Circular colorings of edge-weighted graphs

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Abstract

The notion of (circular) colorings of edge-weighted graphs is introduced. This notion generalizes the notion of (circular) colorings of graphs, the channel assignment problem, and several other optimization problems. For instance, its restriction to colorings of weighted complete graphs corresponds to the traveling salesman problem (metric case). It also gives rise to a new definition of the chromatic number of directed graphs. Several basic results about the circular chromatic number of edge-weighted graphs are derived.

1 Introduction

The theory of circular colorings of graphs has become an important branch of chromatic graph theory with many interesting results, leading to new methods and exciting new results. We refer to the survey article by Zhu [9].

In this paper, circular colorings of edge-weighted graphs are introduced. This notion contains, as special cases, several other optimization problems, e.g., the channel assignment problem. When restricted to complete graphs, it generalizes the traveling salesman problem and the hamiltonicity problem. Edge-weights need not be symmetric. This possibility leads to a new definition of colorings of directed graphs.

A *weighted graph* is a pair $G = (V, A)$, where V is the vertex set and $A : V \times V \to \mathbb{R}^+ \cup \{0\}$ are the edge-weights. For $u, v \in V$, we shall write

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$a_{uv} = A(u,v)$. The set $\vec{E}$ of ordered pairs, $\vec{E} = \{(u,v) \mid a_{uv} > 0\}$ is called the set of *directed edges* of G. The unordered pair $\{u,v\}$, shortly written as uv or vu, is an *edge* of G if $a_{uv} > 0$ or $a_{vu} > 0$. The set of edges is denoted by $E = E(G)$. For $(u,v) \in \vec{E}$, a_{uv} is called the *weight* of the directed edge (u,v). If $a_{uv} = 0$ implies that $a_{vu} = 0$ $(u,v \in V)$, then we say that the weights are *weakly symmetric*. If $a_{uv} = a_{vu}$ for all vertices $u,v \in V$, then the weights are said to be *symmetric*.

Throughout this paper, it will be assumed that G has no loops, i.e., $a_{vv} = 0$ for every $v \in V$.

For a positive real number p, denote by $S_p \subset \mathbb{R}^2$ the circle with radius $\frac{p}{2\pi}$ (hence with perimeter p) centered at the origin of $\mathbb{R}^2$. In the obvious way, we can identify the circle S_p with the set $\mathbb{R}/p\mathbb{Z}$. For $x, y \in S_p$, let us denote by $S_p(x,y)$ the arc on S_p from x to y in the clockwise direction, and let $\mathrm{d}(x,y)$ denote the length of this arc.

Let $G = (V,A)$ be a weighted graph with at least one edge. A *circular p-coloring* of G is a function $c : V \to S_p$ such that for every directed edge $(u,v) \in \vec{E}$, $\mathrm{d}(c(u),c(v)) \geq a_{uv}$. Since $\mathrm{d}(c(u),c(v)) + \mathrm{d}(c(v),c(u)) = p$, a necessary condition for existence of a circular p-coloring is that

$$p \geq \max\{a_{uv} + a_{vu} \mid u,v \in V\}. \tag{1}$$

The *circular chromatic number* $\chi_c(G)$ of the edge-weighted graph G is the infimum of all real numbers p for which there exists a circular p-coloring of G. It will be shown later that in the case of weakly symmetric weights, the infimum is indeed attained, i.e., there exists a circular $\chi_c(G)$-coloring of G.

The circular chromatic number of weighted graphs introduced above generalizes some other graph invariants and can be used as a model for several well-known optimization problems.

(a) If the weights are symmetric and all nonzero edge-weights are equal to 1, then $\chi_c(G)$ is the usual circular chromatic number of G (cf., e.g., [9]).

(b) If there is a function $f : V \to \mathbb{R}^+$, and weights of edges are defined as $a_{uv} = f(u)+f(v)$, then we get the notion of weighted circular colorings that were studied by Deuber and Zhu [1].

(c) Let G be an arbitrary (unweighted) graph with vertex set V. Let K_G be the complete graph with the same vertex set as G and edge-weights 1 (for edges of G) and 2 (for nonedges of G). Then $\chi_c(K_G) = |V|$ if and only if G has a hamiltonian cycle. If G has no hamiltonian cycle, then $\chi(K_G) = |V| + \mathrm{la}(G)$, where $\mathrm{la}(G)$ is the *linear arboricity* of G, i.e., the minimum number of paths whose vertex sets partition $V(G)$. This example shows that computation of the weighted circular chromatic number is NP-hard even for complete graphs with edge-weights 1 and 2 only.

(d) Let $D = [d_{uv}]_{u,v\in V}$ be the cost matrix for a metric traveling salesman problem (TSP), i.e., D satisfies the triangular inequality. Then every circular p-coloring of the weighted complete graph K_V (with edge-weights D) determines a tour of the traveling salesman of cost $\le p$, and vice versa. Therefore, $\chi_c(K_V)$ is the optimum for the considered TSP.

Another closely related area is the channel assignment problem for which we refer to the recent survey article by McDiarmid [6].

The notion of the circular chromatic number thus generalizes several well-known optimization problems and hence introduces the possibility to apply tools from one area into another one. As the edge-weights are not discrete integer values, one may also get use of some tools from continuous optimization. The author of this paper is quite optimistic about such possibilities which may yield better understanding of graph coloring theory. As an example we refer to an extension of Hajós theorem to circular colorings of edge-weighted graphs [8] which sheds some new light to why no nontrivial applications of this celebrated theorem are known.

2 Tight edges

In this section it will be shown that $\chi_c(G)$ can be expressed as an integer fraction a/k, where k is an integer smaller than n and a is the sum of at most n edge-weights. This implies, in particular, that $\chi_c(G)$ is a rational number if all edge-weights are rational. The proof of this result also implies that for every weighted graph G with weakly symmetric weights, there exists a circular p-coloring for $p = \chi_c(G)$.

Let c be a circular p-coloring of G. A directed edge (u, v) is said to be *tight* if $\mathrm{d}(c(u), c(v)) = a_{uv}$. A cycle $C = v_1v_2 \dots v_kv_1$ is *tight* if the directed edges $(v_1, v_2), \dots, (v_{k-1}, v_k)$, and (v_k, v_1) are all tight. If $k = 2$ and the edges (v_1, v_2) and (v_2, v_1) are both tight, then we also consider the 2-cycle $v_1v_2v_1$ to be a tight cycle. If C is a tight cycle, then the weight of C,

$$a(C) := a_{v_1v_2} + \dots + a_{v_{k-1}v_k} + a_{v_kv_1} \qquad (2)$$

is an integer multiple of p, and the number $w(C) = a(C)/p$ is called the *winding number* of C.

Lemma 2.1 *Suppose that the weights of G are weakly symmetric. If $p_0 = \chi_c(G)$, then there is a circular p_0-coloring of G which has a tight cycle.*

Proof. We may assume that G is connected. Suppose that $p_1 \geq p_0$ and that there is a circular p_1-coloring of G. Let $p \leq p_1$ be a real number such that G has a circular p-coloring c with maximum number of tight edges.

Let v_0 be a vertex of G and let $V_0 = \{v_0\}$. For $i = 1, 2, \ldots$, let $V_i = \{v \in V \mid \exists u \in V_{i-1}$ such that (u, v) is tight$\}$. If $V_0 \cup \cdots \cup V_{i-1} \neq V$, and $V_i = \emptyset$, we can shift the colors of $V \setminus (V_0 \cup \cdots \cup V_{i-1})$ counterclockwise until a new tight edge occurs. (Let us observe that the last conclusion uses the fact that the weights are weakly symmetric.) By the maximality of c, this does not happen. Consequently, for each $v \in V$ there is a path from v_0 to v consisting only of tight edges.

Suppose that there is no tight cycle. For $v \in V$, let $l(v)$ be the maximum of $a(P)$ taken over all directed walks P from v_0 to v which consist of tight edges only. Since there are no tight cycles, the values $l(v)$ are finite. By the definition of $l(v)$, if $l(u) > l(v)$, then the edge uv is not tight. This implies that for $p' = p - \varepsilon$, where $\varepsilon > 0$ is small enough, the mapping $c'(v) = l(v) \bmod p' \in \mathbb{R}/p'\mathbb{Z}$ determines a circular p'-coloring of G. By increasing the value of ε as much as possible, a new tight edge occurs. Observe that such maximum increase exists since the edge-weights are weakly symmetric. This contradicts the maximality of c and shows that there exists a tight cycle.

Clearly, if the cycle C is tight, then $p = a(C)/w(C)$. The winding number of C is bounded by the number of edges in C. Therefore, the same cycle can be tight for at most n distinct values of p. Since there are only finitely many cycles of G, this easily implies the statement of the lemma. □

Lemma 2.1 does not hold without the assumption of weak symmetry. For example, the acyclic tournament graph on n vertices with $a_{ij} = 1$ and $a_{ji} = 0$ for every pair of vertices $i < j$, has circular chromatic number equal to 1 but every circular p-coloring of this graph has $p > 1$.

Lemma 2.1 can also be applied to graphs whose weights are not weakly symmetric. This can be achieved by replacing the weight 0 of every directed edge (u, v) for which $a_{uv} = 0$ and $a_{vu} > 0$ by an infinitesimally small weight ε, and then letting ε tend to 0. This gives rise to a weighted graph with weakly symmetric weights, called the *ε-extension* of G. Tight cycles in the ε-extended graph can be described as cyclic sequences of edges, $C = v_1v_2\dots v_kv_1$, where each pair v_iv_{i+1} (indices modulo k) is an (undirected) edge of G. Then C is said to be a *weakly tight cycle* of G. Its weight $a(C)$ is defined by (2). Let us observe that a weakly tight cycle corresponds to an (undirected) cycle in G.

In trees and forests, there are only cycles of length 2. Therefore, Lemma 2.1 implies:

Corollary 2.2 *For every forest F with at least one edge,*

$$\chi_c(F) = \max\{a_{uv} + a_{vu} \mid u, v \in V\}.$$

Let C be a tight cycle with respect to a circular p-coloring. Then $p = a(C)/w(C)$. If the edge-weights are symmetric, then each edge-weight is at most $p/2$. Therefore, $a(C) \le np/2$, so the winding number is at most $n/2$. In the nonsymmetric case, the winding number is at most $n-1$.

Corollary 2.3 *The circular chromatic number of G is of the form $\frac{a(C)}{k}$ where C is a cycle in G (possibly of length 2) and k is an integer which is smaller than $n = |V|$ (and is smaller or equal to $n/2$ if the edge-weights are symmetric). If the weights of G are weakly symmetric, then the infimum in the definition of the circular chromatic number is attained.*

Another interesting corollary of Lemma 2.1 is the following result which shows that the methods of polyhedral combinatorics may be used in graph coloring theory.

Corollary 2.4 *The circular chromatic number of a given graph G, viewed as a function of the edge-wieghts, is a continuous, piecewise linear function with finitely many domains of linearity.*

3 Upper bounds

Let $G = (V, A)$ be a weighted graph. For $v \in V$, let $d^+_G(v) = \sum_{u\in V} a_{uv}$ and $d^-_G(v) = \sum_{u\in V} a_{vu}$, and let $D_G(v) = d^+_G(v) + d^-_G(v)$. The graph G is said to be *weakly p-degenerate* if every subgraph H of G contains a vertex v with $D_H(v) \le p$. The following result is obvious:

Proposition 3.1 *If G is a weakly p-degenerate graph, then $\chi_c(G) \le p$.*

The inequality of Proposition 3.1 has an analogy in the theory of usual graph colorings of unweighted graphs, except that Proposition 3.1 gives a bound which is worse by a factor of 2. The following example shows that the loss of factor 2 in comparison with the usual chromatic number cannot be improved.

Let G_n be the graph obtained from the complete graph K_n with unit edge-weights by adding a new vertex t_{uv} for each edge $uv \in E(K_n)$, and joining t_{uv} with the vertices u and v. The weight of each new edge joining t_{uv} with u or v is equal to $\kappa = \frac{n-1}{2}$.

Proposition 3.2 *The graph G_n is $(n-1)$-degenerate. If n is odd, then $\chi_c(G_n) = 2n - 4 + \frac{4}{n+1}$.*

Proof. It is clear by construction that G_n is $(n-1)$-degenerate.

Suppose now that n is odd. Let $p_0 = 2n - 4 + \frac{4}{n+1} = nr$, where $r = 2 - \frac{4}{n+1}$. Let $v_1, \dots, v_n$ be the vertices of $K_n \subset G_n$. By setting $c(v_i) = (i-1)r \in \mathbb{R}/p_0\mathbb{Z}$, a circular p_0-coloring of K_n is obtained which can be extended to a circular p_0-coloring of G_n. Therefore, $\chi_c(G_n) \le p_0$.

Suppose now that $p < p_0$, and suppose that there is a circular p-coloring c of G_n. Let $x_i = c(v_i)$. We may assume that the cyclic order of these colors on S_p is $x_1, \dots, x_n$. For $x \in S_p$, let $\bar{x}$ be the point of S_p which lies diametrically opposite x on the circle S_p. Let $r_i = \mathrm{d}(x_i, x_{i+1})$, $i = 1, \dots, n-1$, and let $r_n = \mathrm{d}(x_n, x_1)$.

Let α be the minimum distance of a point x_j from some $\bar{x}_i$, $i, j \in \{1, \dots, n\}$. Since the color $c(t_{v_i v_j})$ has distance at least κ from x_i and from x_j, it is necessary that $p/2 + \alpha \ge 2\kappa$. This implies:

$$\alpha \ge 2\kappa - \frac{p}{2} > 1 - \frac{2}{n+1}. \tag{3}$$

Clearly, any two distinct points x_i and x_j are at distance at least 1 on S_p. The same holds for the opposite points $\bar{x}_i$ and $\bar{x}_j$. Therefore, the $2n$ points $x_1, \dots, x_n$ and $\bar{x}_1, \dots, \bar{x}_n$ divide S_p in $2n$ segments, each of length more than $1 - \frac{2}{n+1}$. This implies that $p > 2n(1 - \frac{2}{n+1}) = p_0$, a contradiction.
□

On the other hand, the upper bound of $\Delta(G) + 1$ for the usual chromatic number has a generalization to the weighted case. Such a result, derived for the setting of channel assignment problems, was recently obtained by McDiarmid [5]. His proof can be extended to work also in the case of circular colorings of edge-weighted graphs.

Theorem 3.3 *Let* $\Delta^+(G) = \max\{d_G^+(v) + a_{vu} \mid u, v \in V\}$. *Then* $\chi_c(G) \leq \Delta^+(G)$.

Proof. By applying the ε-extension described after Lemma 2.1, and at the end taking the limit as ε tends to 0, we may assume that the edge weights are weakly symmetric.

Let us first assume that all edge-weights are integers. Let $v_1, \ldots, v_n$ be the vertices of G. We assign them colors $c(v_i) \in \mathbb{N}$ ($i = 1, \ldots, n$) by applying the following "greedy" algorithm. For every value of $\alpha = 0, 1, 2, \ldots, \Delta^+(G)$, traverse all vertices $v_1, \ldots, v_n$. During this traversal, assign color $c(v_i) := \alpha$ to every uncolored vertex v_i whose already colored neighbors do not object to this choice, i.e., every vertex v_j adjacent to v_i which has already received the color $c(v_j)$, must satisfy the condition $\alpha - c(v_j) \geq a_{v_j v_i}$.

We claim that at the end all vertices are colored and that $c(v_i) \leq d_G^+(v_i)$, $i = 1, \ldots, n$. Suppose that v_i has not been colored for $\alpha = 0, 1, \ldots, d$. Then for each such α, there was a vertex $v_{j(\alpha)}$ such that $\alpha - c(v_{j(\alpha)}) < a_{v_{j(\alpha)} v_i}$. If v_j is a neighbor of v_i, then $|\{\alpha \mid j(\alpha) = j\}| \leq a_{v_j v_i}$. This implies that $d + 1 \leq d_G^+(v_i)$.

Using the above conclusion, it is easy to see that c determines a circular $\Delta^+(G)$-coloring of G.

If the edge-weights are not integers, we proceed as follows. Let N be a large positive real number, and let $a'_{uv} = \lceil N a_{uv} \rceil$. Denote by G' the weighted graph thus obtained. Clearly, $\chi_c(G') \geq N \cdot \chi_c(G)$. By the above, $\chi_c(G') \leq \Delta^+(G') = \max\{d_{G'}^+(v) + a'_{vu}\} \leq N\Delta^+(G) + n$. Therefore, $\chi_c(G) \leq \Delta^+(G) + \frac{n}{N}$. Since N is arbitrarily large, $\chi_c(G) \leq \Delta^+(G)$. □

Let G^T be the weighted graph whose weight function a^T is the transpose of a, i.e., $a^T_{uv} = a_{vu}$. Every circular p-coloring of G determines a circular p-coloring of G^T obtained by the reflection of the circle S_p. Hence, $\chi_c(G^T) = \chi_c(G)$. Note that $\Delta^+(G^T) = \Delta^-(G) = \max\{d_G^-(v) + a_{uv} \mid u, v \in V\}$. Therefore, Theorem 3.3 implies that $\chi_c(G) \leq \Delta^-(G)$.

4 Local changes

The following transformation gives a new edge-weighting but preserves the chromatic number. Let t be a real number and let v be a vertex of G. For each neighbor u of v, define new edge-weights $a'_{vu} = a_{vu} + t$ and $a'_{uv} = a_{uv} - t$. If the absolute value of t is small enough so that all new edge-weights remain nonnegative, then the resulting weighted graph has the same circular chromatic number as G.

If u, v are nonadjacent vertices of G, let $G_{u,v}$ denote the graph obtained from G by identifying u and v into a new vertex w. The edge-weights in $G_{u,v}$ are the same as in G except that for each $z \in V(G_{u,v}) \setminus \{w\}$, the weights a'_{zw} of zw and a'_{wz} of wz are equal to $a'_{zw} = \max\{a_{zu}, a_{zv}\}$ and $a'_{wz} = \max\{a_{uz}, a_{vz}\}$, respectively. Then

$$\chi_c(G) \leq \chi_c(G_{u,v}) \tag{4}$$

since every circular p-coloring c of $G_{u,v}$ determines a coloring of G by setting $c(v) = c(u) := c(w)$.

If u, v are adjacent vertices of G, then we define $G_{u,v}$ in the same way except that the weights of edges incident with w are determined differently:

$$a'_{zw} = \max\{a_{zu}, a_{zv} - a_{uv}\} \qquad \text{and}$$

$$a'_{wz} = \max\{a_{uz}, a_{uv} + a_{vz}\}.$$

If c is a circular p-coloring of $G_{u,v}$, then setting $c(u) = c(w)$ and $c(v)$ to be the point on S_p such that $\mathrm{d}(c(u), c(v)) = a_{uv}$ yields a circular p-coloring of G. This shows that

$$\chi_c(G) \leq \chi_c(G_{u,v}). \tag{5}$$

5 Colorings and orientations

In this section we shall assume that the weights of edges of G are weakly symmetric, and we shall point out at the end of the section that Theorems 5.2 and 5.3 also hold without this assumption.

A mapping $T : \vec{E} \to \{-1, 1\}$ is an *(edge-)orientation* of G if for every $(u, v) \in \vec{E}$, $T(u, v) = -T(v, u)$. We say that the edge $uv \in E$ is *oriented* from u to v if $T(u, v) = 1$.

Let T be an orientation. A mapping $t : E \to \mathbb{R}$ is a *tension* if for every cycle $C = v_1 v_2 \ldots v_k v_1$ of G, we have

$$\sum_{i=1}^{k} T(v_i, v_{i+1})\, t(v_i v_{i+1}) = 0. \tag{6}$$

The tension t is *p-admissible* if for every edge $uv \in E$ oriented from u to v, $a_{uv} \leq t(uv) \leq p - a_{vu}$.

Lemma 5.1 *A weighted graph G with weakly symmetric weights has a circular p-coloring if and only if there is an orientation of G for which there exists a p-admissible tension.*

Proof. If c is a circular p-coloring, the following determines an orientation T and a p-admissible tension t. Fix a point $o \in S_p \backslash c(V)$. Suppose that $uv \in E$. If $o \notin S_p(c(u), c(v))$, then we set $T(u,v) = 1$ and $t(uv) = \mathrm{d}(c(u), c(v))$. Otherwise, we set $T(u,v) = -1$ and $t(uv) = \mathrm{d}(c(v), c(u))$.

Conversely, let T be an orientation and t a p-admissible tension. We may assume that G is connected. Let D be a spanning tree of G, and let v_0 be a vertex of G. For $v \in V$, let $P = v_0 v_1 \dots v_{k-1} v_k$ be the path in D from v_0 to $v = v_k$. Set $l(v) = \sum_{i=0}^{k-1} T(v_i, v_{i+1}) t(v_i v_{i+1})$ and $c(v) = l(v) \bmod p \in \mathbb{R}/p\mathbb{Z} \approx S_p$. Consider an arbitrary directed edge $(u,v) \in \vec{E}$ such that $T(u,v) = 1$. If $uv \in E(D)$, then $l(v) = l(u) + t(uv)$. The same relation holds if $uv \notin E(D)$ by (6). Therefore, $a_{uv} \le l(v) - l(u) \le p - a_{vu}$. This implies that $\mathrm{d}(c(u), c(v)) = t(uv) \ge a_{uv}$ and that $\mathrm{d}(c(v), c(u)) = p - t(uv) \ge a_{vu}$. This shows that c is a circular p-coloring of G. □

The following result of Hoffman [4] and Ghouila-Houri [2] gives a necessary and sufficient condition for existence of an admissible tension. Let us mention that a *directed cycle* of G is a cycle in the digraph with edge set $\vec{E}$. Each cycle of G determines two directed cycles (one in each direction), and each edge of G determines a directed cycle of length 2.

Theorem 5.2 *Let G be a weighted graph with a given orientation T, and let $l, u : E \to \mathbb{R}$ be nonnegative functions such that $0 \le l(e) \le u(e)$ for every $e \in E$. Then there exists a tension t such that $l(e) \le t(e) \le u(e)$ for every $e \in E$ if and only if for every directed cycle $C = v_1 v_2 \dots v_k v_1$ ($k \ge 2$) of G, we have*

$$\sum_{T(v_i,v_{i+1})=-1} l(v_i v_{i+1}) \le \sum_{T(v_i,v_{i+1})=1} u(v_i v_{i+1}), \tag{7}$$

where i takes values $1, \dots, k$, and $v_{k+1} = v_1$. Moreover, if l and u are rational (integer) valued, then t can be chosen to be rational (integer) valued.

Observe that Hoffman's condition in Theorem 5.2 must also hold for the reverse directed cycle $C' = v_1 v_k \dots v_2 v_1$. This gives:

$$\sum_{T(v_i,v_{i+1})=1} l(v_i v_{i+1}) \le \sum_{T(v_i,v_{i+1})=-1} u(v_i v_{i+1}). \tag{8}$$

For a directed cycle $C = v_1 \dots v_k v_1$ of G, let C^+ (resp. C^-) be the set of edges of C whose T-orientation is the same (resp. opposite) as on C. For p-admissible tensions we have $l(v_i v_{i+1}) = a_{v_i v_{i+1}}$ if $T(v_i, v_{i+1}) = 1$, and we have $u(v_i v_{i+1}) = p - a_{v_i v_{i+1}}$ if $T(v_i, v_{i+1}) = -1$. Condition (8) is equivalent to the following requirement:

$$a(C) = \sum_{i=1}^{k} a_{v_i v_{i+1}} \leq p|C^-|. \tag{9}$$

This shows that G has no circular p-coloring if and only if for every orientation T there exists a cycle C such that (9) is violated. This implies:

Theorem 5.3 *Let G be a weighted graph. Then*

$$\chi_c(G) = \min_T \max_C \frac{a(C)}{|C^-|} \tag{10}$$

where the minimum is taken over all (acyclic) orientations T of G, and the maximum is over all directed cycles of G.

A version of Theorem 5.3 for usual colorings of graphs was proved by Minty [7]. A version for circular colorings was proved by Goddyn, Tarsi, and Zhang [3] who also pointed out that the same result can be proved in the setting of matroids. Our extension to weighted graphs can also be extended to matroids with weighted elements.

A corollary of Lemma 5.1 and Theorems 5.2 and 5.3 is that the infimum in the definition of the cyclic chromatic number is attained if the weights are weakly symmetric. In fact, $\chi_c(G)$ is of the form $\frac{a}{k}$ where $a = a(C)$ is a sum of at most n edge-weights and k is a positive integer smaller or equal to n. In the case of symmetric edge-weights, $k \leq n/2$ since either C or its inverse C' has at most $n/2$ negatively oriented edges, so that $\max\{\frac{a(C)}{|C^-|}, \frac{a(C')}{|C'^-|}\} \geq \frac{a(C)}{n/2}$. Cf. also Corollary 2.3.

Theorems 5.2 and 5.3 also hold for weighted graphs whose weights are not weakly symmetric. To see this, we replace G by its ε-extension G_ε and take the limit as ε tends to 0. Of course, the maximum in (10) has to be taken over all directed cycles in the extended graph G_ε.

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