch(G_1 * G_2) = n + 1$, i.e., the complete bipartite graph K_{n,n^n} is not n-choosable (indeed, to see this, assign to the i-th vertex on the "left" side (the stable set of size n) of K_{n,n^n} the list $L_i = \{(i-1)n+1, \ldots, (i-1)n+n\}$ $(i = 1, \ldots, n)$. Assign to the vertices on the "right" side, one-to-one, all the lists of size n obtained by picking one element from each L_i, $i = 1, \ldots, n$; clearly there are n^n such possibilities; this produces a list assignment L where all lists have size n and for which there is no L-coloring).

Let us denote by S_k the edgeless graph on k vertices. Since the complete bipartite graph K_{n,n^n} is not n-choosable, if H is any graph on n vertices then $Ch(H * S_{n^n}) > n$. We can therefore define $f(H)$ as the smallest integer k such that $Ch(H * S_k) > |V(H)|$. The fact from [3] that K_{n,n^n} is not n-choosable and is minimal with that property means that $f(S_n) = n^n$. It is easy to see that $f(K) = 1$ for every complete graph K. Obviously, if $e \in E(G)$, then $f(G - e) \geq f(G)$. This implies:

$$\text{If } G \text{ is any graph on } n \text{ vertices, then } 1 \leq f(G) \leq n^n. \tag{1}$$

The definition of $f(G)$ was motivated by the determination of extremal graphs for the inequality $Ch(G) + Ch(\overline{G}) \leq |V(G)| + 1$ (see [1]). Here we would like to examine in more detail the problem of evaluating and computing $f(G)$.

An alternative definition for $f(G)$ can be given as follows. Let $G = (V, E)$ be a graph on n vertices, and let $\mathcal{L}(G)$ be the set of assignments $L : V \to \mathcal{P}(\{1, 2, \ldots\})$ that satisfy:

(i) $|L(v)| \geq n$, $\forall v \in V$, and

(ii) $L(u) \cap L(v) = \emptyset$ if $u, v \in V$, $uv \notin E$.

Clearly, for every $L \in \mathcal{L}(G)$, there exists at least one L-coloring of G, because of (i). Moreover, by (ii) every L-coloring c of G uses exactly n colors; we denote by $c(V)$ the set of n colors used by c. We now write:

$$\mathcal{C}(L) = \{c(V) \mid c \text{ is an } L\text{-coloring of } G\}. \tag{2}$$

Now define $f'(G) = \min\{|\mathcal{C}(L)| : L \in \mathcal{L}(G)\}$.

Lemma 1 *For every graph G, we have $f(G) = f'(G)$.*

Proof. Assume G has n vertices, and write $f(G) = k$. By the definition of $f(G)$, we have $Ch(G * S_k) \geq n + 1$. Thus there exists a list assignment L on $V(G * S_k)$ with $|L(v)| \geq n$ $(\forall v \in V(G * S_k))$ and such that $G * S_k$ is not L-colorable. Suppose there were non-adjacent vertices $u, v \in V(G)$ such that $L(u) \cap L(v) \neq \emptyset$. We could then do the following: assign a color from $L(u) \cap L(v)$ to u and v; for all vertices x of $G - \{u, v\}$ taken successively, assign to x a color from $L(x)$ different from the colors already assigned to the preceding vertices (this is possible because $L(x)$ is large enough); likewise for every vertex y of S_k assign to y a color from $L(y)$ different from the colors assigned to the vertices of G. Thus we would obtain an L-coloring of $G * S_k$, a contradiction. It follows that the restriction of L to G satisfies (i) and (ii). Furthermore, whenever c is an L-coloring of G, the set $c(V(G))$ must appear as $L(s)$ for at least one $s \in S_k$, for otherwise this L-coloring c of G could obviously be extended to an L-coloring of $G * S_k$, a contradiction. Hence $|\mathcal{C}(L)| \leq k$. The definition of f' implies $f'(G) \leq k$, i.e., $f'(G) \leq f(G)$.

Conversely, assume that L is a list assignment on G such that $L \in \mathcal{L}(G)$ and $|\mathcal{C}(L)| = f'(G) = j$. Write $\mathcal{C}(L) = \{C_1, \dots, C_j\}$ and let $S_j = \{s_1, \dots, s_j\}$ be a stable set of size j. Let L' be the list assignment defined by $L'(v) = L(v)$ for all $v \in V(G)$ and $L'(s_i) = C_i$ $(i = 1, \dots, j)$. Observe that, by (ii), $|L'(u)| \geq n$ for all $u \in V(G * S_j)$. Clearly $G * S_j$ is not L'-colorable, so $f(G) \leq j$, i.e., $f(G) \leq f'(G)$. □

Using Lemma 1, it is possible to compute $f(G)$ for some small graphs, but in general the computation is difficult even for graphs with a simple structure. For example, one can establish that $f(C_4) = 36$, but we need a tedious case analysis to show that $f(C_5) = 500$.

Theorem 1 *If G has n vertices and is not a complete graph, then $f(G) \geq n^2$.*

Proof. We will prove, by induction on n, that if u, v are non-adjacent vertices of G and $L \in \mathcal{L}(G)$, then $f'(G) \geq |L(u)||L(v)|$. This statement clearly implies the theorem. For $n = 2$, the statement is obvious. Now, assume that $n \geq 3$, and write $n_1 = |L(u)|$ and $n_2 = |L(v)|$. Pick any $z \in V \setminus \{u, v\}$ and pick any color, say 1, in $L(z)$. We may assume by (ii) that $1 \notin L(v)$. Define:

$$\mathcal{C}_1(L) = \{c(V) \mid c \text{ is an } L\text{-coloring of } G \text{ with } c(z) = 1\},$$

$$\overline{\mathcal{C}}_1(L) = \{c(V) \mid c \text{ is an } L\text{-coloring of } G \text{ with } 1 \notin c(V)\}.$$

Clearly, $\mathcal{C}(L) \supseteq \mathcal{C}_1(L) \cup \overline{\mathcal{C}}_1(L)$ and $\mathcal{C}_1(L) \cap \overline{\mathcal{C}}_1(L) = \emptyset$. Thus $|\mathcal{C}(L)| \geq |\mathcal{C}_1(L)| + |\overline{\mathcal{C}}_1(L)|$. Let us now evaluate these numbers.

On one hand, we have $|\mathcal{C}_1(L)| \geq (n_1 - 1)n_2$ by the induction hypothesis applied to the graph $G - z$ with the list assignment $L_1 \in \mathcal{L}(G - z)$ determined by $L_1(w) = L(w) \setminus \{1\}$ for each $w \in V(G - z)$.

On the other hand, we claim that $|\overline{\mathcal{C}}_1(L)| \geq n_2$. Indeed, fix an L-coloring γ of the subgraph $G \setminus \{u, v\}$ that does not use color 1. Such a coloring exists because that subgraph has $n-2$ vertices while L_1 assigns lists of size at least $n-1$ by (i). Write $t_1 = |L(u) \cap \gamma(V \setminus \{u, v\})|$ and $t_2 = |L(v) \cap \gamma(V \setminus \{u, v\})|$. Write $\lambda_1 = n_1 - (t_1 + 1)$ and $\lambda_2 = n_2 - t_2$. Since color 1 is not in $L(v)$ (but possibly is in $L(u)$), γ can be extended to an L-coloring of G in at least $\lambda_1\lambda_2$ ways, and each of these uses a different set of colors $\gamma(V) \in \overline{\mathcal{C}}_1(L)$. Since $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_1 + \lambda_2 \geq n_2 + 1$, we have $|\overline{\mathcal{C}}_1(L)| \geq \lambda_1\lambda_2 \geq n_2$.

Now, $|\mathcal{C}_1(L)| \geq (n_1 - 1)n_2$ and $|\overline{\mathcal{C}}_1(L)| \geq n_2$ imply $|\mathcal{C}(L)| \geq n_1 n_2$. □

We observe that the bound given in the preceding theorem is tight, i.e., for any $n \geq 2$, there exists a graph G on n vertices with $f(G) = n^2$. Indeed, consider the graph $K_n - E(K_{1,i})$ obtained from a complete graph on n vertices by removing i edges incident to one given vertex u ($1 \leq i \leq n - 1$):

Claim 1 $f(K_n - E(K_{1,i})) = n^2$.

Proof. By Theorem 1, we have $f(K_n - E(K_{1,i})) \geq n^2$, so we need only to prove that $f(K_n - E(K_{1,i})) \leq n^2$. For this purpose, assign to the vertex u the list $\{1, 2, \ldots, n\}$ and to all other vertices of the graph the list $\{n+1, \ldots, 2n\}$. This yields a list assignment $L \in \mathcal{L}(G)$. It is easy to check that $|\mathcal{C}(L)| = n^2$, hence $f(G) \leq n^2$. □

We do not know of any graph G other than $K_n - E(K_{1,i})$ that satisfies $f(G) = |V(G)|^2$.

2 The significance of clique partitions

Given a graph $G = (V, E)$, a *clique partition* of G is a set $Q = \{Q_1, \ldots, Q_p\}$ of pairwise disjoint, non-empty cliques such that $V = Q_1 \cup \cdots \cup Q_p$. Let $n = |V|$ and $q_i = |Q_i|$, $i = 1, \ldots, p$. Then we write

$$w(Q) = \prod_{i=1}^{p} \binom{n}{q_i}$$

and

$$w(G) = \min\{w(Q) \mid Q \text{ is a clique partition of } G\}.$$

Theorem 2 *For every graph G, we have $f(G) \leq w(G)$.*

Proof. Write $n = |V|$. Consider a clique partition $Q = \{Q_1, \ldots, Q_p\}$ of G, and make a list assignment L as follows: to each vertex of Q_i assign a list L_i of n colors, so that $L_i \cap L_j = \emptyset$ whenever $1 \leq i < j \leq p$. Clearly, $L \in \mathcal{L}(G)$. Moreover, any L-coloring of G consists in assigning $|Q_1|$ colors from L_1 to the vertices of Q_1, $|Q_2|$ colors from L_2 to the vertices of Q_2, etc. It follows that $|\mathcal{C}(L)| = w(Q)$. Therefore, $f'(G) \leq w(Q)$. Since Q is an arbitrary clique partition, Lemma 1 implies that $f(G) = f'(G) \leq w(G)$. $\square$

Claim 2 *If G is a disjoint union of cliques, then $f(G) = w(G)$.*

Proof. By the preceding theorem, we need only prove $f(G) \geq w(G)$. Assume G is the union of cliques $Q_1, \ldots, Q_p$. Consider any list assignment $L \in \mathcal{L}(G)$. Let us denote by L^i the restriction of L to the subgraph of G induced by Q_i ($i = 1, \ldots, p$). Note that the colors assigned by L^i to any vertex in Q_i are different from the colors assigned by L^j to any vertex in Q_j whenever $i \neq j$, by (ii). Thus $|\mathcal{C}(L)| = |\mathcal{C}(L^1)| \cdots |\mathcal{C}(L^p)|$. Every L^i-coloring of Q_i can be obtained by choosing among at least n colors for the first vertex of Q_i, then among at least $n-1$ available colors for the second vertex, etc. This way, a given set of $|Q_i|$ colors used in such a coloring occurs at most $|Q_i|!$ times. Thus,

$$|\mathcal{C}(L^i)| \geq \frac{n(n-1)\cdots(n-|Q_i|+1)}{|Q_i|!} = \binom{n}{|Q_i|}.$$

Consequently, $|\mathcal{C}(L)| \geq w(Q) \geq w(G)$. Since L was an arbitrary element of $\mathcal{L}(G)$, the result follows. $\square$

The preceding fact shows that the inequality in Theorem 2 is best possible and motivates the following conjecture.

Conjecture 1 *For every graph G, we have $f(G) = w(G)$.*

We note that if G is a triangle-free graph on n vertices, a clique partition Q consists of some cliques of size two (which form a matching) and some cliques of size one. If p_2 is the number of cliques of size two, we see that $w(Q) = \binom{n}{2}^{p_2} n^{n-2p_2}$; this number is minimized when p_2 is maximized, i.e., when the cliques of size two in Q form a matching of G of maximum size. We denote by $\mu(G)$ the size of a maximum matching. This leads us to:

Conjecture 2 *For every triangle-free graph G, $f(G) = \binom{n}{2}^{\mu(G)} n^{n-2\mu(G)}$.*

This conjecture suggests that the computation of $f(G)$ should be tractable for triangle-free graphs. We have not been able to prove this second conjecture, not even in the case of trees. The following lemma will help us settle a special case.

For a graph $G = (V, E)$ and two adjacent vertices u, v of G, define $\mathcal{L}_{uv}(G) = \{L \in \mathcal{L}(G) \mid L(u) = L(v)\}$.

Lemma 2 *Let G be a graph and uv an edge of G such that u is of degree 1 and v is of degree at most 2 in G. Then, for each $L \in \mathcal{L}(G)$, there exists $L' \in \mathcal{L}_{uv}(G)$ such that $L'(x) = L(x)$, for every $x \in V \setminus \{u, v\}$ and $|\mathcal{C}(L')| \leq |\mathcal{C}(L)|$.*

Proof. Write $U = \bigcup\{L(x) \mid x \in V \setminus \{u, v\}\}$ and observe that $L(u)$ is disjoint from U. If $L(v)$ too is disjoint from U, we set $L'(u) = L'(v) = L(u)$, and we set $L'(x) = L(x)$ for $x \in V \setminus \{u, v\}$. Then it is easy to check that $|\mathcal{C}(L')| \leq |\mathcal{C}(L)|$.

Now assume that $L(v)$ is not disjoint from U. Since L satisfies (ii), this means that v has another neighbour w, and that $L(v) \cap U = L(v) \cap L(w)$. Write $B = L(u) \cap L(v)$ and $C = L(v) \cap L(w)$, and then $A = L(u) \setminus B$, $P = L(v) \setminus (B \cup C)$, and $D = L(w) \setminus C$. Thus we have $L(u) = A \cup B$, $L(v) = B \cup C \cup P$, $L(w) = C \cup D$, with $A \cap B = B \cap C = B \cap P = C \cap P = C \cap D = \emptyset$, and $C \neq \emptyset$.

We can assume that $|A| \leq |C \cup P|$. Indeed, if $|A| > |C \cup P|$, we replace L by the assignment L^* obtained by removing $|A| - |C \cup P|$ elements of A from $L(u)$ and by setting $L^*(x) = L(x)$ for $x \in V \setminus \{u\}$. Clearly, $|\mathcal{C}(L^*)| \leq |\mathcal{C}(L)|$. The corresponding sets A^*, C^*, P^* of L^* satisfy $|A^*| = |C^* \cup P^*|$ so we can work with L^* instead of L.

We fix a mapping $a \mapsto \bar{a}$ from A to $C \cup P$.

Define L' by $L'(u) = L'(v) = L(u) = A \cup B$ and $L'(x) = L(x)$ if $x \in V \setminus \{u, v\}$. We claim that L' satisfies the conclusion of the lemma. Clearly, $L' \in \mathcal{L}_{uv}(G)$.

Let γ' be an L'-coloring of G. We denote elements of A and B by the corresponding lowercase letters, and we write, e.g., $\gamma'(u, v) = (a, b)$ as a shorthand for $\gamma'(u) = a \in A$, $\gamma'(v) = b \in B$. Observe that for $\gamma'(u, v)$, there are four possibilities: (a_1, a_2), (a, b), (b, a), and (b_1, b_2). Define a mapping γ by $\gamma(x) = \gamma'(x)$ for all $x \in V \setminus \{u, v\}$. We extend γ to an L-coloring of G as follows:

If $\gamma'(u, v)$ is either (a, b) or (b, a), set $\gamma(u, v) = (a, b)$.

If $\gamma'(u,v) = (b_1, b_2)$, set $\gamma(u,v) = (b_1, b_2)$.

If $\gamma'(u,v) = (a_1, a_2)$, set $\gamma(u,v) = (a_1, \bar{a}_2)$ if $\bar{a}_2 \neq \gamma'(w)$; otherwise set $\gamma(u,v) = (a_2, \bar{a}_1)$.

Clearly, γ is an L-coloring. Moreover, it is a routine matter to check that whenever γ', δ' are two L'-colorings with $\gamma'(V) \neq \delta'(V)$ then the corresponding L-colorings γ, δ satisfy $\gamma(V) \neq \delta(V)$. This implies that $|\mathcal{C}(L')| \leq |\mathcal{C}(L)|$. □

As an application, consider the class $\mathcal{B}$ of trees obtained from the trees on one or two vertices by iterating the following operation: add a vertex v of degree one, and then add a vertex u adjacent only to v.

Corollary 1 *If G is an n-vertex graph in $\mathcal{B}$, then $f(G) = \binom{n}{2}^{\mu(G)} n^{n-2\mu(G)}$.*

Proof. Let $v_1, u_1, \ldots, v_k, u_k$ be the vertices used in the recursive contruction of G. Note that u_k is pendant in G, hence $v_k u_k$ belongs to a maximum matching of G. Recursively this implies that $M = \{v_1u_1, \ldots, v_ku_k\}$ is a maximum matching of G, hence $k = \mu(G)$. Consider any $L \in \mathcal{L}(G)$. Applying the preceding lemma repeatedly, we obtain an assignment $L' \in \mathcal{L}(G)$ which satisfies $|\mathcal{C}(L')| = \binom{n}{2}^k n^{n-2k} \leq |\mathcal{C}(L)|$. □

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